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On Pseudo-Frobenius Rings

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Abstract. It is proved here that a ring R is right pseudo-Frobenius if and only if R is a right Kasch ring such that the second right singular ideal is injective.

Throughout, rings R are associative with an identity and modules are unitary R-modules. We write $J(M)$ and $\operatorname{soc}(M)$ for the Jacobson radical and the socle of the module M, respectively. For a submodule N of M, $N \leq_e M$ means that N is essential in M. The singular submodule of the module M is defined by $Z(M) = \{m \in M : \exists\, I_R \leq_e R_R \text{ such that } mI = 0\}$. For a ring R, $Z(R_R)$ and $Z({}_RR)$ are called the right singular ideal and the left singular ideal of R, respectively. The second right singular ideal of R, denoted by Z_2^r, is defined by the equality $Z_2^r/Z(R_R) = Z(R_R/Z(R_R))$ and the second left singular ideal Z_2^l of R can be defined analogously. We write $J(R), S_r, S_l$ for the Jacobson radical, right socle and left socle of R, respectively. A ring R is called right pseudo-Frobenius, briefly right PF, if every faithful right R-module is a generator of the category of all right R-modules; and the ring R is called right Kasch if every simple right R-module embeds in R_R. Analogously, one defines left PF and left Kasch rings. It is a well-known result of Osofsky [6] that R is right PF if and only if R is semiperfect, right self-injective with $(S_r)_R \leq_e R_R$ if and only if R is right Kasch, right self-injective (see [1]). It is shown in [8] that a ring R is right PF if and only if $(Z_2^r)_R$ is injective and the dual of every simple left R-module is simple. In [10], it is proved that a ring R is a two-sided PF-ring if and only if R is a two-sided Kasch ring such that $(Z_2^r)_R$ and ${}_R(Z_2^l)$ are both injective. However, it has been left open in [10] whether a right Kasch ring R with $(Z_2^r)_R$ injective is necessarily right PF (also see [8]). In this note, we answer this question affirmatively.

We start by proving the following result.

Theorem 1 *A ring R is right PF if and only if R has a finitely generated projective, quasi-injective right R-module containing a copy of every simple right R-module.*

Proof One direction is clear. Let P be a finitely generated projective, quasi-injective right R-module containing a copy of every simple right R-module. By [3, Theorem 14], P has a finitely generated essential socle. Let $\{S_1, \dots, S_n\}$ be a complete set of representatives of the isomorphism classes of simple submodules of P; it is actually a complete set of representatives of the isomorphism classes of simple right

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R-modules since P contains a copy of every simple right R-module. Since P is quasi-injective, there exists a summand $Q_1 \oplus \cdots \oplus Q_n$ of P such that $S_i \leq_e Q_i$ for all $i = 1, \ldots, n$. Then, clearly Q_i is Q_j-injective for all $1 \leq i, j \leq n$. Since Q_i is indecomposable quasi-injective, it has a local endomorphism ring. Thus, $J(Q_i)$ is maximal and small in Q_i for each i by [5, Lemma 1.54]. Since Q_i is projective, it follows that Q_i is a projective cover of the simple module $Q_i/J(Q_i)$. Since every module has at most one projective cover up to isomorphism, we have $Q_i/J(Q_i) \cong Q_j/J(Q_j)$ if and only if $Q_i \cong Q_j$. On the other hand, $Q_i \cong Q_j$ implies clearly that $S_i \cong S_j$ and the converse holds because of [4, Corollary 2.32]. Hence $\{Q_1/J(Q_1), \ldots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple right R-modules. Now one concludes that every simple right R-module has a projective cover, so R is semiperfect; moreover, $\{Q_1, \ldots, Q_n\}$ is a complete set of representatives of the isomorphism classes of indecomposable projective right R-modules. Since R is semiperfect, write $R_R = e_1R \oplus \cdots \oplus e_mR$ where each e_iR is indecomposable. So, for each i, $1 \leq i \leq m$, $e_iR \cong Q_k$ for some k, $1 \leq k \leq n$. It follows that e_iR is e_jR-injective for all $1 \leq i, j \leq m$. This shows by [4, Proposition 1.17] that R_R is quasi-injective, and hence is injective. So, being right Kasch, R is right PF. ■

Remark 2 The proof of Theorem 1 shows that if R has a finitely generated projective, quasi-injective right module P containing a copy of every simple right R-module, then P_R is injective (hence a cogenerator) and is a generator. In Theorem 1, "projective" cannot be weakened to "quasi-projective". To see this, let $R = \{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{Z}_2, x \in \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$ and let $P = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) : x \in (0) \oplus \mathbb{Z}_2\}$. It can easily be seen that P_R is a finitely generated quasi-projective, quasi-injective module containing a copy of every simple R-module, but R is not PF. The proof of Theorem 1 also shows that a ring R is a right Kasch, right continuous ring if and only if R has a finitely generated projective, continuous right R-module containing a copy of every simple right R-module; in this case R is semiperfect (for the definitions of continuous modules, right continuous rings, and right CS-rings below, see [4]). We refer to [7] for the discussion of right Kasch right continuous rings. It is a result of Gómez Pardo and Guil Asensio that every right Kasch, right CS-ring has a finitely generated essential right socle, but it is unknown whether a right Kasch right CS-ring is always semiperfect [3].

Corollary 3 *Let $R_R = I \oplus K$ where I_R is injective and $\operatorname{soc}(K_R) = 0$. If R is right Kasch then R is right PF.*

Proof Clearly, I_R is a finitely generated projective, injective module containing a copy of every simple right R-module; so R is right PF by Theorem 1. ■

Theorem 4 *If $(Z_2^r)_R$ is injective such that every simple singular right R-module embeds in $(Z_2^r)_R$, then R is semiperfect.*

Proof This proof uses the same idea of the proof of Theorem 1. It follows from [3, Theorem 14] that $(Z_2^r)_R$ has a finitely generated essential socle. Then, by hypothesis, there exist simple submodules $S_1, \ldots, S_n$ of $(Z_2^r)_R$ such that $\{S_1, \ldots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple singular right R-modules. Since $(Z_2^r)_R$ is injective, there exist submodules $Q_1, \dots, Q_n$ of $(Z_2^r)_R$ such that $Q_1 \oplus \cdots \oplus Q_n$ is a direct summand of $(Z_2^r)_R$ and $(S_i)_R \leq_e (Q_i)_R$ for $i = 1, \dots, n$. Since Q_i is an indecomposable injective R-module, it has a local endomorphism ring, and since Q_i is projective, $J(Q_i)$ is maximal and small in Q_i by [5, Lemma 1.54]. Then Q_i is a projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$, and the converse also holds because every module has at most one projective cover up to isomorphism. But it is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$ if and only if $i = j$. Moreover, every $Q_i/J(Q_i)$ is singular. Thus, $\{Q_1/J(Q_1), \dots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple singular right R-modules. Hence every simple singular right R-module has a projective cover. Since every non-singular simple right R-module is projective, we conclude that R is semiperfect. ■

Theorem 5 *A ring R is right PF if and only if R is a right Kasch ring such that $(Z_2^r)_R$ is injective.*

Proof One direction is clear. Suppose that R is right Kasch and $(Z_2^r)_R$ is injective. Then every simple singular right R-module embeds in $(Z_2^r)_R$. So it follows from Theorem 4 that R is semiperfect. Let $\{S_1, \dots, S_n\}$ be a complete set of representatives of the isomorphism classes of non-singular simple right R-modules. Then all $(S_i)_R$ are projective. Write $R_R = Z_2^r \oplus N$. There is an epimorphism f_1 from R_R to $(S_1)_R$. By the choice of S_1, $f_1|_N \colon N \to S_1$ is epic, so $N = X_1 \oplus N_1$ where $X_1 \cong S_1$. There is an epimorphism f_2 from R_R to $(S_2)_R$. By the choice of S_1 and S_2, $f_2|_{N_1} \colon N_1 \to S_2$ is epic, so $N_1 = X_2 \oplus N_2$ where $X_2 \cong S_2$. Continuing this process, we have $R_R = Z_2^r \oplus X_1 \oplus \cdots \oplus X_n \oplus Y$ where $X_i \cong S_i$ for $i = 1, \dots, n$. Let $Q = Z_2^r \oplus X_1 \oplus \cdots \oplus X_n$. Then Q_R is a finitely generated projective, quasi-injective module containing a copy of every simple right R-module. So R is right PF by Theorem 1. ■

Remark 6 An example is provided in [8] of a left Kasch ring R with $(Z_2^r)_R$ injective which is not right PF. However, it is still an open question whether a left Kasch, right self-injective ring is necessarily right PF (see [2]).

Corollary 7 *A ring R is two-sided PF if and only if R is a right Kasch ring such that $(Z_2^r)_R$ and ${}_R(Z_2^l)$ are both injective.*

Proof A right PF-ring must be left Kasch; so the claim follows from Theorem 5. ■

Corollary 8 *Let $R_R = I \oplus K$ where I_R is injective and K_R is semisimple. If R is right Kasch then R is right PF.*

Proof If $R_R = I \oplus K$ where I_R is injective and K_R is semisimple, then $Z_2^r \subseteq I$. So $(Z_2^r)_R$ is a direct summand of I_R since $(Z_2^r)_R$ is a closed submodule of I_R (*i.e.*, $(Z_2^r)_R$

has no proper essential extensions within I_R). So $(Z_2^r)_R$ is injective, and hence R is right PF by Theorem 5. ■

The next corollary gives an answer to a question of R. Yue Chi Ming [9].

Corollary 9 *If R is a right Kasch ring containing an injective maximal right ideal, then R is right PF.*

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