TRIANGULATION OF FIBRE BUNDLES

H. PUTZ

In this paper we consider the following problem. Let (E, M, N, π) be a differentiable fibre bundle, where E is the total space, M the base space, N the fibre, and $\pi\colon E \to M$ the projection map. Then, given a C^r triangulation (f, D) of M, can one obtain a C^r triangulation (F, K) of E such that the induced map $f^{-1}\pi F\colon K \to D$ is linear? R. Lashof and M. Rothenberg **(3)** have obtained this result for vector bundles.

Using methods quite different from theirs, we obtain a solution in the general case. The methods we use are the geometric methods developed by J. H. C. Whitehead. **(7)** in his triangulation of differentiable manifolds, as found in **(5)**. In fact, our solution consists of generalizing his techniques in a fibre bundle setting. As a corollary of our main result, we are able to triangulate vector bundles whose base space is a finite-dimensional, locally finite, simplicial complex.

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1. Introduction. All manifolds will be understood to be submanifolds of some Euclidean space. All simplicial complexes will be understood to be locally finite and to lie in some Euclidean space. All maps of simplicial complexes will be understood to be piecewise C^r ($1 \leqslant r \leqslant \infty$ and r fixed throughout this paper), that is, C^r on each simplex.

1.1. *Definition.* Let K be a simplicial complex, and $f\colon K \to R^q$. Then f is an *immersion* if $df_b\colon \overline{\mathrm{St}}(b, K) \to R^q$ is one-to-one, where $df_b(x) = Df(b)\cdot(x - b)$ (we denote the Jacobian of f at b by $Df(b)$ and the closed star of b in K by $\overline{\mathrm{St}}(b, K)$). It is an *embedding* if it is also a homeomorphism. In addition, if $f(K) = E$, a differentiable manifold, it is a C^r *triangulation* of E.

1.2. *Definition.* Let $f\colon K \to R^q$, and δ map K continuously to the positive reals. The map $g\colon K' \to R^q$ (K' a subdivision of K) is a $\delta(x)$-*approximation* to f if $||f(b) - g(b)|| < \delta(b)$ and $||df_b(x) - dg_b(x)|| < \delta(b)||x - b||$ for each b in K and each x in $\overline{\mathrm{St}}(b, K')$.

1.3. THEOREM. *For $f\colon K \to R^q$ an immersion (embedding) there is a $\delta(x)$ such that any $\delta(x)$-approximation to f is an immersion (embedding).*

This is Theorem 8.8 of **(5)**.

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1.4. *Remark.* Let K_1 be a subcomplex of K and K_1' a subdivision of K_1. We obtain the *standard subdivision* K' *of* K *induced by* K'_1 by subdividing only those simplices whose interiors lie in $\mathrm{St}(K_1, K) - K_1$ as follows. First, barycentrically subdivide all 1-simplices. Assuming $\mathrm{Bd}\,\sigma$ is subdivided, join the barycentre of σ to this subdivision to get a subdivision of σ.

1.5. *Definition.* Let

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ {\scriptstyle \pi_K}\downarrow & f & \downarrow{\scriptstyle \pi} \\ \mathrm{D} & \longrightarrow & \mathrm{M} \end{array}$$

be a commutative array. The tuple (E, M, π) is a differentiable fibre bundle with fibre N, where E and M are embedded as submanifolds of some Euclidean spaces. (We ignore the group of the bundle since it will play no role in our results.) K and D are simplicial complexes and $\pi_K: K \to D$ is a linear map. The maps F and f are C^r maps. Such an array will be called a *diagram*.

1.6. *Remark.* In the special case where $E = R^m \times R^n$, $M = R^m$, and $\pi = p_1$ an easy computation shows that $dF_b(x) = (df_{\pi_K(b)}(\pi_K(x)), d(p_2F)_b(x))$, where x lies in $\overline{\mathrm{St}}(b, K)$ and p_1, p_2 are the projection maps.

We now give a definition of what it means for the images of two complexes in Euclidean space to fit together nicely.

1.7. *Definition.* Let $F_1: K_1 \to R^q$ and $F_2: K_2 \to R^q$ be homeomorphisms whose images are closed subsets of $F_1|K_1| \cup F_2|K_2|$. Then (F_1, K_1) and (F_2, K_2) *intersect in a subcomplex* if $L_i = F_i^{-1}(F_1|K_1| \cap F_2|K_2|)$ is a subcomplex of K_i for $i = 1, 2$, and if $F_2^{-1}F_1: L_1 \to L_2$ is a linear isomorphism. They intersect in a *full subcomplex* if L_i is a full subcomplex of K_i, i.e., if σ is a simplex in K_i, all of whose vertices lie in L_i, then σ is in L_i. We may then construct a complex K and a homeomorphism $F: K \to R^q$ such that the following is commutative:

$$\begin{array}{ccc} K_1 & \searrow^{F_1} & \\ {\scriptstyle i_1}\downarrow & F & \searrow \\ K & \longrightarrow & R^q \\ {\scriptstyle i_2}\uparrow & & \nearrow \\ K_2 & \nearrow_{F_2} & \end{array}$$

where i_1 and i_2 are linear isomorphisms of K_1 and K_2 respectively, with subcomplexes of K whose union equals K. We call (F, K) the *union* of (F_1, K_1) and (F_2, K_2).

1.8. *Remark.* Let

$$\begin{array}{ccc} K_1 & \xrightarrow{F_1} & E \\ {\scriptstyle \pi_{K_1}}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} K_2 & \xrightarrow{F_2} & E \\ {\scriptstyle \pi_{K_2}}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

be two diagrams. Suppose that f is a one-to-one map and that (F_1, K_1) and (F_2, K_2) intersect in a full subcomplex. Let (F, K) be their union. Then we can, in an obvious manner, construct a diagram

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ {\scriptstyle \pi_{K}}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

1.9. *Definition.* Let

$$\begin{array}{ccc} P & \xrightarrow{F} & R^m \times R^n \\ {\scriptstyle \pi_{P}}\downarrow & f & \downarrow{\scriptstyle p_1} \\ C & \longrightarrow & R^m \end{array}$$

be a diagram. Let $f'\colon C \to R^m$. We define a map $F'\colon P \to R^m \times R^n$ as follows:

$$F'(x) = (f'\pi_P(x), p_2F(x)).$$

We call F' the *shift of* F *over* f' and obtain the *shift diagram*

$$\begin{array}{ccc} P & \xrightarrow{F'} & R^m \times R^n \\ {\scriptstyle \pi_{P}}\downarrow & f' & \downarrow{\scriptstyle p_1} \\ C & \longrightarrow & R^m \end{array}$$

1.10. *Remark.* Using Remark 1.6, it is easily seen that, if $F, f,$ and f' are immersions, embeddings, or closed embeddings, then F' is an immersion, embedding, or closed embedding, respectively.

2. The triangulation of fibre bundles.

2.1. THEOREM. *Suppose that we have two diagrams*

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ {\scriptstyle \pi_{K}}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array} \quad \textit{and} \quad \begin{array}{ccc} L & \xrightarrow{G} & E \\ {\scriptstyle \pi_{L}}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

where E and M are non-bounded, and F, G, and f are closed embeddings, and that $\delta(x) > 0$ (defined on the disjoint union $K \cup L$). Then there are two diagrams

$$\begin{array}{ccc} K' & \xrightarrow{F'} & E \\ \pi_K \Big\downarrow & f & \Big\downarrow \pi \\ D & \longrightarrow & D \end{array} \quad \textit{and} \quad \begin{array}{ccc} L' & \xrightarrow{G'} & E \\ \pi_L \Big\downarrow & f & \Big\downarrow \pi \\ D & \longrightarrow & M \end{array}$$

where K' and L' are subdivisions of K and L. The maps F' and G' are $\delta(x)$-approximations to F and G; (F', K') and (G', L') intersect in a full subcomplex and their union is a closed embedding.

Proof. See §4.

2.2. COROLLARY. *Let (E, M, π) be a differentiable fibre bundle, with E and M non-bounded. Let $f: D \to M$ be a C^r triangulation. Then there is a diagram*

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ \pi_K \Big\downarrow & f & \Big\downarrow \pi \\ D & \longrightarrow & M \end{array}$$

where $F: K \to E$ is a C^r triangulation.

Proof. M^m has $m + 1$ coordinate neighbourhoods (U_i, h_i), $i = 1, \ldots, m + 1$, such that $(U_i \times N, \phi_i)$ is a local product structure for E, where N is the fibre of E. Let $D = \cup K_i$, $i = 1, \ldots, m + 1$, where K_i is a subcomplex of D, such that $f: K_i \to U_i$. Now, since N is a differentiable manifold, there is (**5**, pp. 101–103) a C^r triangulation $g: L \to N$. Let F_i be the composite map

$$\phi_i(f \times g): K_i \times L \to E.$$

Then we have the diagrams

$$\begin{array}{ccc} K_i \times L & \xrightarrow{F_i} & E \\ p_1 \Big\downarrow & f & \Big\downarrow \pi \\ D & \longrightarrow & M \end{array}$$

By Theorem 2.1, we have diagrams

$$\begin{array}{ccc} (K_i \times L)' & \xrightarrow{G_i} & E \\ p_1 \Big\downarrow & f & \Big\downarrow \pi \\ D & \longrightarrow & M \end{array}$$

where G_i is a $\delta(x)$-approximation to F_i, $i = 1, \ldots, m + 1$ such that $\{(G_i, (K_i \times L)')\}$ intersect in a full subcomplex and their union is a closed C^r embedding. This union covers E and, therefore, is a C^r triangulation of E if $\delta(x)$ is chosen small enough.

2.3. COROLLARY. *Corollary 2.2 holds in the case where the fibre N has a boundary, but M is still assumed to be non-bounded.*

Proof. We can choose a product neighbourhood of the boundary of E, H_1: Bd $E \times [0, 1) \rightarrow E$ such that the following is commutative:

$$\begin{array}{ccc} \mathrm{Bd}\, E \times [0,1) & \xrightarrow{H_1} & E \\ {\scriptstyle \pi p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{1} & M \end{array}$$

(see 5.2). Applying Corollary 2.2 to (Bd E, M, π), we have the diagram

$$\begin{array}{ccc} J & \xrightarrow{H_2} & \mathrm{Bd}\, E \\ {\scriptstyle \pi_J}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{f} & M \end{array}$$

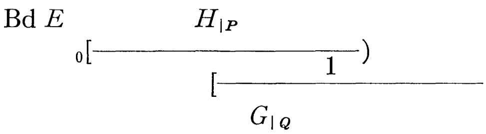
Hence, we have the diagram

$$\begin{array}{ccccc} J \times [0,1) & \xrightarrow{H_2 \times 1} & \mathrm{Bd}\, E \times [0,1) & \xrightarrow{H_1} & E \\ {\scriptstyle \pi_J p_1}\downarrow & & \downarrow{\scriptstyle \pi p_1} & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{f} & M & \xrightarrow{1} & M \end{array}$$

Let $H = H_1(H_2 \times 1)$: $J \times [0, 1) \rightarrow E$. By Corollary 2.2 we also have the diagram

$$\begin{array}{ccc} L & \xrightarrow{G} & \mathrm{Int}\, E \\ {\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{f} & M \end{array}$$

where G is a C^r triangulation of the interior of E. By suitably subdividing, we may obtain subcomplexes P of $J \times (0, 1)$ and Q of L such that $H_{|P}$ and $G_{|Q}$ are closed embeddings and they cover E. A schematic diagram is given below:



We now may apply the proof of Theorem 2.1 (see 4.4), since the process of fitting together takes place sufficiently far away from the boundary so that

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we are in the non-bounded case. Furthermore, with $\delta(x)$ chosen small enough, $H'_{|P'}$ and $G'_{|Q'}$ will cover E. Their union (F, K) will be the desired C^r triangulation of E. We also note that, by staying away from the boundary, the subcomplex J of P will not be subdivided, nor H changed there. Hence, $F^{-1}H\colon J \to K$ will be a linear isomorphism of J with a subcomplex of K. Thus, the triangulation F of E is an extension of the triangulation H_2 of Bd E.

2.4. COROLLARY. *Corollary 2.2 holds with no assumptions on the boundedness of E or M.*

Proof. By Corollaries 2.2 and 2.3 we need only consider the case where M has a boundary. We form the double of M, M_*, and the double of D, D_*. (We assume that the subcomplex of D which triangulates Bd M is a full subcomplex.) Let $f_*\colon D_* \to M_*$ be the obvious "double" of f. Let $r\colon M_* \to M$ be the obvious retract, and let E^* be the induced bundle over M_*. Then, by using Corollary 2.2 or Corollary 2.3, depending on whether the fibre is with or without boundary, we obtain a diagram

$$\begin{array}{ccc} L & \xrightarrow{G} & E^* \\ {\scriptstyle \pi_L}\downarrow & f_* & \downarrow{\scriptstyle \pi} \\ D_* & \longrightarrow & M_* \end{array}$$

where the map G is a C^r triangulation. The diagram is commutative; therefore, if we let $K = \pi_L^{-1}(D)$ (K is a complex, since π_L is linear), $G_{|K} = F$, $\pi_{L|K} = \pi_K$, we obtain our desired result.

2.5. COROLLARY. *Let (E, M, π) be a differentiable fibre bundle and $f\colon D \to M$ a C^r triangulation. Let N_1 be a closed, non-bounded submanifold contained in the interior of N (the fibre of E), which is compatible with the fibre bundle structure. Let E_1 denote the sub-bundle of E whose fibre is N_1 (e.g. sphere bundles, etc.); then we have a diagram*

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ {\scriptstyle \pi_K}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

where $F\colon K \to E$ is a C^r triangulation and $F^{-1}(E_1)$ is a subcomplex of K.

Proof. By Corollary 2.4 we have two diagrams

$$\begin{array}{ccc} L & \xrightarrow{G} & E \\ {\scriptstyle \pi_L}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \xrightarrow{H} & E_1 \subset E \\ {\scriptstyle \pi_P}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

We apply the proof of Theorem 2.1 (see 4.4). Note that all of our approximations, i.e. secant maps and extensions, were done locally in $R^m \times R^n$ (where n is the dimension of N). Now, H maps simplices into $R^m \times R^q \subset R^m \times R^n$ (where q is the dimension of N_1), and note that, if H' is a change in H obtained by secant approximation and extension, then H' also maps simplices into $R^m \times R^q \subset R^m \times R^n$ (see the proof of Theorem 3.3). Thus, there are approximations G': $L' \rightarrow E$ and H': $P' \rightarrow E_1$ which intersect in a full subcomplex and their union is the desired C^r triangulation of E.

2.6. Lemma. *Let $\xi = (E, M, \pi)$ be a vector bundle, M a non-bounded C^r manifold. Then (E, M, π) is a differentiable C^r vector bundle.*

Proof. Let $\gamma = (E_2, G_{p,n}, \pi_2)$ denote the classifying (C^∞) bundle for n-dimensional vector bundles. By **(4**, p. 50**)** we have a continuous map g: $M \rightarrow G_{p,n}$ such that $g^*(\gamma)$ is isomorphic to ξ. By **(5**, p. 39**)** we may construct a map g': $M \rightarrow G_{p,n}$ which is C^r and which is homotopic to g. By **(6**, p. 53**)**, g' is a C^r map so $g'^*(\gamma)$ has a natural C^r structure induced by γ. The isomorphism of $g'^*(\gamma)$ with ξ gives ξ a C^r structure.

2.7. Theorem. *Given a vector bundle $\xi = (E, K, \pi)$, where K is a finite-dimensional locally finite simplicial complex, there is a triangulation G: $L \rightarrow E$ such that πG is linear.*

Proof. There is a subdivision K' of K which may be embedded as a subcomplex of a rectilinear triangulation of some Euclidean space R^q (for finite K, see **(5**, p. 71**)**, and for infinite K proceed locally). Let $N^2(K')$ denote the second regular neighbourhood of K'. There is a retraction map r: $N^2(K') \rightarrow K'$; see **(2**, p. 70; the proofs there go through for infinite complexes**)**. Now, $N^2(K')$ is an open subset of R^q and has, therefore, by **(1**, p. 143**)**, a (rectilinear) simplicial structure such that K' is a subcomplex. Also, as an open subset, $N^2(K')$ is a differentiable manifold which we shall denote by M. The map r: $M \rightarrow K'$ induces a vector bundle E_1 over M which by Lemma 2.6 is a differentiable vector bundle (the bundle E_1 over K' is our original bundle E over K'). By Corollary 2.2 we have the diagram

$$\begin{array}{ccc} C & \xrightarrow{F} & E_1 \\ {\scriptstyle \pi_C}\downarrow & 1 & \downarrow{\scriptstyle \pi} \\ N^2(K') & \longrightarrow & M \end{array}$$

where F is a C^r triangulation of E_1. Let L be the subcomplex $\pi_C^{-1}(K')$. Then $F_{|L}$: $L \rightarrow E$ is a triangulation of E, and $\pi F_{|L} = \pi_{C|L}$: $L \rightarrow K'$ is linear and, therefore, $\pi F_{|L}$: $L \rightarrow K$ is linear.

2.8. Theorem. *Suppose that we have two diagrams*

$$\begin{array}{ccc} K & \xrightarrow{F} & E \\ {\scriptstyle \pi_K}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} L & \xrightarrow{G} & E \\ {\scriptstyle \pi_L}\downarrow & f & \downarrow{\scriptstyle \pi} \\ D & \longrightarrow & M \end{array}$$

where F is a closed C^r embedding, and f and G are C^r triangulations, and assume that $F(K) = \pi^{-1}(\pi(F(K)))$. Then there exist approximations F' and G' to F and G respectively, with the usual commutative diagrams, such that G' is a C^r triangulation and $F'(K') = \pi^{-1}(\pi(F'(K')))$. Furthermore, $(G')^{-1}F'$ is an isomorphism of K' with a subcomplex of L'.

Proof. When M and the fibre N of E are non-bounded, this is just Theorem 2.1, with $\delta(x)$ taken small enough that G' is a C^r triangulation and

$$F'(K') = \pi^{-1}(\pi(F'(K'))).$$

We consider now the case where the fibre N of E has a boundary but M is non-bounded. Let N_* denote the double of N, and E_* the induced double of E. We apply Theorem 2.1 to the bundle $\pi: E_* \to M$. As in the proof of Corollary 2.5, we see that the approximation G' to G will carry Bd L into Bd E. By choosing $\delta(x)$ small enough we can ensure that $G'(\text{Bd } L) = \text{Bd } E$. Hence we have $G'(L')$ contained in E. Again if $\delta(x)$ is small enough we shall get $G'(L') = E$. If we now consider Bd $E \cap F(K)$ instead of Bd E, then similar reasoning yields $F'(K') = \pi^{-1}(\pi(F'(K')))$ if $\delta(x)$ is chosen small enough.

Finally we consider the case where M has a boundary and the fibre N of E is with or without boundary. Let M_* denote the double of M and D_* the double of D (we assume that the subcomplex of D which triangulates Bd M is a full subcomplex). By $r: M_* \to M$ we denote the obvious retract and by E^* we denote the induced bundle over M_*. Similarly we form K^*, L^* and the "doubles" f_*, F_*, G_*. We now have the hypotheses for the theorem for the bundle $\pi: E^* \to M_*$. By one of the previous two cases and by restriction to the bundle $\pi: E \to M$ we obtain the conclusions of the theorem for the induced bundle.

As an immediate corollary we obtain the following.

2.9. Corollary. *Given the hypotheses of the theorem above and assuming that $F: K \to E$ is a C^r triangulation, then there are subdivisions of K and L that are isomorphic.*

Theorem 2.8 also gives us the following result due to Hirsch and Mazur (unpublished).

2.10. Corollary. *Let $G: L \to E$ and $f: D \to M$ be C^r triangulations such that the composite map $\pi_L = f^{-1}\pi G$ is piecewise linear. Then $\pi_L: L \to D$ is a piecewise linear fibre bundle (i.e. piecewise linearly locally trivial) with fibre a complex P smoothly triangulating the fibre of $\pi: E \to M$.*

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Proof. Fix a C^r triangulation $h\colon P \to N$ of the fibre N of E. By subdividing D, if necessary, we can find about each point of D an open set W contained in a subcomplex Q such that $f(Q)$ is contained in U, where $(U \times N, \phi)$ is a local product neighbourhood for E. Thus we have the two diagrams

$$\begin{array}{ccc} K = Q \times P & \xrightarrow{F} & E \\ \pi_K \downarrow & f & \downarrow \pi \\ D & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} L & \xrightarrow{G} & E \\ \pi_L \downarrow & f & \downarrow \pi \\ D & \longrightarrow & M \end{array}$$

where π_K is projection into the first factor and $F = \phi(f \times h)$. We note that $F\colon K \to E$ is a closed C^r embedding and $F(K) = \pi^{-1}(\pi(F(K)))$. We apply Theorem 2.8 and obtain subdivisions K' and L' and an isomorphism $\theta = (G')^{-1}F'$ of K' with a subcomplex of L' together with the associated commutative diagram. Furthermore, $\theta(K') = \pi_L^{-1}(\pi_L(\theta(K')))$. By **(1**, p. 143**)** the open sets W and $\pi_L^{-1}(W)$ have simplicial structures so that the inclusion maps into D and L' respectively are piecewise linear. Also the simplicial structure on W induces a simplicial structure on $W \times P$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc} (W \times P)' & \xrightarrow{i} & K' & \xrightarrow{\theta} & L' & \xleftarrow{i} & \pi_L^{-1}(W) \\ \pi_K \downarrow & i & \downarrow \pi_K & 1 & \downarrow \pi_L & i & \downarrow \pi_L \\ W & \longrightarrow & D & \longrightarrow & D & \longleftarrow & W \end{array}$$

where i denotes the obvious inclusion map. Now by **(5**, Corollary 7.9**)** any two simplicial complexes which have the same polytope have a common simplicial subdivision. Therefore we obtain a piecewise linear homeomorphism $i^{-1}\theta i\colon (W \times P)' \to \pi_L^{-1}(W)$. Thus $\pi_L\colon L \to D$ is a piecewise linear fibre bundle with fibre P.

As an application of our triangulation of vector bundles we have the following theorem.

2.11. THEOREM. *Let M be a non-bounded C^r $(1 \leqslant r \leqslant \infty)$ submanifold contained in the interior of the C^r manifold V. Then any C^r triangulation of M may be extended to a C^r triangulation of V. (If $f\colon D \to M$ is a C^r triangulation of M, an extension of f is a C^r triangulation $j\colon J \to V$ of V such that $j^{-1}f$ is a linear isomorphism of some subdivision of D with a subcomplex of J.)*

Proof. Let (N, M, π) be the normal bundle of the embedding $i\colon M \to V$. By the tubular neighbourhood theorem we may choose a Riemannian metric on N such that if $N([0, 1])$ denotes the subset of N of all vectors of length less than or equal to 1, then there is a diffeomorphism P of $N([0, 1])$ with a closed neighbourhood of $i(M)$ in V. (M need not be compact because there exists a C^r map δ: $M \rightarrow R^+$ such that the subset of N,

$$N(\delta(x)) = [v| \, ||v|| \leqslant \delta(x), \pi(v) = x],$$

is diffeomorphic with a neighbourhood of $i(M)$ in V. Using the diffeomorphism of N with itself taking the vector v to $\delta(x)v$ yields the desired result.) By Corollary 2.2 there is a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{F} & N \\ \pi_K \downarrow & f & \downarrow \pi \\ D & \longrightarrow & M \end{array}$$

where π_K is linear and F: $K \rightarrow N$ is a C^r triangulation. The zero section of K is isomorphic with a subdivision D' of D. (The rest of the proof now follows the proof of Theorem 10.6 of **(5)** where it is shown that a C^r triangulation of the boundary of a given manifold can be extended to a triangulation of the manifold.) By subdividing K, if necessary, let K_0 be the subcomplex of K whose image under F contains $N([0, 5/6])$ and is contained in $N([0, 1])$. Define $g = PF$: $K_0 \rightarrow V$. Let h: $L \rightarrow V$ be a C^r triangulation of V. Let L_0 be the subcomplex of L whose image under h contains $V - P(N([0, 4/5]))$. We assume, by subdividing L if necessary, that the image of L_0 is disjoint from $P(N([0, 3/4]))$. By Theorem 10.4 of f**(5)** we obtain approximations g': $K_0' \rightarrow V$ and h': $L_0' \rightarrow V$ to g and h respectively, which intersect in a full subcomplex whose union j: $J \rightarrow V$ is a C^r triangulation of V. Furthermore, $g' = g$ and $K_0' = K_0$ on D', the zero section of K. (The reader is referred to Chapter II of **(5)** for the relevant details.) Thus we have the commutative diagram

$$\begin{array}{ccc} K_0' & & \\ j_1 \downarrow & \searrow g' & \\ J & \xrightarrow{j} & V \\ j_2 \uparrow & \nearrow h' & \\ L_0' & & \end{array}$$

where j_1 and j_2 are linear isomorphisms with subcomplexes of J. Therefore $j_1 = j^{-1}g' = j^{-1}g = j^{-1}f$: $D' \rightarrow J$ is a linear isomorphism with a subcomplex of J.

3. Secant approximations and extensions.

3.1. *Definition.* Let f: $K \rightarrow R^q$ and K' be a subdivision of K. The linear map $L_{K'}$, defined by $L_{K'}(v) = f(v)$ for each vertex v of K', is called the *secant map induced by* f.

3.2. THEOREM. *Let $f\colon K \to R^q$ and K_1 be a finite subcomplex of K. Given $\delta > 0$, there is a δ-approximation $g\colon K' \to R^q$ to f such that* (1) *g equals the secant map induced by f on K_1',* (2) *$g = f$ and $K' = K$ outside* $\mathrm{St}(K_1, K)$.

This is Theorem 9.7 of **(5)**.

3.3. THEOREM. *Let*

$$\begin{array}{ccc} P = Q \cup A & \xrightarrow{F} & R^m \times R^n \\ \pi_P \downarrow & & \downarrow p_1 \\ C & \xrightarrow{f} & R^m \end{array}$$

be a diagram, where A is a finite subcomplex. Let $F_{|Q}$, $F_{|A}$, and f be closed embeddings. Let $S = F^{-1}F(A)$, $P_0 = \overline{\mathrm{St}}(S, P)$, $P_1 = \overline{\mathrm{St}}(P_0, P)$, and assume f is linear on $\pi_P(P_1)$. Then given $\delta > 0$, *there is a δ-approximation $G\colon P' \to R^m \times R^n$ to F which preserves commutativity and which is such that $(*)G_{|Q'}$ and $G_{|A'}$ are closed embeddings which intersect in a full subcomplex and whose union is a closed embedding. Furthermore, $G = F$ and $P' = P$ outside* $\mathrm{St}(P_1, P)$.

Proof. Let G be the δ-approximation given in the previous theorem. By the proof of Lemma 10.2 of **(5)** $(*)$ holds. Thus, it only remains to prove commutativity, i.e. that $f\pi_P = p_1 G$. If a point x lies in $P - \mathrm{St}(P_1, P)$, then $G(x) = F(x)$, so there is no problem.

(1) Let x lie in P_1, say $x = \sum \alpha_i v_i$. Then, since G and p_1 are linear,

$$p_1 G(x) = \sum \alpha_i p_1 G(v_i) = \sum \alpha_i p_1 F(v_i) = \sum \alpha_i f\pi_P(v_i).$$

Now, since f is linear on $\pi_P(P_1)$, we have $f\pi_P(x) = \sum \alpha_i f\pi_P(v_i)$.

(2) Let x lie in $\mathrm{St}(P_1, P) - P_1$. Assume x lies in the interior of σ and G has been defined on $\mathrm{Bd}\,\sigma$ so that commutativity holds there. For a point x of σ, $x = ty + (1-t)\sigma_0$, where y is in $\mathrm{Bd}\,\sigma$, σ_0 is the barycentre of σ, and $0 \leqslant t \leqslant 1$. Let $\alpha(t)$ be a monotonic C^∞ function, which is 0 for $t \leqslant 1/3$ and 1 for $t \geqslant 2/3$. We now recall (proof of Theorem 9.7 of **(5)**) how G is extended over σ:

$$G(x) = F(x) + \alpha(t(x))(G(y(x)) - F(y(x))).$$

Since p_1 is linear and

$$p_1 G(y(x)) = p_1 F(y(x)) = f\pi_P(y(x)),$$

we have

$$p_1 G(x) = p_1 F(x) = f\pi_P(x).$$

3.4. LEMMA. *Let*

$$\begin{array}{ccc} P & \xrightarrow{F} & E \\ \pi_P \downarrow & & \downarrow \pi \\ D & \xrightarrow{f} & M \end{array}$$

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be a diagram. Suppose that E and M are non-bounded, Let P_1 be a finite subcomplex of P. Given $\delta > 0$, there is an $\epsilon > 0$ such that (*) any ϵ-approximation $G: P'_1 \to E$ to $F_{|P_1}$ can be extended to a δ-approximation $H: P' \to E$ to F such that $H = F$ and $P' = P$ outside $\mathrm{St}(P_1, P)$. *Furthermore, if G preserves commutativity so does H.*

Proof. We extend H over $\mathrm{St}(P_1, P) - P_1$ first on 1-simplices, then on 2-simplices, and so on. Without loss of generality, we may assume that F and f map simplices into coordinate neighbourhoods, so we may replace (E, M, π) by $(R^m \times R^n, R^m, p_1)$; then (*) follows by Lemma 9.8 of **(5)**. The commutativity assertion is proved as in the preceding theorem.

4. Proof of Theorem 2.1. Our first lemma is similar to Theorem 3.3, except that in Theorem 3.3 we required f to be linear on a subcomplex and we do not require this here.

4.1. Lemma. *Let*

$$\begin{array}{ccc} P = Q \cup A & \xrightarrow{F} & R^m \times R^n \\ \pi_P \downarrow & & \downarrow p_1 \\ C & \xrightarrow{f} & R^m \end{array}$$

be a diagram, where A is a finite subcomplex. Let $F_{|Q}$, $F_{|A}$, and f be closed embedding. Then given $\delta > 0$, there is a δ-approximation $G: P''' \to R^m \times R^n$ to F which preserves commutativity and which is such that (*) $G_{|Q'''}$ and $G_{|A'''}$ are closed embeddings which intersect in a full subcomplex and whose union is a closed embedding. Furthermore, $G = F$ and $P''' = P$ outside $\mathrm{St}^5(F^{-1}F(A), P)$.

Proof. Let $S = F^{-1}F(A)$, $P_0 = \overline{\mathrm{St}}(S, P), P_1 = \overline{\mathrm{St}}(P_0, P)$; therefore, $\pi_P P_1 \subset C_1 = \overline{\mathrm{St}}(\pi_P(P_0), C)$. Let $f': C' \to R^m$ be the map of Theorem 3.2 which is the secant approximation to f on C_1'. Now C' induces a subdivision P' of P so that $\pi_P: P' \to C'$ is still linear. Let $F': P' \to R^m \times R^n$ be the shift of F over f'. Using Remark 1.10, we are in a position to apply Theorem 3.3. Let $F'': P'' \to R^m \times R^n$ be the δ-approximation to F' of that theorem. The map F'' preserves commutativity and (*) holds for this map. Now let $G: P'' \to R^m \times R^n$ be the shift of F'' over f. Since G is merely a shift of F'', it is easily seen that (*) holds for the map G. Using Remark 1.6, one sees that G is a δ-approximation to F. By letting $P_2 = \overline{\mathrm{St}}^4(F^{-1}F(A), P)$ and P_2'' be the subdivision induced by P'' we obtain P''', the standard subdivision of P induced by P_2''. One checks that for $G: P''' \to R^m \times R^n$, (*) holds; hence, since $\pi_P: P''' \to C$ is linear, the theorem is proved.

4.2. *Remark.* The following corollary is our main tool and is used in induction arguments. It enables us to fit together a finite subcomplex (A) with what we have already embedded nicely (Q). For induction arguments we only wish

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to change maps and subdivisions locally; hence in the above proof we replace P'' by P'''.

4.3. COROLLARY. *The preceding lemma is true if we replace* $(R^m \times R^n, R^m, p_1)$ *by* (E, M, π) *where* E *and* M *are non-bounded manifolds.*

Proof. Without loss of generality, we may assume that simplices are mapped into coordinate neighbourhoods, which allows us to apply the lemma.

4.4. *Proof of Theorem* 2.1. Without loss of generality, we assume that simplices are mapped into coordinate neighbourhoods. We order the simplices of L: $A_1, A_2, \ldots, A_i, \ldots$ such that each simplex is preceded in the ordering by all of its faces. Let $F_0 = F$, $G_0 = G$, $K_0 = K$, and $L_0 = L$.

Induction Hypothesis
Suppose that we have diagrams

$$\begin{array}{ccc} K_i & \xrightarrow{F_i} & E \\ {\scriptstyle\pi_K}\downarrow & f & \downarrow{\scriptstyle\pi} \\ D & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} L_i & \xrightarrow{G_i} & E \\ {\scriptstyle\pi_L}\downarrow & f & \downarrow{\scriptstyle\pi} \\ D & \longrightarrow & M \end{array}$$

where K_i and L_i are subdivisions of K and L. The maps F_i and G_i are $(1 - 1/2i)\delta(x)$-approximations to F and G. Furthermore, if J_i is the subcomplex of L_i whose polytope is $A_1 \cup \ldots \cup A_i$, we suppose that (F_i, K_i) and (G_i, J_i) intersect in a full subcomplex and that their union is a closed embedding.

We now establish the induction hypothesis for $i + 1$. If (H, Q) is the union of (F_i, K_i) and (G_i, J_i) we have the usual diagram (Remark 1.8). Let $Q = K_i \cup J_i$. Let $P = Q \cup A_{i+1}$ formed by identifying points in Bd A_{i+1} with J_i and extend H over P by letting H equal G_i on A_{i+1}. The map π_P on A_{i+1} is π_L. Given $\epsilon > 0$, we apply Corollary 4.3 to obtain a diagram

$$\begin{array}{ccc} P' & \xrightarrow{H'} & E \\ {\scriptstyle\pi_H}\downarrow & f & \downarrow{\scriptstyle\pi} \\ D & \longrightarrow & M \end{array}$$

where H' is an ϵ-approximation to H. The maps $H'_{|Q'}$ and $H'_{|A_{i+1}'}$ intersect in a full subcomplex and their union is a closed embedding. This union is the same as the union of $H'_{|K_i'}$ and $H'_{|J_i' \cup A_{i+1}'}$. Furthermore, if ϵ is small enough, $H' : K'_i \to E$ will be a $\delta(x)/2^{i+1}$ approximation to H. We let $K_i' = K_{i+1}$ and $F_{i+1} = H'_{|K_i'}$. Also, if ϵ is small enough by Lemma 3.4, we may extend

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H': $J_i' \cup A_{i+1}' \to E$ to a $\delta(x)/2^{i+1}$-approximation of G_{i+1}: $L_{i+1} \to E$. Thus, the induction hypothesis is established for $i + 1$.

Now, since maps and subdivisions are changed only locally at each stage i, we may form $K' = \lim K_i$, $F' = \lim F_i$, $L' = \lim L_i$, $G' = \lim G_i$. The maps F' and G' are automatically $\delta(x)$-approximations to F and G. Thus, Theorem 2.1 is proved.

5. Appendix.

5.1. LEMMA. *Let* (E, M, π) *be a differentiable fibre bundle, where* M *is non-bounded, but the fibre* N *has a boundary, denoted* Bd N. *Then there exists a neighbourhood* W *of the boundary of* E, Bd E, *and a smooth retraction* ρ: $W \to$ Bd E, *such that* $\pi\rho = \pi$, *i.e. the retraction is along the fibre.*

Proof. We recall (Theorem 5.9 of **(5)**) that there is a diffeomorphism g of a neighbourhood V of Bd N with Bd $N \times [0, 1)$, such that $g(x) = (x, 0)$ for x in Bd N. Let f_t: Bd $N \times [0, 1) \to$ Bd $N \times [0, 1)$ be defined by

$$f_t(x, s) = (x, ts).$$

Let $g_t = g^{-1}f_t g$: $V \to V$; $g_1 = 1$, g_0 is a retraction of V onto Bd N, and $g_{t|\text{Bd}\,N} = 1$.

Now let $\{(U_i, k_i)\}$ be a locally finite coordinate cover of M such that $k_i(U_i) = B(2)$ (the ball of radius 2) and $\{k_i^{-1}(B(1))\}$ covers M.

Let $\alpha(s)$ be a monotonic smooth function, which is 0 for $s \leqslant 1 + 1/3$ and 1 for $s > 1 + 2/3$. We now define a smooth map which retracts $B(1) \times V$ onto B(1) $\times$ Bd N. Let h_i: $U_i \times V \to U_i \times V$ where $h_i(x, y) = (x, g_0(y))$ if $||k_i(x)|| \leqslant 1$ and $h_i(x, y) = (x, g_{\alpha(s)}(y))$ if $1 \leqslant ||k_i(x))|| = s \leqslant 2$. Let $\{(U_i \times N, \phi_i)\}$ be the local product structure for E. Choose a neighbourhood W of Bd E such that $W_i = W \cap \pi^{-1}(U_i) \subset \phi_i(U_i \times V)$ and $\rho_i = \phi_i h_i \phi_i^{-1}$: $W_i \to W_i$. By the definition of ρ_i, it is obvious that we can extend it to a smooth function on all of W. We let $\rho = \lim \rho_i$: $W \to$ Bd E. Because each ρ_i is fibre preserving, so is ρ, i.e. $\pi\rho = \pi$.

5.2. COROLLARY. *We can choose a product neighbourhood of the boundary of* E, F_1: Bd $E \times [0, 1) \to E$ *such that the following is commutative:*

$$\begin{array}{ccc} \text{Bd}\,E \times [0,1) & \xrightarrow{F_1} & E \\ \pi p_1 \big\downarrow & & \big\downarrow \pi \\ M & \xrightarrow{1} & M \end{array}$$

Proof. The proof is that of Theorem 5.9 of **(5)**, using in that proof the retraction of the previous lemma. The choice of this particular retraction yields the desired commutativity.

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Temple University,
Philadelphia