NOTE ON SOME LOW-ORDER PERFECT SQUARED SQUARES

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1. Introduction. In the Mathematical Games department of the *Scientific American* for November 1958 **(12)**, William H. Tutte gave an entertaining account of the researches of himself and three colleagues (C. A. B. Smith, A. H. Stone, and R. L. Brooks) concerning "squaring the square." The problem is to subdivide a square into a finite number of non-overlapping smaller squares of which no two are the same size. At first suggested as impossible, a number of different solutions have been published. The first one, by Sprague in 1939 **(9)**, showed a square divided into 55 smaller unlike squares. The following year, the four collaborators published solutions of one square divided into 26, two squares into 28, and others into higher numbers of, squares **(5, 10)**. *Science Service* listed the solution of the problem as among the important achievements in mathematics for the year. In 1948 T. H. Willcocks published a dissection into 24 squares **(13)**, which is still the lowest one known. Bouwkamp **(1, 2)**, Tutte **(11)**, and Willcocks **(14)** have described general methods for the construction of squares dissected into a finite number of smaller different squares. Another general empirical method is here described, by means of which 24 perfect squares of order below 29 were constructed. These squares are listed at the end of this note, with the addition of others previously described in the literature, making a total of 35. The term "low order" is used for squared squares of order 28 and lower. This limit was chosen to avoid too long a list. While only one 29-order perfect square has been published, and some twenty of orders from 33 to 85, numerous perfect squares of orders above 28 can be obtained by some of the methods described in this note. Twenty-nine perfect squares of order 29 were collected without attempting to apply fully the methods to this and higher orders. However, some 29-order and a 30-order square are mentioned to illustrate some methods when no lower-order square was available, and the smallest-size 29- and 30-order squares obtained have been added at the end of the list.

The empirical approach has been through the problem of building up rectangles from unequal squares. The empirical construction of numbers of such rectangles is comparatively simple, and a theoretical treatment developed by the above-mentioned group **(5)** has led to the exhaustive construction and enumeration of such (simple) squared rectangles up to a certain order and the publication of a number of tables **(1, 3, 4)**. From such rectangles, pairs

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can be found having certain relationships enabling them to be fitted together, with added squares, to form squared squares.

The terminology generally used is as follows. The term "squared rectangle" is used for a rectangle divided into squares. The "order" is the number of squares into which the rectangle is divided, and the component squares are the "elements" of the dissection. The squared rectangle is "perfect" if all the squares into which it is divided are different; otherwise it is "imperfect." The squared rectangle is "compound" if it contains a smaller squared rectangle as a part, and "simple" if it does not. A compound squared rectangle is trivially compound if it can be divided into a square and a squared rectangle **(1)**; obviously two squared rectangles can be formed from any one by simply adding a square to either of the two sides, and squares can be added to these to form other rectangles, and so on. Rectangles compounded in this manner can be used in many of the constructions which follow.

A dissection is usually expressed by giving the order, the sides of the rectangle, and the sides of each of the square elements, the elements being expressed in integers not having any common factor. The notation introduced by C. J. Bouwkamp **(1)** lists the elements from left to right, beginning with those at the top of the rectangle and proceeding down, enclosing within brackets or parentheses the contiguous elements whose upper sides lie on the same horizontal line. A simple perfect squared rectangle may be represented by $P_\alpha(x, y)$, where x and y are the lengths of the sides and α is the order, which, however, may be omitted; an individual element may be represented by the length of its side enclosed in parentheses. A perfect squared square may be represented by $S_\alpha(x)$.

Two different rectangles having proportional sides are "conformal" and when brought to the same size, if necessary, are "congruent." Two conformal rectangles have none, one, two, etc., elements in common if such is the case when the rectangles are the same size or are brought to the same size by multiplying the respective elements of each by the appropriate factor.

The most extensive catalogue of squared rectangles is the one recently published by C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema **(4)**, which lists in various tables the simple perfect squared rectangles from order 9 (the lowest possible) through 15. The number of such rectangles is 3,663. Imperfect simple squared rectangles of orders 9 through 15, numbering 431, are also listed. Goldberg **(8)** estimated that it would require 16,000 volumes of the same size to carry the tables through order 23. The tables were constructed by electronic computer; Duijvestijn **(7)** has published a thesis describing the method and giving the programme.

2. Review of empirical methods. While theoretical methods of constructing perfect squared squares have been developed, these have resulted only in squares of fairly high order; as stated by Tutte, "If the merit of a perfect square is measured by the smallness of its order, then the empirical method of cataloguing the perfect rectangles has proved superior to our beautiful theoretical method" **(12)**. But the squares constructed by the empirical method have all been compound, that is, they contain a squared rectangle as part of the figure. A simple squared square of order 38 was published in 1950 by Tutte **(11)**; Duijvestijn **(7)** in 1962 published one of order 37 constructed by Willcocks in 1959, and has shown that a simple perfect square below order 20 is not possible. The 37-order simple perfect square is given at the end of the list, since its description is not readily available.

Methods of forming low-order squared squares, with some minor extensions, will be briefly reviewed **(1, 2, 11, 14)**. The various methods, or types, are numbered and given a catch phrase. In what follows it is to be understood that any two squared rectangles are brought to the appropriate size and then do not have any elements the same, unless the contrary is indicated.

Method 1. *Two rectangles.* If two simple perfect squared rectangles are found, the sides of which are so proportioned that the sum of the two short sides is equal to the common long side, they can be put together to form a square which will be perfect if no elements are duplicated. One such pair can be found in the tables:

$$P_{15}(451, 914) + P_{15}(463, 914) = S_{30}(914).$$

The next two methods which follow use this same principle with compounded rectangles.

Method 2. *Two rectangles and one square.* Figure 1 shows two simple perfect squared rectangles so proportioned that they form a square with the addition

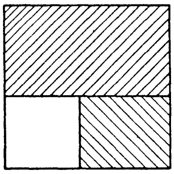

FIGURE 1

of the supplemental square shown at the lower left corner. If the sides of the smaller rectangle are x, y, the sides of the larger one are $y, x + y$ and the added square is of side x. The 26-order perfect square numbered 6 in the list and the two 28-order squares numbered 28 and 30 were obtained by this method **(5, 10, 11)**.

Method 3. *Two rectangles and two squares.* In Figure 2, two conformal perfect rectangles of sides x and y, not having any elements the same, form a perfect square with the addition of two supplemental squares of sides x and y. The 28-order square numbered 27 in the list was obtained by this method **(5, 10)**. The other twelve pairs of conformal simple squared rectangles not having

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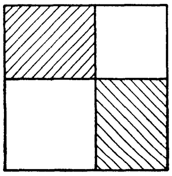

FIGURE 2

any elements the same, listed in the Bouwkamp *et al.* tables, result in twelve squares of order 30, 31, and 32; cf. **(4)**.

Method 4. *Two rectangles and three squares.* Two rectangles having the relative proportions indicated in the following equation can form squares with three added elements:

$$P(2x, 2y) + P(3x - y, y + x) + (y - x) + (x + y) + (2x) = S(3x + y), \quad x < y < 3x.$$

The 28-order perfect square numbered 33 in the list is of this type.

This process can be continued with more added squares, but low-order perfect squares do not appear to have been obtained.

Method 5. *Modified simple rectangle.* If one of the corner squares of either of the two conformal perfect rectangles of Figure 2 is removed and the other rectangle fitted in as shown in Figure 3, a perfect square, of smaller size and order diminished by one, is formed if the two added squares are different

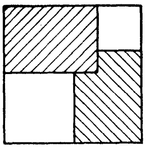

FIGURE 3

from any component square already present. Since there are eight different corner squares there are eight possible combinations for each pair of conformal rectangles but these may not all yield perfect squares. Only 22 perfect squares, of orders 29, 30, and 31, can be obtained by this method from the thirteen pairs of conformal perfect rectangles listed by Bouwkamp *et al.* in **(4)**. The smallest of these are:

$$S_{29}(820 - a) = 4.P_{13}(99, 106) + P_{15}(396, 424) - (a) + (396 - a) + (424 - a),$$

where a is in turn 116 and 96.

Either of the rectangles, but not both, may be trivially compound, that is made up of a squared rectangle with a square added to one side, etc. If one of the rectangles is compounded once, the two simple rectangles used have sides (x, y) for one and $(y, x + y)$ for the other, the same relationship as in Method 2. The results in such a case are illustrated by the four 27-order squares numbered 14, 15, 17, 18 in the list **(14)** which utilize the same two squared rectangles as the 26-order square numbered 6 in the list.

If one of the rectangles is compounded twice, the two simple rectangles used have sides (x, y) for one and $(x + y, x + 2y)$ for the other. An example of six squares is:

$$S_{29}(1570 - a) = P_{12}(304, 329) + P_{14}(633, 937) - (a) + (304) + (633) + (633 - a) + (937 - a),$$

where a is in turn 229, 193, 171, 158, 113, and 54. The square formed when a is 171 is given by Tutte **(11)**.

Since a corner square of one of the two congruent rectangles is discarded, this method can be used with two congruent simple rectangles having a corner element of one the same as any element of the other, but no low-order results appear to have been obtained.

This method of combining two squared rectangles is usually called "overlapping," since it was first suggested only for two rectangles having a corner square of one the same as a corner square of the other, but the term is somewhat unsuitable for the general method.

Method 6. *Modified imperfect rectangle.* The same method described above can also be used with two congruent simple rectangles, one perfect and the other imperfect, with only two elements the same, one of these being the corner element that is removed. This combination does not appear to have yielded any low-order results. However, low-order results have been obtained with two conformal rectangles, one perfect and the other imperfect, and also compound, with two elements the same, one a corner element and the other an adjacent element. Such a squared rectangle is compound as the two equal squares form a rectangle; they have been designated as "trivially imperfect" rectangles **(5)**.

Compound rectangles are not included in the Bouwkamp *et al.* tables and the use of the type mentioned depends upon constructing numbers of them and then looking for conformal perfect rectangles. They were used by Willcocks **(14)** in obtaining the lowest-order perfect square yet known, number 1 in the list, and also numbers 5 and 20 in the list. Either rectangle can be trivially compound. As will be indicated in the next section, the squares numbered 8, 13, 16, 23, and 24 in the list were also obtained indirectly by this method.

The methods reviewed above utilize two squared rectangles, one of which may be modified by removal of a square; combinations of three rectangles, etc., can be devised, but these are not apt to yield low-order perfect squares. Sprague's 55-order perfect square (side 4205) was formed by adding together five perfect rectangles with five additional squares.

3. Another empirical method. The first approach was as follows. Construct what might be called a deficient perfect square. This is a square which is divided into smaller squares, all different, and one rectangle. The same simple method, described in **(12)** and elsewhere, for constructing perfect rectangles can be used. First sketch a rectangle cut up into smaller rectangles. Imagine all of these rectangles except a corner one to be squares and work out what the relative sizes of the squares must be. Thus in Figure 4 the side of one small square is designated by x and of an adjoining square by y; the

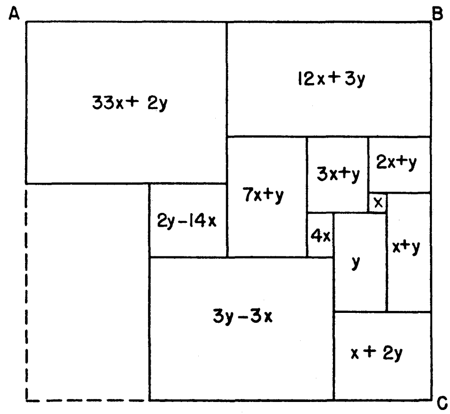

FIGURE 4

sides of the other assumed squares can be computed from these two, except the lower left-hand corner, shown in dotted lines, is not assumed to be a square. Then the sum of the sides of the component squares along AB is set equal to the sum of the sides along BC in order to make the whole figure a square. This gives the equation $2y = 29x$, and x can be taken equal to 2 and y to 29. The result is a square of side 235 divided into 12 different squares and one rectangle 94×111, discernible in Figure 5. In the same manner the upper left-hand corner of Figure 4 and the upper right-hand corner can be assumed to remain rectangles and two more deficient perfect squares obtained.

After a deficient perfect square is found, the next step is to see whether there is a perfect squared rectangle (including compounded rectangles) of the same shape as the single rectangle of the deficient square, to be fitted in to form a complete squared square. This is done from the tables of simple perfect squared rectangles. For the example given, there is a 94×111 simple squared rectangle that fits; no component squares are duplicated and a perfect square of order 25 and side 235 is formed (number 2 in the list). Incidentally, it may be noted that the included squared rectangle (which also appears in other squares) is the same as that in the 24-order perfect square.

This method requires the construction of large numbers of deficient squared squares. As patterns for such constructions, any one of the simple squared

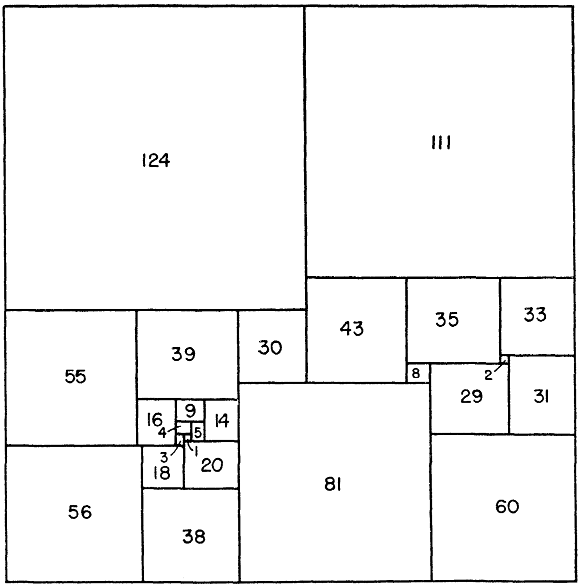

FIGURE 5

rectangles in the Bouwkamp *et al.* tables can be used, the imperfect ones as well as the perfect ones (the example given is patterned after the simple perfect rectangle 528×545 of order 13). This means that 4,094 different patterns are available. Less than a third of these were tried. Most of the patterns resulted in one or more of these deficient squares.

At first blush it would seem that the chances of finding a match would be small, but the "law of unaccountable recurrence" mentioned by Tutte **(12)** seems to operate. A substantial proportion of the deficient perfect squares that were found contained a rectangle of the same shape as some tabulated perfect squared rectangle. However, the same law of unaccountable recurrence saw to it that in most of these cases some squares were duplicated, leaving a much smaller number of perfect squares. The over-all result was that nearly 2 per cent of the patterns tried resulted in perfect squares.

Four common forms emerged in this construction of deficient squared squares, the reasons for which can be readily seen on studying the figures produced from the various patterns. In one form the figure obtained on removing the rectangle and the two largest squares was nothing but a simple perfect rectangle with a corner element removed; as to these, the method merely duplicated what was labelled Method 5 in the preceding section and became trivial. In another form the figure so obtained was a squared rectangle with a corner element removed, this element being the same as an adjacent element; the compound rectangles mentioned under Method 6. As to this form, the method turned into one of constructing such compound rectangles and then applying method 6, and hence is not distinct. The squares numbered 8, 13, 16, 23, and 24 in the list resulted from these. The other two forms were distinct and non-trivial and are listed as separately numbered methods or types.

Method 7. *Simple deficient perfect squares.* Some of the deficient perfect squares could not be reduced to the other figures mentioned in the preceding paragraph, or to the figures mentioned in the following paragraph; these might be called simple deficient perfect squares. The 25-order perfect square numbered 3 in the list, and shown in Figure 6, is a result. The rectangular space to be filled is 97×193, which is reduced to 96×97 by inserting a square of side 97 and filled by $P_{11}(96, 97)$.

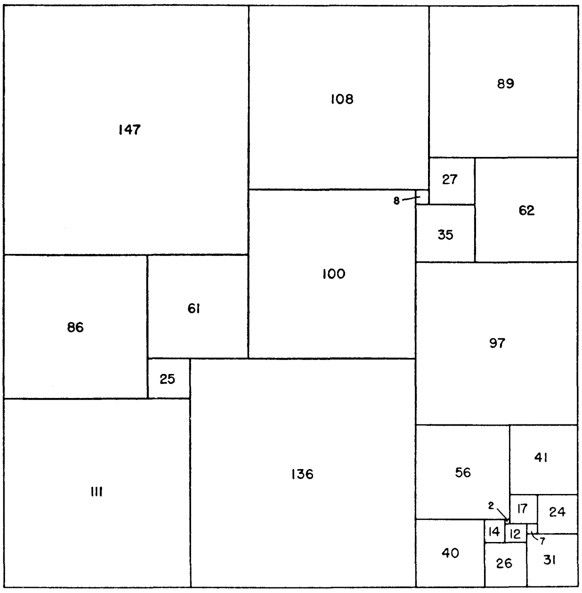

FIGURE 6

Method 8. *Squared hexagons with two opposite sides equal.* The last form was the most numerous and resulted in the majority of the low-order perfect squares obtained. Referring to Figure 5, removal of the rectangle and the two large squares results in a figure of a shape redrawn in Figure 7. In this figure, which is divided into unequal squares, the two sides lettered x are equal and the two re-entrant sides, a (the smaller) and b, are unequal. But the side a

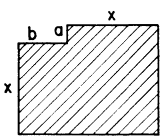

FIGURE 7

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may project outwardly instead of inwardly, in which case a is negative in what follows. The figure is a squared hexagon (necessarily having five right angles and one angle of 270°), with two opposite sides equal. Squares divided into squares and one rectangle can be formed from these squared hexagons in three different ways. If a square of side x is placed on the upper side x, and a square of side $x + a$, adjacent to it, a deficient squared square of side $2x + a$ with a rectangle of sides x and $x + a - b$ is formed. Also if a square of side x is placed to the left of the vertical side x, and a square of side $x + b$ above it, a deficient squared square of side $2x + b$ with a rectangle of sides x and $x - a + b$ is formed. Third, if a square of side b is placed on the side b of the hexagon, a square of side $x + b$ containing a rectangle of sides $b - a$ and x is formed. For some hexagons one or the other of these types of squares is not possible owing to the appearance of negative sides. If none of the squares added to the hexagon is the same as a square already present, if a squared rectangle of the same shape as the rectangular space can be found, and if none of the elements of this rectangle are the same as a square already present, a perfect squared square is formed. Seventeen of the squares in the list were formed in this manner: those numbered 2, 4, 9, 10, 12, 21, 26, 29, and 31 are of the first type; those numbered 7, 11, 32, 34, and 35 of the second type; and those numbered 19, 22, and 25 of the third type. In some instances the rectangular space is filled by a compounded rectangle.

The method which began as the direct construction of deficient perfect squares as first described, in large part, turned into the construction, in the same manner, of squared rectangles with a corner element and an adjacent element the same, and of squared hexagons with two opposite sides equal. The patterns of squared rectangles can be used in manners additional to that described at the beginning of this section to obtain additional figures of these two kinds. Referring to Figure 4, if the three larger corner quadrilaterals are discarded at once, the remainder of the figure is a right-angled hexagon which can be used directly twice, once to obtain a squared hexagon with two opposite sides equal and again to obtain a squared rectangle with a corner element (removed) and an adjacent element the same. Also, if the quadrilateral in the lower right-hand corner is removed from Figure 4, the rest of the figure is a right-angled hexagon, which can be used to obtain another squared hexagon with opposite sides equal. Other patterns obtained from the tables of squared rectangles can be treated in the same or an analogous manner, with an even greater variety of choices in some instances.

As has been stated, this general method was not carried through to the end of the available patterns which could be tried; only a few, and mainly the lower-order, patterns were used, to the extent practical without a computer. Furthermore, the method can be extended to form deficient perfect squares with a side or an internal rectangle, instead of a corner rectangle; to form doubly deficient perfect squares, that is, a squared square containing two rectangular spaces; and to a wider use of squared hexagons. Hence, it is apparent that more perfect squares of low order can be obtained, and the possibility of obtaining one below order 24, while remote, is not yet ruled out. It may also be noted that two trivially imperfect rectangles having certain relationships, and several different pairs of appropriately related squared hexagons with two opposite sides equal, may be put together, with added squares, to form simple squared squares of low order, but those obtained in this way were all imperfect.

4. List of perfect squares. The following list tabulates the perfect squares to order 28 arranged by order and length of side, with a description or identification, and giving references in the case of those previously described in the literature.

	Order	*Side*	*Description*
1.	24	175	(55, 39, 81) (16, 9, 14) (4, 5) (3, 1) (20) (56, 18) (38) (30, 51) (64, 31, 29) (8, 43) (2, 35) (33). Willcocks **(13, 14)**.
2.	25	235	(124, 111) (43, 35, 33) (55, 39, 30) (2, 31) (8, 29) (81) (16, 9, 14) (4, 5) (60) (3, 1) (20) (56, 18) (38).
3.	25	344	(147, 108, 89) (27, 62) (100, 8) (35) (86, 61) (97) (25, 136) (111) (56, 41) (17, 24) (40, 14, 2) (12, 7) (31) (26).
4.	26	384	(179, 205) (99, 80) (54, 63, 88) (125, 9) (47, 25) (41, 58) (22, 91) (69) (24, 17) (7, 20, 48) (18, 13) (5, 28) (23).
5.	26	492	(57, 59, 56, 95, 225) (3, 14, 39) (55, 2) (53, 11) (25) (17, 142) (125) (24, 60, 141) (255, 12) (36) (96) (15, 126) (111). Willcocks **(14)**.
6.	26	608	(209, 205, 194) (11, 183) (44, 172) (168, 41) (1, 43) (42) (85) (231, 95, 61, 108, 113) (34, 27) (7, 20) (136) (123, 5) (118). Brooks *et al.* **(3)**. Method 2. $P_{12}(231, 377) + P_{13}(377, 608) + (231)$. Four perfect squares of order 27 can be obtained from these same two rectangles. First add a square of side 377 to the smaller rectangle to make it conformal with the larger one, then remove in succession the corner squares 136, 118, 113, and 95, and add two squares of sides $377 - a$ and $608 - a$, where a is the side of the removed square. This results in four different perfect squares of order 27 and sides 849, 867, 872, and 890. Willcocks **(14)**.
7.	27	325	(196, 60, 69) (36, 15, 9) (6, 35, 37) (21) (49, 8) (41, 2) (39) (71, 58) (129, 67) (13, 45) (52, 32) (77) (62, 5) (57).
8.	27	408	(55, 36, 53, 264) (19, 17) (70) (74) (6, 64) (80) (22, 42) (82, 20) (62) (27, 30, 63, 144) (117, 51, 3) (33) (15, 81) (66).
9.	27	600	(333, 267) (66, 201) (85, 60, 95, 159) (25, 35) (100, 10) (62, 78) (69, 132) (46, 16) (108, 51) (94) (82, 18) (64) (6, 63) (57).
10.	27	618	(327, 291) (105, 186) (105, 154, 69) (99, 75) (66, 38) (28, 10) (24, 51) (45, 141) (7, 3) (1, 16, 137) (4) (123) (11) (121) (96).

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	Order	*Side*	*Description*
11.	27	645	(354, 128, 163) (50, 32, 46) (18, 14) (11, 152) (71) (68) (1, 70) (69) (39, 33, 60, 159) (6, 27) (291, 108) (87) (183, 12) (171).
12.	27	648	(333, 315) (18, 42, 108, 147) (114, 123, 90, 24) (66) (225, 39) (80, 34) (25, 22, 76) (186) (3, 19) (46, 16) (35) (121, 5) (116).
13.	27	825	(200, 253, 372) (102, 43, 55) (31, 12) (2, 251) (22, 47) (28, 3) (25) (202) (132, 240) (321, 156, 108) (48, 87, 213) (165, 39) (126).
14.	27	849	Method 5. See note under item 6 in this list. Willcocks **(13, 14).**
15.	27	867	Method 5. See note under item 6 in this list. Willcocks **(13, 14)**.
16.	27	869	(281, 264, 324) (26, 61, 177) (160, 68, 44, 9) (35) (170, 154) (24, 116) (92) (428, 117) (16, 138) (64, 122) (123, 58) (65, 253) (188).
17.	27	872	Method 5. See note under item 6 in this list. Willcocks **(14)**.
18.	27	890	Method 5. See note under item 6 in this list. Willcocks **(14)**.
19.	28	374	(146, 109, 119) (66, 43) (33, 86) (117, 29) (23, 53) (88, 30) (169) (55, 39, 111) (16, 9, 14) (4, 5) (3, 1) (20) (56, 18) (38).
20.	28	577	(224, 123, 129, 101) (28, 73) (117, 6) (111, 52) (7, 66) (59) (113, 51, 21, 23, 16) (32, 337) (19, 2) (25) (11, 8) (65) (62) (240). Willcocks **(13, 14)**.
21.	28	714	(423, 128, 163) (50, 32, 46) (18, 14) (11, 152) (71) (68) (1, 70) (69) (120, 171) (291, 96, 36) (24, 45, 51) (60) (39, 6) (228) (195).
22.	28	732	(273, 192, 106, 76, 85) (30, 46) (37, 48) (86, 34, 16) (18, 70, 11) (59) (52) (168, 291) (183, 90) (3, 165) (93) (42, 249) (276) (207).
23.	28	741	(134, 259, 348) (81, 53) (28, 25) (7, 18) (105, 4) (11) (29) (61, 227) (166) (138, 210) (255, 108, 69, 99) (29, 30) (27, 183) (156) (147).
24.	28	765	(148, 236, 381) (60, 88) (32, 28) (4, 40, 106, 202) (36) (76) (10, 96) (86) (57, 84, 81, 159) (327, 114) (3, 78) (87) (12, 255) (213).
25.	28	765	(237, 243, 128, 76, 81) (52, 24) (19, 62) (43) (115, 45, 20) (5, 100) (25) (70) (231, 6) (225, 309) (297, 159) (96, 213) (138, 21) (117).
26.	28	824	(436, 488) (140, 248) (143, 201, 92) (132, 100) (88, 55) (32, 68) (33, 19, 3) (60, 88) (17, 187) (14, 5) (4, 13) (9) (164) (157) (128).
27.	28	1015	(593, 192, 230) (154, 38) (116, 67, 85) (49, 18) (103) (247, 72) (175) (222, 164, 207, 422) (80, 41, 43) (39, 2) (37, 215) (200, 22) (178). Stone **(10)**. Method 3. $P_{13}(422, 593) + P_{13}(422, 593) + (422) + (593)$.

	Order	*Side*	*Description*
28.	28	1015	(280, 372, 363) (188, 92) (93, 270) (119, 261, 84) (177) (165, 23) (142) (163, 120, 167, 183, 382) (43, 30, 47) (13, 17) (219) (215) (199). Tutte **(10, 11)**. Method 2. $P_{13}(382, 633) + P_{14}(633, 1015) + (382)$.
29.	28	1071	(588, 483) (225, 258) (134, 253, 201) (81, 144) (111, 147) (65, 69) (43, 22) (282) (19, 50) (21, 1) (20) (84) (73, 230) (219, 36) (183) (157).
30.	28	1073	(244, 153, 248, 169, 259) (91, 62) (79, 90) (29, 33) (364) (360) (349) (465, 252, 156, 89, 111) (67, 22) (133) (135, 88) (221) (213, 39) (174). Tutte **(11)**. Method 2. $P_{13}(465, 608) + P_{14}(608, 1073) + (465)$.
31.	28	1089	(585, 504) (213, 126, 165) (200, 253, 132) (87, 39) (48, 156) (372, 108) (102, 43, 55) (264) (31, 12) (2, 251) (22, 47) (28, 3) (25) (202).
32.	28	1113	(639, 199, 275) (123, 76) (113, 238) (80, 43) (20, 23) (17, 3) (14, 125) (111) (204, 270) (474, 165) (87, 51, 66) (36, 15) (21, 330) (309).
33.	28	1137	(544, 593) (145, 122, 228, 49) (332, 310) (23, 99) (168) (92, 7) (85, 150) (280, 65) (22, 54, 234) (215) (212, 110, 32) (86) (102, 8) (94). Method 4. $2 \cdot P_{12}(272, 321) + P_{13}(495, 593) + (49) + (544) + (593)$.
34.	28	1166	(700, 466) (180, 286) (74, 106) (247, 223, 176, 62) (42, 32) (10, 414) (114) (290) (52, 68, 103) (219, 28) (80) (33, 35) (31, 2) (140) (111).
35.	28	1166	(700, 466) (180 ,286) (74, 106) (267, 203, 176, 62) (42, 32) (10, 414) (114) (290) (80, 123) (199, 68) (52, 28) (13, 15) (11, 2) (140) (131).
	29	468	(146, 109, 119, 55, 39) (16, 9, 14) (4, 5) (3, 1) (20) (56, 18) (38) (66, 43) (94) (33, 86) (117, 29) (23, 53) (88, 30) (263) (205).
	30	566	(128, 114, 149, 81, 94) (68, 13) (107) (14, 24, 22, 54) (106, 36) (2, 20) (26) (85, 80, 52) (8, 12) (70) (66) (28, 131) (5, 103) (332) (234).
	37	1947	(728, 378, 406, 435) (350, 28) (405, 29) (464) (648, 347, 83) (184, 206, 98) (10, 454) (108) (162, 22) (336) (245, 102) (20, 142) (122) (210, 54) (56, 189) (250, 594) (571, 133) (438, 94) (344). Willcocks **(7)**.

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