NOZZLE FLOWS FOUND BY THE HODOGRAPH METHOD. II

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1. Introduction

This is a sequel to a recent paper [1] on the construction by the hodograph method of trans-sonic nozzle-flows of a perfect gas. At the end of that paper it was shown how we can obtain regular flows that are ultimately uniform (as required in the test section of a supersonic wind tunnel), and the object now is to give some quantitative examples of such flows. The gas is supposed to have the polytropic equation of state $p\rho^{-\gamma} =$ constant, and the calculations have been made for the case $\gamma = 1.4$, with the Mach number $M = 2.25$ at the test section. The results, which are exhibited graphically, are indicative of what may be expected for other supersonic values of M, and it is hoped that they may be significant for the design of wind tunnels.

In the hodograph method we regard the stream function ψ and the position coordinates x, y as functions of the magnitude q and the direction-angle θ (from Ox) of the velocity. The field equations are linear, and a solution of them determines three related functions $\psi(q, \theta)$, $x(q, \theta)$, $y(q, \theta)$. The solutions which are to be considered are to have $\theta = 0$ as an axis of symmetry, so that x is an even function of θ and y, ψ odd, with the properties that near the axis

(i) x increases steadily as q increases from 0 to a supersonic value q_0,
(ii) as q decreases to 0, x decreases to $-\infty$,
(iii) as q increases to q_0, x tends to a limit which is $+\infty$, or (exceptionally) finite.

A family of such solutions, depending on two parameters a, b, may be specified in the form

$$(1) \quad \psi = \psi_T - a\psi_R + b\psi_U, \; x = x_T - ax_R + bx_U, \; y = y_T - ay_R + by_U,$$

where the suffixes T, R, U indicate three standard solutions with the distinctive properties that T gives a *trans-sonic* flow, R *radial flow* and U a flow that is *ultimately uniform*. Our object is to survey the flow patterns given by this family.

2. The basic solutions

R is the solution for radial flow. Its Legendre potential is

$$\Omega_R = \int q^{-1}(1 - q^2)^{-\beta} dq, \tag{2}$$

and

$$\psi_R = -\theta, \quad x_R = \frac{\cos\theta}{q(1-q^2)^\beta}, \quad y_R = \frac{\sin\theta}{q(1-q^2)^\beta},$$

where $\beta = 1/(\gamma - 1)$ and q is the dimensionless measure of speed such that $q = 1$ gives the limiting speed at which $p = 0$, $M = \infty$.

The specification of the solutions T, U involves a fair amount of detail, which is available in [2] and is suppressed from what follows: Trans-sonic nozzle flows are best handled with the aid of a transformation

$$\theta = \phi - 2\alpha \arctan \frac{q \sin\phi}{1 - q\cos\phi}, \tag{3}$$

where α is the positive root of $2\alpha(1 + \alpha) = \beta = 1/(\gamma - 1)$. This transformation has a branch locus

$$\theta = \pm\, \omega(q),$$

where $\omega(q)$ is a function which is real positive for supersonic q and imaginary for subsonic q; to a given (q, θ) correspond three points (q, ϕ), which are all real if $|\theta| \leqq \omega(q)$ but of which only one is real if $|\theta| > \omega(q)$ or if q is subsonic. The locus is characteristic for the hodograph equation, and hence (essentially) a trans-sonic nozzle flow in the xy-plane is in one-one correspondence with the $q\phi$-plane near the axis, whereas the correspondence with the $q\theta$-plane is $3:1$ for $|\theta| < \omega(q)$. The loci θ = constant in the $q\phi$-plane are sketched in figure 2R.

In [2] a set of single valued functions $\Omega_p(q, \phi)$, $p = 0, 1, 2, \cdots$, has been defined.

T denotes the solution whose Legendre potential is $\operatorname{Re} \Omega_2(q, \phi)$, where ϕ is the real root of (3). It has been tabulated in [2], and its general character can be seen from figures 1 and 2T below; it is regular for all q, ϕ that here come into question, and $\partial x_T/\partial q$ is strictly positive for $\phi = 0$ and $0 < q < 1$.

To define the solution U we start with the function $\Omega_1(q, \phi')$ where, for subsonic q or supersonic q with $\omega(q) > |\theta|$, ϕ' is the unreal root of (3) whose imaginary part is positive. By (3) we can write

$$\Omega_1(q, \phi') = f(q, \theta),$$

a function with lines of singularity $\theta = \pm\omega(q)$. Then if q_0 is the chosen ultimate speed for the nozzle flow, we define $\theta_0 = \omega(q_0)$, and the solution U

is the one whose Legendre potential is

$$\Omega_U = -\operatorname{Re}\{f(q, \theta_0 + \theta) + f(q, \theta_0 - \theta)\} + c\Omega_R, \tag{4}$$

where the constant c is so chosen that $\partial^3 \Omega_U / \partial q^3 = 0$ at the axial sonic point $q = q_S$, $\theta = 0$; the adjunction of the term $c\Omega_R$ is for the purposes of the superposition (1) trivial, but it serves to give the solution U the clear-cut properties exhibited in figures 1 and 2U, where the level curves of $\psi_U(q, \theta)$ have been transferred via (3) to the $q\phi$-plane. The curve $\psi_U = 0$ has 3 branches through the axial sonic point, and in this neighbourhood we have the approximation $\psi_U = C(q - q_S)\theta$, which is connected with the approximation $x_U = C'(q - q_S)^3$ by the formulae of [1], § 2. On the characteristic loci $q_0 AB$ and CD, whose equation is $\omega(q) - \theta = \theta_0$, the function ψ_U becomes positive infinite like $\{\theta + \theta_0 - \omega(q)\}^{-\frac{1}{2}}$; and on BC, $\omega(q) + \theta = \theta_0$, it becomes negative infinite; and x_U, y_U have similar singularities.

3. Singularities of the flow field

The equations (1) will specify a regular flow-field in the xy-plane in so far as they give a regular one-one mapping between the $q\phi$- and xy-planes. This condition is satisfied, for some strip of the $q\phi$-plane centred on the axis $\phi = 0$, provided that $\partial x / \partial q > 0$ for $0 < q < q_0$, $\phi = 0$; and from figure 1 it is seen that this will be so for a, $b \geqq 0$, provided a is not too large.

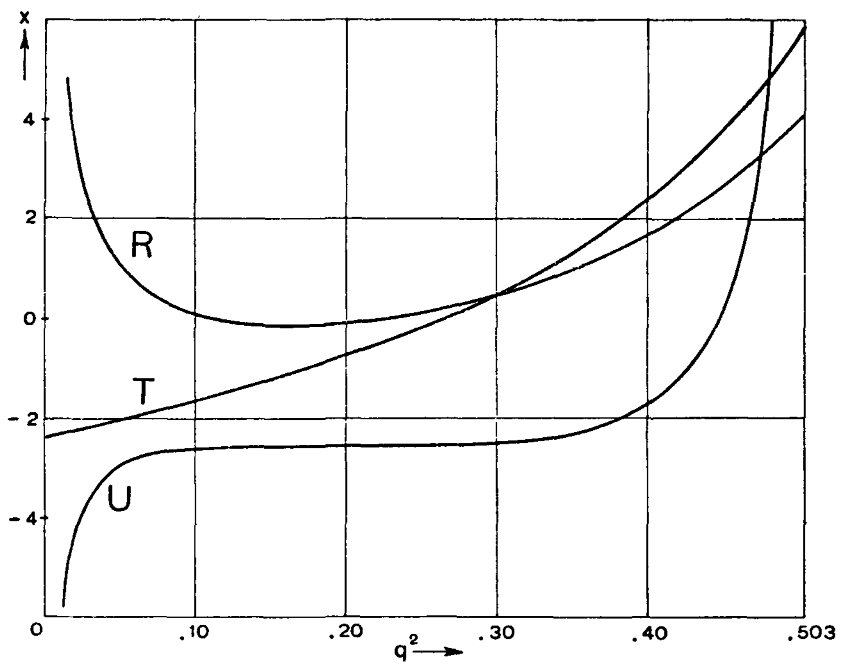

Fig. 1. x on axis, as function of q^2, for solutions T, R, U.

A 'natural boundary' of the flow field will be given, in the $q\phi$-plane, by the nearest level curve ψ = constant to the axis on which the mapping becomes singular. In a region where ψ is regular (which is always here the

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case) two sorts of singularities may be encountered:

(i) *a point P where* $\psi_q = \psi_\phi = 0$, which will be a minimax or extremum of ψ according as q_P is subsonic or supersonic; in the present context only the former case occurs;

(ii) *a point P where a level curve* $\psi =$ *constant has inflexional contact with a characteristic curve* (i.e. the two curves touch and cross); this can occur only when q_P is supersonic; the level curves 'beyond' P map on to the xy-plane as curves with two cusps, and the loci of these cusps form *limit-lines* which come together in a cusp at the map of P.

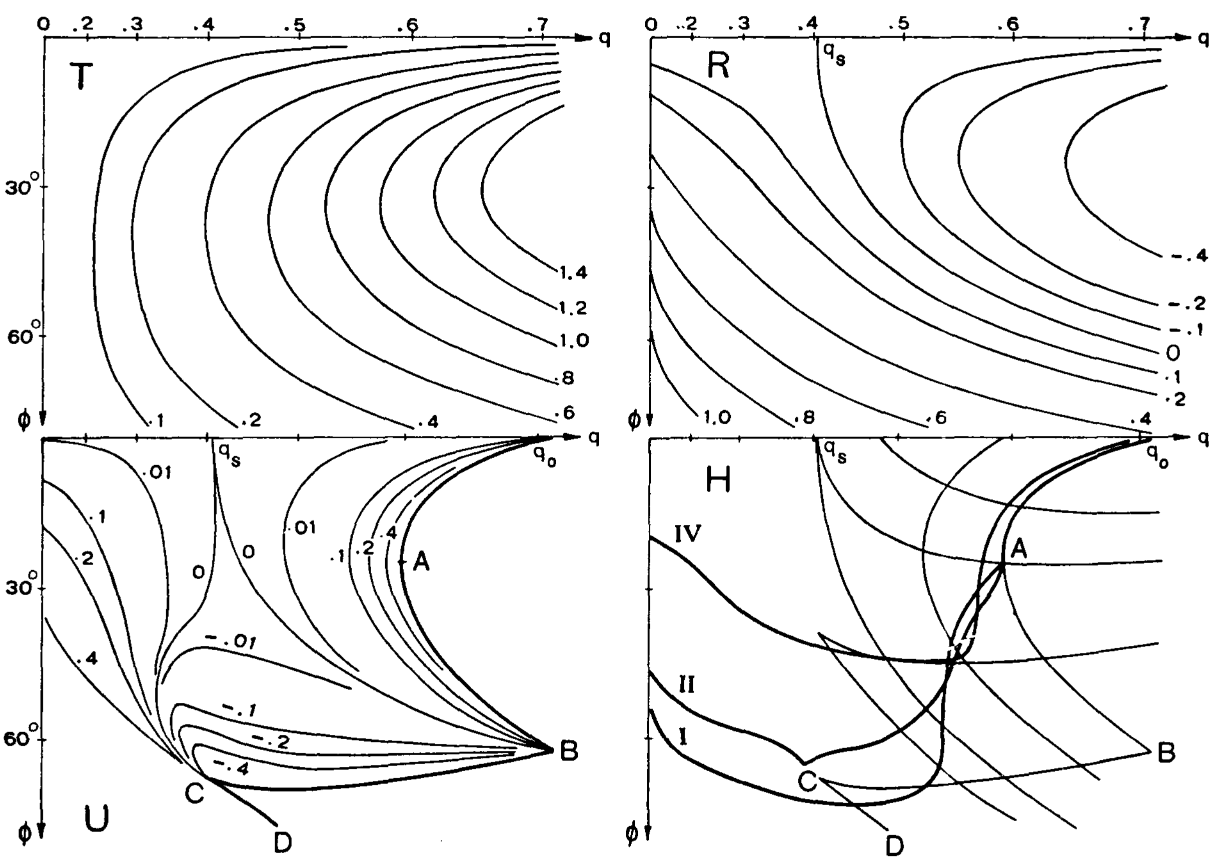

Fig. 2. Hodograph plane, with q^2, ϕ as cartesian coordinates.

T: Level curves of ψ_T

R: Level curves of $-\psi_{R'} = \theta$

U: Level curves of ψ_U

H: Representative characteristics (thin), and hodographs of natural boundaries of solutions I, II, IV.

Apart from these possible singularities it is not acceptable physically to have, on the ideal fluid approximation to a real flow, a stretch of a bounding wall where the acceleration is negative. It is therefore required that no level curve should cross any locus $q =$ constant more than once, so we reckon amongst the singularities

(iii) *a point P where a level curve* $\psi =$ *constant has inflexional contact with a locus* $q =$ *constant.*

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The general character of the level curves of $\psi = \psi_T - a\psi_R + b\psi_U$ can be seen on considering figure 2. For $b > 0$ there must always be a col (i.e. a minimax of ψ), which is near C when b is small and near the axial point q_S when b is large. Also there must be an inflexional singularity near A, when b is positive and not too large. For the 'accurate' placing of these singularities we must of course plot the level curves (for which a prior task is to tabulate $\psi(q, \phi)$), and this serves also to place the inflexional contact with a characteristic, which in practice always occurs. In this way the natural boundary to the flow field is found, and from (1) the streamlines can be plotted in the xy-plane. Examples are shown in figures 4, 5, and for three of them the hodographs are shown in figure 2H.

We call the singularity which determines the natural boundary of the flow field the *dominant* one. Numerical exploration shows that, according to the values of a, b, the dominant singularity may be of any of the types (i), (ii), (iii) set out above; the results in this matter are shown in figure 6 — where the following comment on the case $b = 0$ is required.

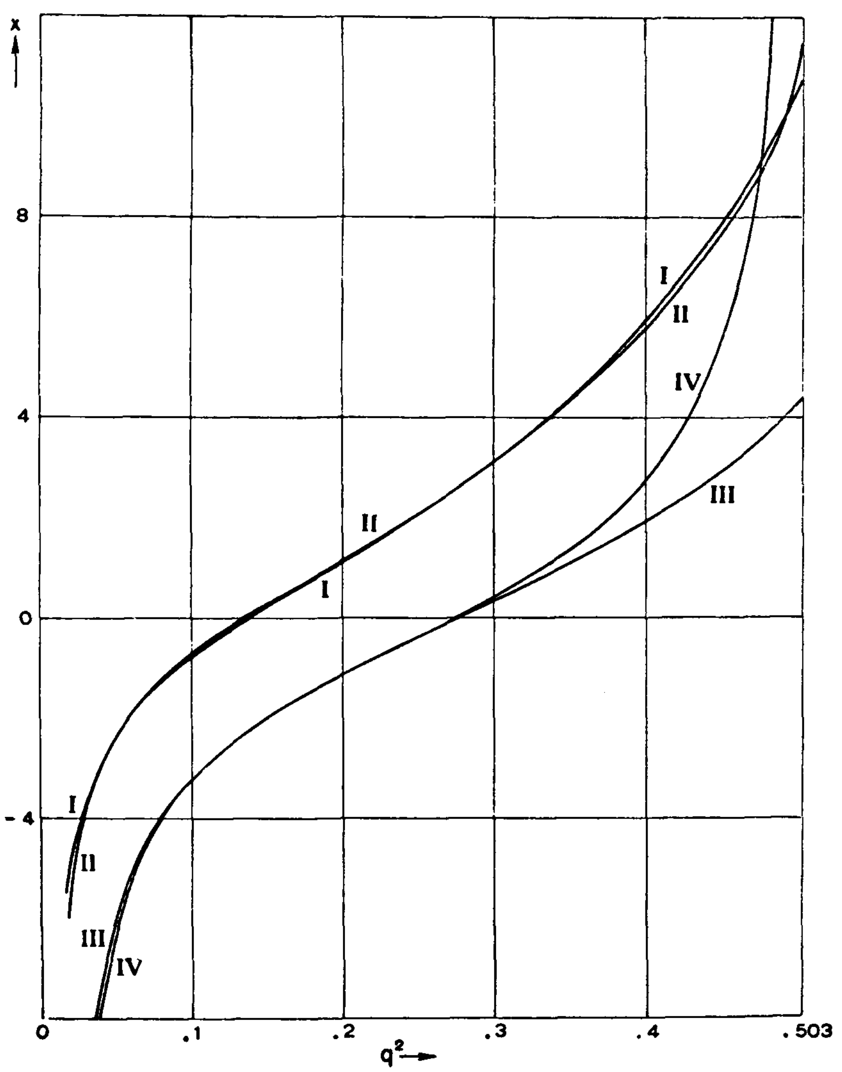

Fig. 3. x on axis, as function of q^2, for solutions I, II, III, IV.

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4. The case $b = 0$

On account of the infinity of ψ_U on the fixed curve q_0ABCD the case $b = 0$ is to be distinguished from the limit of the case $b > 0$ as $b \to 0$; the two cases can be symbolically distinguished as $T - aR$, $T - aR + (0+)U$. If we think of the level curves of ψ as contour lines of a surface, the surface $\psi_T - a\psi_R$ is regular across the curve q_0ABCD; but the surface $\psi_T - a\psi_R + (0+)\psi_U$ runs smoothly up to vertically ascending cliffs along q_0AB and CD and to a descending cliff along CB, and where a regular contour line abuts on one of these cliffs it is to be regarded as continued horizontally across the cliff. This picture is instructive because there are limiting forms of the formulae (1) which give a one-one mapping of the cliff face on to the xy-plane — formulae which in fact give the 'simple wave' continuation of the flow up to the cliff. For each streamline, so continued, the limits of x, y as $q \to q_0$ are finite, and $\theta \to 0$, so finally each streamline can be smoothly continued parallel to the axis. The procedure hence gives the merging of a throat-flow into a uniform supersonic stream via a simple wave, usually performed by the method of characteristics.

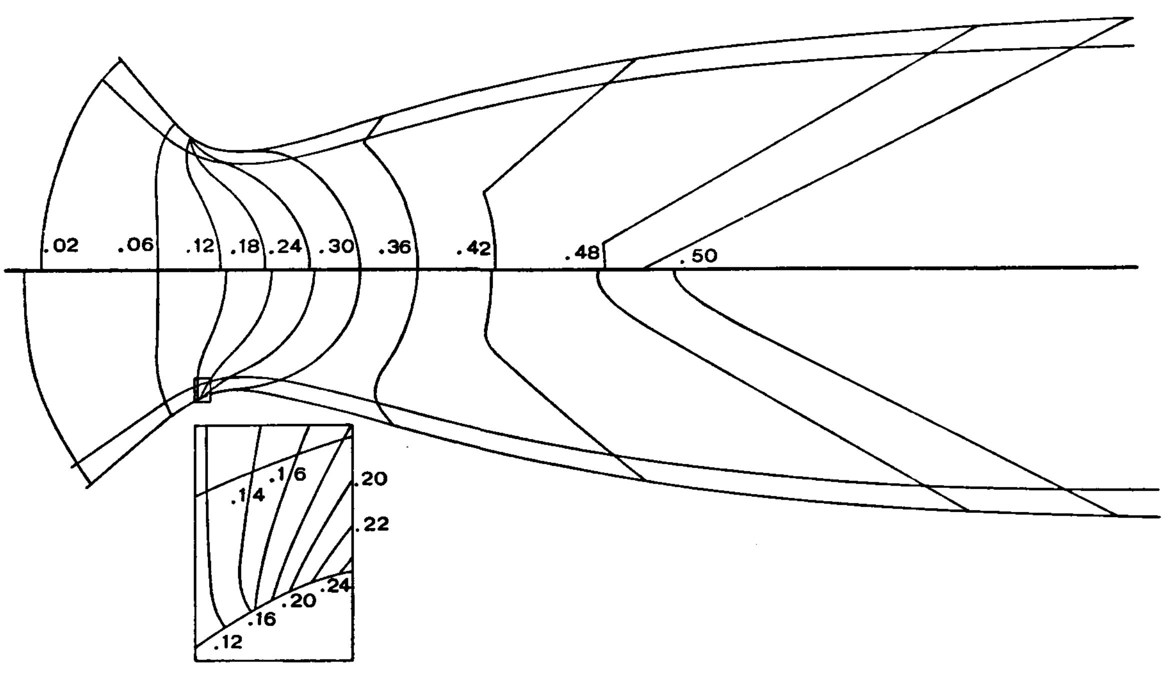

Fig. 4. Streamlines and isovels for solutions I (above) and II (below).

It is of course at our choice whether a set of level curves of $\psi_T - a\psi_R$, on meeting a part of the singular curve, are to be continued regularly across the curve or singularly along it. In the present context we naturally take the regular continuation across CD and CB but the singular continuation along BAq_0, and the symbolism $T - aR + (0+)U$ is to be taken as implying this. For this solution it is clear that we must reckon the point A to be an inflexional singularity (iii), and a level curve abutting on BA will be outside the natural boundary.

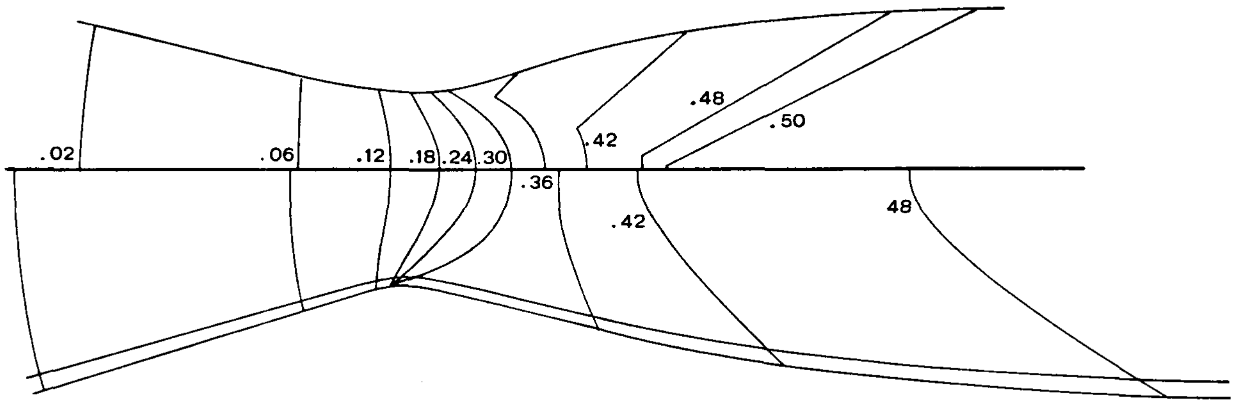

Fig. 5. Streamlines and isovels for solutions III (above) and IV (below).

An interesting corollary follows. At A the loci $\theta = -\omega(q)$ and $\omega(q) - \theta = \omega(q_0) = \theta_0$ intersect, so $\theta = -\frac{1}{2}\theta_0$. Also $\theta > \theta_A$ throughout the region of regularity of ψ_U, as is seen from figures 2U and 2R. Hence *in the supersonic expansion of a stream to the limiting speed* q_0 *no streamline can have a point where* $|\theta| > \frac{1}{2}\omega(q_0)$. *

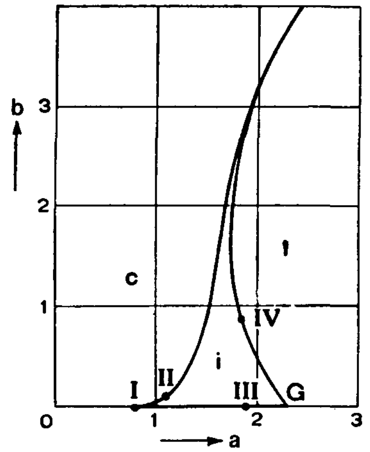

Fig. 6. Dependence of dominant singularity on a, b: (c) col, (i) inflexion (l) incipient limit line. (For $b > 1$ the boundaries are not placed with precision).

Returning now to figure 6, the points where $b = 0$ are to be taken as referring to the solution $T - aR + (0+)U$. For $a = 0.895$ the natural boundary passes through both C and A; for $0.859 < a < 0.895$ the natural boundary passes through A but below C; while for $0 < a < 0.859$ it passes

* By 'expansion' we mean to imply a restriction to points downstream from the locus $\theta = 0$. The vital fact is that no streamline can cross the characteristic q_0AB and this (and the assertion in the text) appears to hold irrespective of restriction to flows of the family (1); for if a streamline finishing at q_0 crossed q_0AB there would be tangencies between streamlines and characteristics, and hence limit-lines in the field.

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above A, and is determined by a (proper) inflexional singularity which the solution $T - aR$ 'happens' to have. For $a > 2.292$ the natural boundary is determined by a limit-line singularity (ii) and passes above A.

The limiting forms of (1) for $b = 0+$, referred to above, are obtained from the formulae (24) of [1], which, writing

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta, \tag{5}$$

give the principal parts of X_U, Y_U, ψ_U near the locus $\theta + \theta_0 = \omega(q)$ on which $f(q, \theta + \theta_0)$ is infinite. Since

$$2(1+\alpha)q\cos\sigma = 1 + (1+2\alpha)q^2$$

on the singular locus, these formulae are equivalent to

$$\begin{aligned} b\psi_U &= -\frac{(1-q^2)^{(1+\beta)/2}}{(1+\alpha)\sqrt{(q^2/q_S^2 - 1)}}\,\frac{b}{\zeta} + O(b) \\ bY_U &= -q^{-1}(1-q^2)^{-\beta}b\psi_U + O(b) \\ bX_U &= \frac{\sqrt{(q^2/q_S^2-1)}}{q(1-q^2)^{(1+\beta)/2}}\,b\psi_U + O(b), \end{aligned} \tag{6}$$

where $\zeta \to 0$ as the singular locus is approached. Thus, before letting $b \to 0$, (1) are equivalent to

$$\begin{cases} \psi = \psi_T - a\psi_R + b\psi_U, \qquad Y = Y_T - aY_R - \dfrac{b\psi_U}{q(1-q^2)^\beta} + O(b), \\ \qquad\qquad X = X_T - aX_R + \dfrac{\sqrt{(q^2/q_S^2-1)}\,b\psi_U}{q(1-q^2)^{(1+\beta)/2}} + O(b). \end{cases} \tag{7}$$

Now for following an assigned streamline along the singular locus (i.e. the 'cliff'), in the limiting case $b = 0+$, ψ has an assigned value and ψ_T, ψ_R have the determinate values appropriate to the current point on the locus. Hence $b\psi_U$ has a determinate limit (which determines the limit of b/ζ in (6) as $b, \zeta \to 0$). The limiting values of X, Y now follow from $(7)_{2,3}$ by omitting the $O(b)$ terms, and x, y follow from (5).

It is of mathematical interest that the uniform stream with $q = q_0$, $\theta = 0$, which is the final singular continuation of the flow, may be obtained from limits (as $b \to 0$) at the axial point $q = q_0$, $\theta = 0$ where both terms $f(q, \theta_0 + \theta)$, $f(q, \theta_0 - \theta)$ in (4) are singular.

5. Discussion

The flow fields considered in this paper belong to the ideal fluid approximation. We may expect such an ideal field to give a good approximation to a flow of a real fluid in a nozzle except near the bounding walls (where the real flow will have a boundary layer), provided that the first and second spatial derivatives of the velocity do not become too large in the interior of the field.

To minimize the disturbance from the boundary layer the relative length of the nozzle, i.e. the ratio of its length to the breadth of its test section, should be as small as possible. Since our ideal fields extend to infinity a significant definition of relative length must depend essentially upon the cut-off points chosen for the physical walls. For any likely choice of these points it is plain (from the behaviour of x_U, figure 1) that the relative length will in general increase with the parameter b, so that the relatively shortest nozzles will be those with $b = 0+$. However, for $b = 0+$ the first spatial derivatives of the velocity field have finite discontinuities at the entrance to the simple-wave region, whereas for $b > 0$ the field is everywhere regular, with second derivatives which decrease in magnitude as b increases. The solution $T - 1.012R + 0.03U$ suggests that by suitably choosing b we may obtain adequately small second derivatives at the cost of very little increase in relative length.

As regards the dependence of relative length on the parameter a we must expect that, *for the natural boundary*, the relative length will show a non-smooth variation as (a, b) crosses from one to another of the regions (c), (i), (l) of figure 6. Moreover when (a, b) is in the region (c) or (l) the natural boundary fails 'grossly' to attain the slope $\frac{1}{2}\theta_0$ in the expanding part of the nozzle, and this to an increasing extent as (a, b) moves away from the region (i). On this account we may guess that the relative length will be least at one of the points I, G in figure 6. To confirm this guess we must crystallize the concept by a 'reasonable' quantitative definition, which I have chosen as

$$(8) \qquad l = \text{relative length} = \frac{x_Q - x_E}{y_Q}$$

where E is the axial point where $q = 0.141$, $M = 0.32$ and Q is the point on the wall streamline, a little downstream from the test section, where $\theta = 0^\circ.34$; the value of q_Q is about 0.707 (depending slightly on the values of a, b), which is to be compared with the limiting speed $q_0 = 0.709$. Some values of l are shown in the table below, and from these, together with a detailed investigation of the cases $b = 0+$ which will not here be reproduced, it is verified that l is least at the point I in figure 6, with the value 4.36.

As regards relative length, therefore, the most favourable flows of our family are those for which a is near 0.9 and b is small, two of which are illustrated in figure 4.

Data for figures 3, 4, 5

Fig.	Sol^n.	ψ	Rel. length, equ^n. (8) $x_Q - x_E$	y_Q	l	$\lvert\theta\rvert$ at 'entrance' $q^2 = 0.02$	max $\lvert\theta\rvert$ in expansion sec^n.	throat ($\theta = 0$) at q^2	M
4 upper	I	0.7827	27.67	6.34	4.36	50°.11	16°.5	0.298	2.09
		0.7	26.34	5.67	4.65	44°.36	16°.2	0.250	1.67
4 lower	II	0.7837	27.75	6.34	4.38	41°.47	16°.4	0.301	2.15
		0.7	26.41	5.67	4.66	36°.76	16°.1	0.255	1.71
5 upper	III	0.5003	22.29	4.05	5.50	14°.81	16°.5	0.204	1.28
5 lower	IV	0.7558	64	6.0	10.7	16°.89	15°.0	0.285	1.99
		0.7	62	5.5	11.3	15°.64	13°.3	0.254	1.70

Notes:

(i) The limiting speed in each case is $q_0 = 0.709$, $M_0 = 2.25$, giving $\theta_0 = \omega(q_0) = 33°$.

(ii) Solution I is $T - 0.859R + (0+)U$. Natural boundary $\psi = 0.7827$ determined by two inflexions, one of them at A; hodograph in figure 2H.

Solution II is $T - 1.012R + 0.03U$. Natural boundary $\psi = 0.7837$ determined by an inflexion at A and a col; hodograph in figure 2H; the loci $q =$ const. tend to lines $x =$ const. as q increases from 0.707 to 0.709, but the curtailment of the figure on the right hides this.

Solution III is $T - 1.84R + (0+)U$. Natural boundary $\psi = 0.5003$ determined by an inflexion at A.

Solution IV is $T - 1.84R + 0.93U$. Natural boundary $\psi = 0.7558$ determined by an inflexion and an incipient limit line; hodograph in figure 2H.

(iii) In figures 4 and 5 the constant-speed loci are shown for

q^2	.02	.06	.12	.18	.24	.30	.36	.42	.48	.50
q	.14	.24	.35	.42	.49	.55	.60	.65	.69	.71
M	.32	.56	.83	1.05	1.26	1.46	1.68	1.90	2.15	2.24

In figure 4 lower the enlargement shows the 'col' singularity on the natural boundary, with the loci $q^2 = .12, .14, .16, .18, .20, .22, .24$.

6. Extensions

We may obtain a greater variety of flow fields by including further basic solutions in the superposition (1), and two specific possibilities here may be noted.

(i) For the 'Chaplygin set' of solutions, [1] § 3, the stream function is

$$\psi_n(q) \sin n\theta \tag{9}$$

where n is an arbitrary parameter and $\psi_n(q)$ essentially a hypergeometric function. Starting from a solution (1) whose dominant singularity is at a point P, we can make this singularity move away from the axis by superposing a small multiple of (9), with n remaining largely at our choice; and by suitably choosing n we can do the same to two singularities in the case where they are simultaneously dominant. In this way we can reduce the relative length of a flow (1).

(ii) A new basic solution V analogous to U can be formed as in [2] by starting with the potential $\Omega_2(q, \phi')$ instead of $\Omega_1(q, \phi')$ and symmetrizing it as in (4). Since Ω_2 is a θ-integral of Ω_1, ψ_V, x_V, y_V will have on the locus q_0AB of figure 2U a singularity whose principal part varies as $\{\theta + \theta_0 - \omega(q)\}^{1/2}$. Accordingly, a combination $\psi = \psi_T - a\psi_R + c\psi_V$ will give level curves which *abut* on the singular locus, and when $c \neq 0$ the θ- and q-derivatives of ψ will be dominated by those of $c\psi_V$, so the level curves will abut *tangentially* on the singular locus. Hence the solution

$$T - aR + cV + (0+)U$$

will merge the flow $T - aR + cV$ via a simple wave continuation into a strictly uniform flow, such that the velocity field in the plane has continuous first-order spatial derivatives. This method of smoothing the transition might be preferable, in practice, to the use of a solution $T - aR + bU$ where b is small positive, as suggested in § 5.

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