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A VARIANT OF HOFSTADTER'S SEQUENCE AND FINITE AUTOMATA

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In memory of Alf van der Poorten: colleague, connoisseur, raconteur, friend

Abstract

Following up on a paper of Balamohan *et al*. ['On the behavior of a variant of Hofstadter's q-sequence', *J. Integer Seq.* **10** (2007)], we analyze a variant of Hofstadter's Q-sequence and show that its frequency sequence is 2-automatic. An automaton computing the sequence is explicitly given.

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1. Introduction

In his 1979 book *Gödel, Escher, Bach* [7], Douglas Hofstadter introduced the sequence $Q(n)$ defined by the recursion

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$$

for $n \geq 2$ and $Q(1) = Q(2) = 1$. Although it has been studied extensively (for example, [9]), still little is known about its behavior, and it is not mentioned in standard books about recurrences (for example, [6]). It is sequence A005185 in Sloane's *Encyclopedia* [11].

Twenty years later, Hofstadter and Huber introduced a family of sequences analogous to the Q-sequence, and defined by the recursion

$$Q_{r,s}(n) = Q_{r,s}(n - Q_{r,s}(n - r)) + Q_{r,s}(n - Q_{r,s}(n - s))$$

for $n > s > r$ (private communication cited in [4]). The case $r = 1$, $s = 4$ is of particular interest.

TABLE 1. The sequence V.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$V(n)$	1	1	1	1	2	3	4	5	5	6	6	7	8	8	9	9	10

TABLE 2. The sequence F.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$F(n)$	4	1	1	1	2	2	1	2	2	1	3	2	1	2	2	1	3

Recently, Balamohan *et al.* [4] gave a nearly complete analysis of the sequence $Q_{1,4}$ (called V in their paper). It is defined by

$$V(1) = V(2) = V(3) = V(4) = 1$$
$$V(n) := V(n - V(n-1)) + V(n - V(n-4))$$

for $n > 4$. Table 1 is a short table of the sequence V (sequence A063882 in Sloane's *Encyclopedia* [11]).

Among the results of Balamohan, Kuznetsov, and Tanny is a precise description of the 'frequency' sequence $F(n)$ defined by

$$F(a) := \#\{n : V(n) = a\}.$$

Table 2 is a short table of the sequence F (sequence A132157 in Sloane's *Encyclopedia* [11]). In particular, they proved the following theorem [4, Lemmas 13–19 and Table 5].

THEOREM 1.1 (Balamohan, Kuznetsov, Tanny). *There exist two (explicit) maps g, h, with $g, h : \{1, 2, 3\}^4 \to \{1, 2, 3\}$, such that, for all $a > 3$*

$$F(2a) = g(F(a-2), F(a-1), F(a), F(a+1)),$$
$$F(2a+1) = h(F(a-2), F(a-1), F(a), F(a+1)).$$

(We note that in [4, Lemma 13], the quantifiers $a \geq 3$ for the equality $F(2a) = 2$ and $a \geq 4$ for the equality $F(2a+1) = 2$ should have been mentioned.)

In this paper we prove that the sequence $(F(n))_{n\geq 1}$ is 2-automatic, which means essentially that $F(n)$ can be computed 'in a simple way' from the base-2 representation of n. In particular, it can be computed in $O(\log n)$ time. Furthermore, we give the automaton explicitly. For definitions and properties of automatic sequences, the reader is referred to [2]. For some recent related papers, see [5, 8, 10].

2. The main result

We begin this section with a general result on automatic sequences. Before stating the theorem we need some notation.

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Definition 2.1. Let $U = (U(n))_{n\geq 0}$ be a sequence. Let α be an integer. We let U^α denote the sequence defined by

$$U^\alpha(n) := U(n+\alpha),$$

where $n \geq -\alpha$. For positive integers q, i, j, let $U_{q,i,j}$ be the subsequence of U defined by

$$U_{q,i,j}(n) = U(q^i n + j),$$

where $n \geq 0$.

For integers $q \geq 2$ and $t \geq 1$, let $m(q,t) = q^{t+1} - 1$ and $s(q,t) = (q^{t+1} - 1)/(q-1)$, so that $s(q,t)$ counts the number of integer pairs (i, j) with $0 \leq i \leq t$ and $0 \leq j \leq q^i - 1$. In the proof below it will be convenient to fix some ordering of such pairs (i, j) and then enumerate the corresponding sequences $U_{q,i,j}$ as $U_1 = U, U_2, \ldots, U_{s(q,t)}$.

Theorem 2.2. *Let $(U(n))_{n\geq 0}$ be a sequence with values in a finite set $\mathcal{A}$ and let $q \geq 2$ be an integer. Then $(U(n))_{n\geq 0}$ is q-automatic if there exist nonnegative integers t, a, b, n_0 and a family $\{f_j\}_{j=0}^{m(q,t)}$ of functions from the set $\mathcal{A}^{a+b+s(q,t)}$ to $\mathcal{A}$ such that for all $j \in [0, m(q,t)]$ and all $n \geq n_0$,*

$$\begin{aligned}&U(q^{t+1}n + j)\\ &= f_j(U^{-a}(n), \ldots, U^{-1}(n), U^0(n), U^1(n), \ldots, U^b(n), U_2(n), U_3(n), \ldots, U_{s(q,t)}(n)).\end{aligned}$$

Proof. To prove that the sequence $U = (U(n))_{n\geq 0}$ is q-automatic, it suffices to find a finite set of sequences $\mathcal{E}$ that contains U, such that if $V = (V(n))_{n\geq 0}$ belongs to $\mathcal{E}$, then, for any $r \in [0, q-1]$, the sequence $(V(qn + r))_{n\geq 0}$ also belongs to $\mathcal{E}$. Fix two positive integers K and L such that $K \geq \max(n_0, q(a+1)/(q-1))$ and $L \geq q(b+1)/(q-1)$. Recall that the sequence U_ℓ^α is defined by $U_\ell^k(n) := U_\ell(n+k)$.

Let $\mathcal{E}$ be the (finite) set of sequences defined by

$$V \in \mathcal{E} \Longleftrightarrow \exists \ell \in [1, s(q,t)], \exists k \in [-K, L], \forall n \geq A, V(n) = U_\ell^k(n).$$

Now let V be a sequence in $\mathcal{E}$. Take $r \in [0, q-1]$. There exist $\ell \in [1, s(q,t)]$ and $k \in [-K, L]$ such that for all $n \geq A$,

$$V(qn + r) = U_\ell^k(qn + r) = U_\ell(qn + r + k).$$

Hence, for some $i \leq t$ and $j \in [0, q^i - 1]$,

$$V(qn + r) = U(q^i(qn + r + k) + j).$$

By the division algorithm, write $q^i(r + k) + j = q^{i+1}x + y$, with $x \in \mathbb{Z}$ and $0 \leq y \leq q^{i+1} - 1$ (so that, in particular, $y \leq m(q,t)$). Then

$$V(qn + r) = U(q^{i+1}(n + x) + y).$$

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Note that

$$q^{i+1}x \le q^{i+1}x + y = q^i(r+k) + j < q^i(r+k+1)$$

and

$$q^{i+1}x = q^i(r+k) + j - y > q^i(r+k) - q^{i+1}.$$

Hence,

$$(r+k-q)/q < x < (r+k+1)/q.$$

We distinguish two cases.

Case 1. $i < t$. Then $i+1 \le t$. Thus there exists $\ell' \in [1, s(q,t)]$ such that, for $n \ge A$,

$$V(qn+r) = U(q^{i+1}(n+x)+y) = U_{\ell'}(n+x) = U^x_{\ell'}(n).$$

Now

$$x > (r+k-q)/q \ge (r-K-q)/q \ge (-K-q)/q \ge -K$$

(since $K \ge q(a+1)/(q-1) \ge q/(q-1)$), and

$$x < (r+k+1)/q \le (q+L)/q \le L$$

(since $L \ge q(b+1)/(q-1) \ge q/(q-1)$). This shows that the sequence $(V(qn+r))_{n\ge 0}$ belongs to $\mathcal{E}$.

Case 2. $i = t$. Then $i+1 = t+1$. From the hypothesis and the condition $K \ge n_0$, we can write, for $n \ge K$,

$$\begin{aligned} V(qn+r) &= U(q^{t+1}(n+x)+y) \\ &= f_y(U^{x-a}(n), \ldots, U^x(n), U^{x+1}(n), \ldots, U^{x+b}(n), U^x_2(n), U^x_3(n), \ldots, U^x_{s(q,t)}(n)). \end{aligned}$$

To prove that the sequence $(V(qn+r))_{n\ge 0}$ belongs to $\mathcal{E}$, it suffices to prove that all sequences U^β for $\beta \in [x-a, x+b]$ and all sequences U^x_ℓ for $\ell \in [1, (q^{t+1}-1)/(q-1)]$ belong to $\mathcal{E}$, and to use composition of maps. But

$$\beta \ge x - a > (r+k-q)/q - a \ge (-K-q)/q - a \ge -K$$

(recall that $K \ge q(a+1)/(q-1)$) and

$$\beta \le x + b < (r+k+1)/q + b \le (q+L)/q + b \le L$$

(recall that $K \ge q(b+1)/(q-1)$). This implies that all sequences occurring in the arguments of f_y above belong to $\mathcal{E}$. □

Remark 2.3. Theorem 2.2 above is similar to (but different from) [3, Theorem 6, p. 5] on k-regular sequences. That theorem implies Theorem 2.2 above in the case where the maps f_j are linear.

Corollary 2.4. *The sequence $F = (F(n))_{n\ge 0}$ is 2-automatic.*

Proof. It suffices to use the theorem recalled in the first section, after having extended the sequence F by $F(0) = 0$. □

3. An explicit automaton

In this section we provide an explicit automaton[1] to calculate the sequence F.

The automaton is constructed in two stages. First, we give an automaton A with the property that reading n in base 2 takes us to a state q with the property that the four values $F(n + a)$ for $-2 \leq a \leq 1$ are completely determined by q. Next, we show that A can be minimized to give an automaton B computing $F(n)$. We remark that we assume throughout that the automaton reads the ordinary base-2 representation of n from 'left to right', ending at the least significant digit, although we do allow the possibility of leading zeros at the start.

Let us start with the description of $A = (Q, \Sigma, \Delta, \delta, q_0, \tau)$. The machine A has 33 states with strings as names; $\Sigma = \{0, 1\}$; $\Delta = \{0, 1, 2, 3, 4\}^4$, $q_0 = \epsilon$. The transition function δ and the output map τ are given in Table 3.

We introduce some notation. Let $[w]$ denote the integer represented by the binary string w in base 2. Thus, for example, $[00110] = [110] = 6$. Note that $[\epsilon] = 0$, where ϵ denotes the empty string. If F is our sequence defined above, then by $F(a..a + i - 1)$ we mean the string of length i given by the values of the function F at $a, a + 1, \ldots, a + i - 1$.

Our intent is that if w is a binary string, then $\tau(\delta(q_0, w))$ is the string of length four given by $F(n - 2..n + 1)$, where $n = [w]$. (Note: we define $F(0) = F(-1) = F(-2) = 0$.)

To prove that this automaton computes $F(n)$ correctly, it suffices to show that:

(a) for each state q we have $\tau(q) = F([q] - 2)F([q] - 1)F([q])F([q] + 1)$; and

(b) if $p = \delta(q, a)$ for two states $p, q \in Q$ and $a \in \{0, 1\}$, then $F([px]) = F([qax])$ for all strings x.

Part (a) can be verified by a computation, which we omit. For example, since $[111001111] = 463$, the claim $\tau(111001111) = 2133$ means $F(461..464) = 2133$, which can easily be checked.

Part (b) requires a tedious simultaneous induction on all the assertions, by induction on $|x|$. Not surprisingly, we omit most of the details and just prove a single representative case.

Consider the transition $\delta(100, 1) = 110$. Here we must prove that

$$F([1001x]) = F([110x]) \tag{3.1}$$

for all strings x. We do so by induction on x. The base case is $x = \epsilon$, and we have $F([1001]) = F(9) = 2$ and $F([110]) = F(6) = 2$.

For the induction step, we use the fact that [4, Table 5] shows that $F(2a)$ and $F(2a + 1)$ are completely determined by $F(a - 2)$, $F(a - 1)$, $F(a)$, and $F(a + 1)$. It thus suffices to check that $F([1001x] + a) = F([110x] + a)$ for $-2 \leq a \leq 1$; doing so will then prove (3.1) for $x0$ and $x1$, thus completing the induction.

The only cases that require any computation are when $[x] = 0$ and $a = -1, -2$, or $[x] = 1$ and $a = -2$, or x is a number of the form $2^j - 1$ for some $j \geq 1$ and $a = 1$.

[1] In honor of Alf van der Poorten, we cannot resist quoting Voltaire: 'Impuissantes machines/Automates pensants mus par des mains divines'.

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Table 3. The automaton A.

q	$\delta(q,0)$	$\delta(q,1)$	$\tau(q)$
ϵ	ϵ	1	0004
1	10	11	0041
10	100	101	0411
11	110	111	4111
100	1000	110	1112
101	1010	1011	1122
110	1100	1101	1221
111	1110	110	2212
1000	1010	1011	2122
1010	1110	10101	2213
1011	10110	10111	2132
1100	1101	1110	1321
1101	11010	11011	3212
1110	11100	11101	2122
10101	101010	101011	1223
10110	10110	10111	2232
10111	1101	1110	2321
11010	101010	110101	1222
11011	111	1000	2221
11100	11010	111001	2213
10111	111010	10111	2132
101010	1010100	11101	1322
101011	101010	101011	3223
110101	1100	1101	3221
111001	1010	1110011	2223
111010	110100	1110101	2232
1010100	11010	111001	3213
1110011	10110	11100111	2133
1110100	11101000	10111	1332
1110101	1101	1110	3321
11100111	1010100	111001111	2323
11101000	110100	1110101	3232
111001111	111010	11100111	2133

Case 1. $x = 0^j$ for some $j \geq 0$. If $j = 0$ then this is the assertion that $F([1001] + a) = F([110] + a)$ for $-2 \leq a \leq 1$, which is the same as the claim that $F(7..10) = F(4..7)$. But $F(7..10) = 1221 = F(4..7)$.

Otherwise $j \geq 1$. Then $[1001x] - 1 = [10010^j] - 1 = [10001^j]$ and $[110x] - 1 = [1100^j] - 1 = [1011^j]$. Now by induction we have $F([10001^j]) = F([100011^{j-1}]) = F([10111^{j-1}]) = F([1011^j])$, as desired.

Similarly, $[1001x] - 2 = [10010^j] - 2 = [1001^{j-1}0]$. Also, $[110x] - 2 = [1100^j] - 2 = [101^j0]$. Then by induction we have $F([1001^{j-1}0]) = F([10011^{j-2}0]) = F([10111^{j-2}0]) = F([101^j0])$, as desired.

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TABLE 4. The automaton B.

q	$\delta'(q,0)$	$\delta'(q,1)$	$\tau'(q)$
ϵ	ϵ	1	0
1	10	11	4
10	100	101	1
11	110	111	1
100	101	110	1
101	1010	1011	2
111	1110	110	1
1010	1110	10101	1
1011	1011	1100	3
1100	1101	1110	2
1101	11010	11011	1
1110	11100	1011	2
10101	1110	10101	2
11010	1110	110	2
11011	111	101	2
11100	11010	111001	1
111001	1010	1110011	2
110011	1011	11100111	3
11100111	11100	1110011	2

Case 2. $x = 0^j1$ for some $j \geq 0$. Then $[1001x] - 2 = [10010^j1] - 2 = [10001^{j+1}]$. Also $[110x] - 2 = [1100^j1] - 2 = [101^{j+2}]$. By induction we have $F([10001^{j+1}]) = F([100011^j]) = F([10111^j]) = F([101^{j+2}]$, as desired.

Case 3. $x = 1^j$ for some $j \geq 1$. Then $[1001x] + 1 = [1010^{j+1}]$. Similarly $[110x] + 1 = [1101^j] + 1 = [1110^j]$. By induction we have $F([1010^{j+1}]) = F([101000^{j-1}]) = F([11100^{j-1}]) = F([1110^j])$, as desired.

This completes the proof of correctness of a single transition.

Ultimately, we are not really interested in computing $\tau(q)$, but only the image of $\tau(q)$ formed by extracting the third component, which is the one corresponding to $F(n)$. This means that we can replace τ by τ', which is the projection of τ along the third component. In doing so some of the states of A become equivalent to other states. We can now use the standard minimization algorithm for automata to produce the 20-state minimal automaton $B = (Q', \Sigma, \Delta, \delta', q_0, \tau')$ computing $F(n)$. Table 4 gives the names of the states of Q, and δ' and τ' for these states.

4. Concluding remarks

It would be interesting to know whether the first difference sequence of the variant of Hofstadter's, that is, the sequence $(V(n+1) - V(n))_{n\geq 0}$, is also 2-automatic. We already know that it takes only finitely many values [4, Theorem 1, p. 5]. Of course it might well be the case that this sequence is *not* automatic: in a very different context, think of the classical Thue–Morse sequence which is 2-automatic, but whose run-length sequence is not [1]. It would be also interesting to determine for which sequences $Q_{r,s}$ (with the notation in the introduction) the frequency sequence is automatic.

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