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DUALIZABILITY OF GRAPHS

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Abstract

We investigate natural dualities for classes of simple graphs. For example, we give a natural duality for the class consisting of all n-colourable graphs and show that, for all $n \geq 3$, there is no natural duality for the class consisting of all freely n-colourable graphs. We also prove that there exist arbitrarily long finite chains of 3-colourable graphs that alternate between being dualizable and nondualizable.

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1. Introduction

The dualizability problem for algebras has been studied over the past 30 years [6, 8, 11, 14, 21, 22]. It was realized in recent years that much of the general theory of natural dualities for algebras works for more general types of structures (see [7, 13]). In particular, we can consider the dualizability problem for relational structures.

One of the earliest examples of a natural duality for a class of relational structures is Banaschewski's duality for ordered sets [2]. Banaschewski showed that the category of ordered sets is dually equivalent to the category of Boolean topological distributive lattices. More recently, natural dualities have been found for the classes of quasi-ordered sets, equivalence-relationed sets, and reflexive, symmetric graphs [17]. This paper aims to broaden the known examples of dualizable relational structures, and give examples of nondualizable relational structures. We focus on the dualizability of *graphs* (that is, sets with a symmetric binary relation). More specifically, most of this paper concerns the dualizability of *simple graphs* (that is, sets with a symmetric, anti-reflexive binary relation).

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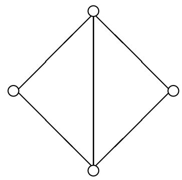

FIGURE 1. The graph $\mathbf{O}_3'$.

We use techniques introduced by Clark *et al.* [5] to obtain a duality for the class consisting of all n-colourable graphs (Theorem 3.4). In contrast, we prove that, for all $n \geq 3$, the class consisting of all freely n-colourable graphs is nondualizable (Corollary 3.8). We then show that every finite bipartite graph is dualizable (Corollary 4.3).

Answering the dualizability problem for 3-colourable graphs seems to be more complicated than the bipartite case. We give a complete characterization of dualizability for 3-colourable graphs that contain a 3-cycle by showing that such a graph is dualizable if and only if the graph $\mathbf{O}_3'$ depicted in Figure 1 is an induced subgraph (Theorem 5.2). To illustrate the difficulty of the dualizability problem for 3-colourable graphs in general, we construct arbitrarily long finite chains of 3-colourable graphs, under the induced subgraph order, that alternate between being dualizable and nondualizable (Theorem 5.5).

Each graph is naturally associated with an algebra; see Section 6. The dualizability problem for these graph algebras has been solved by Davey *et al.* [8] using graph-theoretic techniques. We prove that if a finite graph algebra is dualizable, then the corresponding graph is also dualizable (Theorem 6.4). We show that the converse of this statement does not hold by proving that the 4-element path is a dualizable graph with a nondualizable associated graph algebra.

2. Preliminaries

In this section, we summarize the background theory and prove a useful lemma.

A *relational structure* $\mathbf{M} := \langle M; R\rangle$ consists of an underlying set M and a set R of finitary relations on M. We refer to $\mathbf{M}$ as a *graph* if R consists of a single symmetric binary relation, and we refer to $\mathbf{M}$ as a *simple graph* if R consists of a single symmetric, anti-reflexive binary relation. Note that if $\mathbf{M}$ is a graph, then by a *subgraph* of $\mathbf{M}$ we mean an induced subgraph.

We now give the background theory on natural dualities that will be required here (see [4, 7] or [10]). Let $\mathbf{M} = \langle M; R\rangle$ be a finite relational structure. Suppose that $\underset{\sim}{\mathbf{M}} = \langle M; G, S, \mathcal{T}\rangle$ is a topological structure on the same underlying set, where G is a set of finitary total operations on M, S is a set of finitary relations on M and $\mathcal{T}$ is the discrete topology on M. We allow the operations in G to be nullary but the relations in S have positive arities. We say that $\underset{\sim}{\mathbf{M}}$ is an alter ego of $\mathbf{M}$ if, for each $n \geq 0$, every n-ary operation in G is a homomorphism from $\mathbf{M}^n$ to $\mathbf{M}$.

Let $\underset{\sim}{\mathbf{M}}$ be an alter ego of $\mathbf{M}$. The relational structure $\mathbf{M}$ generates the class $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ consisting of all isomorphic copies of substructures of powers of $\mathbf{M}$. The alter ego $\underset{\sim}{\mathbf{M}}$ generates the topological class $\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}})$ consisting of all isomorphic copies of topologically closed substructures of powers of $\underset{\sim}{\mathbf{M}}$. (In both cases, we take nonempty substructures and powers over nonempty index sets. This is the setting used in [10].) A natural duality gives a dual category equivalence between $\mathcal{A}$ and a full subcategory of $\mathcal{X}$. It provides a means of representing each object in $\mathcal{A}$ as a structure arising from a special set of morphisms in $\mathcal{X}$. We now give the technical definition of a natural duality.

Let $\mathbf{A} \in \mathcal{A}$ and let $\mathcal{A}(\mathbf{A}, \mathbf{M})$ be the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}$. The *dual* of $\mathbf{A}$ is the topologically closed substructure $\mathrm{D}(\mathbf{A})$ of $\underset{\sim}{\mathbf{M}}^A$ with underlying set $\mathcal{A}(\mathbf{A}, \mathbf{M})$. Suppose that $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ is a morphism. We say that α is an *evaluation map* if there exists some $a \in A$ such that $\alpha(\varphi) = \varphi(a)$, for all $\varphi \in \mathcal{A}(\mathbf{A}, \mathbf{M})$. If, for each $\mathbf{A} \in \mathcal{A}$, every morphism $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ is an evaluation, then we say that $\underset{\sim}{\mathbf{M}}$ *dualizes* $\mathbf{M}$. (In this case, each $\mathbf{A} \in \mathcal{A}$ is isomorphic to its double dual—the substructure of $\mathbf{M}^{\mathrm{D}(\mathbf{A})}$ formed by the set of all morphisms from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$.) A finite structure $\mathbf{M}$ is *dualizable* if there exists an alter ego that dualizes $\mathbf{M}$, and is *nondualizable* if no alter ego dualizes $\mathbf{M}$.

Independence of the generator is a fundamental theorem in duality theory. The proof given by Davey and Willard [12] for algebras can be generalized to structures that can have total operations, partial operations and relations in the type (see also [23]). We state the theorem for relational structures, as this is all we need.

THEOREM 2.1 (Independence of the generator). *Let $\mathbf{M}$ and $\mathbf{N}$ be finite relational structures. If $\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{N})$, then $\mathbf{M}$ is dualizable if and only if $\mathbf{N}$ is dualizable.*

The duality compactness theorem [4, Theorem 2.2.11] (see also [7, 13]) is a very useful tool for proving dualizability. It gives a sufficient condition for a duality to lift from the finite level to the infinite level. We state a specific case of the duality compactness theorem suited to our needs.

THEOREM 2.2 (Duality compactness). *Let $\mathbf{M}$ be a finite relational structure and let $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$. Let $\underset{\sim}{\mathbf{M}}$ be an alter ego of $\mathbf{M}$ with only finitely many operations and relations in its type. If, for each finite $\mathbf{A} \in \mathcal{A}$, every morphism $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ is an evaluation, then $\underset{\sim}{\mathbf{M}}$ dualizes $\mathbf{M}$.*

We will prove that, for all $n \geq 3$, the complete simple graph $\mathbf{K}_n$ and the cycle graph $\mathbf{O}_n$ are nondualizable (Lemmas 3.7 and 5.3). We will use the following special case of the nondualizability lemma for relational structures (see [4, Theorem 10.5.1] and [9, Lemma 3.4.1]).

LEMMA 2.3 (Nondualizability). *Let $\mathbf{M}$ be a finite relational structure. Assume that there is a substructure $\mathbf{A}$ of $\mathbf{M}^S$, for some set S, and an infinite subset A_0 of A such that the following hold.*

(i) *For each homomorphism* $x : \mathbf{A} \to \mathbf{M}$, *the equivalence relation* $\ker(x{\restriction}_{A_0})$ *has a unique nontrivial block.*

(ii) *The set* A *does not contain the element* g *of* M^S *defined by* $g(s) = \pi_s(a_s)$, *where* a_s *is any element of the unique nontrivial block of* $\ker(\pi_s{\restriction}_{A_0})$.

Then $\mathbf{M}$ *is nondualizable.*

We require some background material from Clark *et al.* [5]. We remark that the definitions of gst-functions and gst-elements given in Definition 2.5 are stronger than those given in [5]. (The string 'gst' is not an abbreviation, but comes from the symbols used in [5, Lemma 18].) Also, we state [5, Theorem 19] for relational structures in Theorem 2.6. It is an easy exercise to check that the proof of [5, Theorem 19] for algebras can be adapted for relational structures. (In fact, we can allow $\mathbf{M}$ to have both total operations and relations in its type.)

DEFINITION 2.4. Let $\mathbf{M}$ be a finite relational structure and let $g : \mathbf{M}^2 \to \mathbf{M}$ be a homomorphism. We say that $s \in M$ is a *strong idempotent* of $\mathbf{M}$ and that g is a *strong idempotent function* for s if $g^{-1}(s) = \{(s, s)\}$.

DEFINITION 2.5. Let $t \in M$. If there exist $g : \mathbf{M}^2 \to \mathbf{M}$ and $s \in M$ satisfying

$$g(s, v) = s \iff v = t,$$

then we refer to g as a *gst-function* for t and to s as a *gst-element* for t.

Given a relational structure $\mathbf{M}$, we denote by $\mathrm{End}(\mathbf{M})$ the set of endomorphisms of $\mathbf{M}$. Also, for $n \in \mathbb{N}$, we denote the set of all n-ary relations on M by R_n.

THEOREM 2.6. *Let* $\mathbf{M}$ *be a finite relational structure, let* S *be a set of strong idempotents of* $\mathbf{M}$ *and let* G *be a collection of homomorphisms from* $\mathbf{M}^2$ *to* $\mathbf{M}$. *Assume that* G *includes a strong idempotent function for each* $s \in S$. *Assume also that, for each* $t \in M \setminus S$, *there is a gst-function for* t *in* G *and an associated gst-element for* t *in* $S \cap \{f(t) \mid f \in \mathrm{End}(\mathbf{M})\}$. *Then* $\mathbf{M}$ *is dualized by* $\underset{\sim}{\mathbf{M}} := \langle M; G \cup \mathrm{End}(\mathbf{M}), R_{|S|}, \mathcal{T}\rangle$.

The following key lemma will be used to prove a number of our dualizability results. If $\mathbf{A} = \langle A; r\rangle$ is a simple graph and $x \in A$, then we denote by $N_{\mathbf{A}}(x)$ the *neighbourhood of* x in $\mathbf{A}$, that is,

$$N_{\mathbf{A}}(x) := \{y \in A \mid (x, y) \in r\}.$$

LEMMA 2.7. *Let* $\mathbf{M} = \langle M; r\rangle$ *be a finite simple graph. Assume that there exist* $\mathbf{A} \leq \mathbf{M}$ *and distinct* $x, y \in A$ *with* $N_{\mathbf{A}}(x) \subseteq N_{\mathbf{A}}(y)$. *Assume also that there exists a homomorphism* $\varphi : \mathbf{M} \to \mathbf{A}$ *with* $\varphi^{-1}(x) = \{x\}$. *Then* x *is a strong idempotent of* $\mathbf{M}$. *Furthermore,* x *is a gst-element for each* $z \in M \setminus \{x\}$.

PROOF. Let $z \in M$. Let $g_z : M^2 \to M$ be the map defined by

$$g_z(u, v) = \begin{cases} \varphi(u) & \text{if } u \neq x \text{ or } (u, v) = (x, z), \\ y & \text{otherwise.} \end{cases}$$

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We want to prove that g_z is a homomorphism. Let $(s, t), (u, v) \in r$. We will show that $(g_z(s, u), g_z(t, v)) \in r$.

We know that $g_z(s, u) \in \{\varphi(s), y\}$ and $g_z(t, v) \in \{\varphi(t), y\}$. Suppose that $g_z(s, u) = \varphi(s)$ and $g_z(t, v) = \varphi(t)$. Then $(g_z(s, u), g_z(t, v)) \in r$ since φ is a homomorphism. Suppose now that $g_z(s, u) \neq \varphi(s)$. Therefore, $g_z(s, u) = y$ and $y \neq \varphi(s)$. Then $s = x$. So $t \neq x$, since $(s, t) \in r$ and r is anti-reflexive. Therefore, $g_z(t, v) = \varphi(t)$. Using the fact that φ is a homomorphism with $\varphi(x) = x$,

$$\begin{aligned}
& (s, t) \in r \\
\Longrightarrow\ & (\varphi(s), \varphi(t)) \in r \\
\Longrightarrow\ & (x, \varphi(t)) \in r \\
\Longrightarrow\ & (y, \varphi(t)) \in r \quad \text{since } N_{\mathbf{A}}(x) \subseteq N_{\mathbf{A}}(y) \\
\Longrightarrow\ & (g_z(s, u), g_z(t, v)) \in r.
\end{aligned}$$

The case $g_z(t, v) \neq \varphi(t)$ follows by symmetry. Therefore, g_z is a homomorphism.

Consider the homomorphism $g_x : \mathbf{M}^2 \to \mathbf{M}$. Suppose that $g_x(u, v) = x$. Then $\varphi(u) = x$. Using the fact that $\varphi^{-1}(x) = \{x\}$, it follows that $(u, v) = (x, x)$. Therefore, x is a strong idempotent of $\mathbf{M}$. Now let $z \in M \setminus \{x\}$. If $v \in M$, then

$$g_z(x, v) = x \iff v = z,$$

so x is a gst-element for z. □

3. Graph colourings

In this section, we will give a natural duality for the class consisting of all n-colourable graphs. As our finite generator for the class of all n-colourable graphs, we use the following graph due to Wheeler [25].

DEFINITION 3.1. Let $n \in \mathbb{N}$ and let $\Gamma_n := \{d_{i,j} \mid i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}\}$. Let $\mathbf{\Gamma}_n = \langle \Gamma_n; \gamma_n \rangle$, where γ_n is a symmetric, binary relation on Γ_n defined in the following way. Up to symmetry, $(d_{i,j}, d_{r,s}) \in \gamma_n$ if and only if:

(i) $j = s = 1$ and $i \neq r$; or
(ii) $j = 1$, $s = 2$ and $i \neq r$; or
(iii) $j = s = 2$, $i \neq r$, $i \neq 1$ and $r \neq 1$.

THEOREM 3.2 (See [25]). *Let $n \in \mathbb{N}$. Then $\mathbb{ISP}(\mathbf{\Gamma}_n)$ is the class of all n-colourable graphs.*

REMARK 3.3. It is shown by Nešetřil and Pultr [19] that the complete simple graph on $n + 2$ vertices with three particular edges removed also generates the $\mathbb{ISP}$-class of all n-colourable graphs. We could use this generator to prove that there is a natural duality for the class of n-colourable graphs, however, the generator due to Wheeler leads to a simpler proof.

We will prove that $\mathbf{\Gamma}_n$ is dualizable, for all $n \in \mathbb{N}$, by using Lemma 2.7 and Theorem 2.6.

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THEOREM 3.4. *There is a natural duality for the class consisting of all n-colourable graphs, for all $n \in \mathbb{N}$.*

PROOF. We will prove that, for every $i \in \{1, \ldots, n\}$:

(i) $d_{i,2}$ is a strong idempotent of $\mathbf{\Gamma}_n$;
(ii) $d_{i,2}$ is a gst-element for $d_{i,1}$; and
(iii) $d_{i,2} \in \{f(d_{i,1}) \mid f \in \mathrm{End}(\mathbf{\Gamma}_n)\}$.

Let $i \in \{1, \ldots, n\}$. If $d_{r,s} \in \Gamma_n$ then

$$(d_{i,2}, d_{r,s}) \in \gamma_n \Longrightarrow i \neq r \Longrightarrow (d_{i,1}, d_{r,s}) \in \gamma_n,$$

so $N_{\mathbf{\Gamma}_n}(d_{i,2}) \subseteq N_{\mathbf{\Gamma}_n}(d_{i,1})$. Therefore, (i) and (ii) hold by Lemma 2.7. To prove (iii), let $f : \Gamma_n \to \Gamma_n$ be defined by

$$f(d_{r,s}) = \begin{cases} d_{r,1} & \text{if } r \neq i, \\ d_{r,2} & \text{if } r = i. \end{cases}$$

Suppose that $(d_{r,s}, d_{t,u}) \in \gamma_n$. Then $r \neq t$, so $r \neq i$ or $t \neq i$. Hence,

$$(f(d_{r,s}), f(d_{t,u})) \in \{(d_{r,1}, d_{t,1}), (d_{r,1}, d_{t,2}), (d_{r,2}, d_{t,1})\} \subseteq \gamma_n.$$

Therefore, f is an endomorphism of $\mathbf{\Gamma}_n$ with $f(d_{i,1}) = d_{i,2}$. It follows from Theorem 2.6 that $\mathbf{\Gamma}_n$ is dualizable. □

DEFINITION 3.5. For each $n \in \mathbb{N}$, let $\mathbf{K}_n = \langle K_n; \delta_n \rangle$ be the complete simple graph on the set $K_n = \{1, \ldots, n\}$. Thus, $N_{\mathbf{K}_n}(i) = \{1, \ldots, n\} \setminus \{i\}$, for all $i \in \{1, \ldots, n\}$.

A simple graph $\mathbf{A} = \langle A; r \rangle$ is *freely n-colourable* if it is n-colourable and, for every $u, v \in A$ with $u \neq v$ and $(u, v) \notin r$, there exist homomorphisms $\varphi_1, \varphi_2 : \mathbf{A} \to \mathbf{K}_n$ such that $\varphi_1(u) = \varphi_1(v)$ and $\varphi_2(u) \neq \varphi_2(v)$. Thus, an n-colourable graph is freely n-colourable if, for every pair of distinct elements that are not connected by an edge, there is an n-colouring that gives the elements the same colour and an n-colouring that gives the elements different colours. (Freely n-colourable graphs have been studied in a different, though related context by Trotta [24].)

LEMMA 3.6. *For each $n \in \mathbb{N}$, the class $\mathbb{ISP}(\mathbf{K}_n)$ consists of all freely n-colourable graphs.*

PROOF. Let $\mathbf{A} = \langle A; r \rangle$ be freely n-colourable. Let $x, y \in A$ with $x \neq y$. Suppose that $(x, y) \in r$. Since $\mathbf{A}$ is n-colourable, there exists a homomorphism $\varphi : \mathbf{A} \to \mathbf{K}_n$ with $\varphi(x) \neq \varphi(y)$. Suppose instead that $(x, y) \notin r$. Since $\mathbf{A}$ is freely n-colourable, there is a homomorphism $\varphi : \mathbf{A} \to \mathbf{K}_n$ with $\varphi(x) \neq \varphi(y)$. Now let $x, y \in A$ with $(x, y) \notin r$. There is a homomorphism $\varphi : \mathbf{A} \to \mathbf{K}_n$ with $\varphi(x) = \varphi(y)$ so $(\varphi(x), \varphi(y)) \notin \delta_n$. Therefore, $\mathbf{A} \in \mathbb{ISP}(\mathbf{K}_n)$.

Let $\mathbf{A} \in \mathbb{ISP}(\mathbf{K}_n)$. Then $\mathbf{A}$ is n-colourable. Let $x, y \in A$. If $x \neq y$, then there exists a homomorphism $\varphi : \mathbf{A} \to \mathbf{K}_n$ with $\varphi(x) \neq \varphi(y)$. If $(x, y) \notin r$, then there exists a homomorphism $\varphi : \mathbf{A} \to \mathbf{K}_n$ with $(\varphi(x), \varphi(y)) \notin \delta_n$, so $\varphi(x) = \varphi(y)$. It follows that $\mathbf{A}$ is freely n-colourable. □

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The following technical lemma will show that $\mathbf{K}_n$ is nondualizable, for each $n \geq 3$, and will also be used in Section 5.

LEMMA 3.7. *Let $\mathbf{M} = \langle M; r\rangle$ be a finite simple graph and let $m \in \mathbb{N}$ be such that $\mathbf{K}_m$ embeds into $\mathbf{M}$ but $\mathbf{K}_{m+1}$ does not embed into $\mathbf{M}$. Assume that $m \geq 3$. Let $\mathbf{K}^*_{m+1}$ be the complete graph $\mathbf{K}_{m+1}$ with one edge removed. If $\mathbf{K}^*_{m+1}$ does not embed into $\mathbf{M}$, then $\mathbf{M}$ is nondualizable.*

PROOF. Assume $\mathbf{K}^*_{m+1}$ does not embed into $\mathbf{M}$. We will use Lemma 2.3 (nondualizability). We can assume that $M = \{1, \ldots, p\}$, for some $p \geq m \geq 3$. We can also assume that $\delta_m \subseteq r^{\mathbf{M}}$.

Suppose that $u, v \in M$ and $F \subseteq \mathbb{N}$. We define $u^v_F \in M^{\mathbb{N}}$ in the following way:

$$u^v_F(w) = \begin{cases} v & \text{if } w \in F, \\ u & \text{if } w \notin F, \end{cases}$$

where $w \in \mathbb{N}$. If $F = \{i\}$ or $F = \{i, j\}$, then we write u^v_i and u^{vv}_{ij} instead of $u^v_{\{i\}}$ and $u^v_{\{i,j\}}$, respectively. If $F = \varnothing$, then we write $\hat{u}$ instead of $u^v_{\varnothing}$.

Let $A_0 := \{3^1_i \mid i \in \mathbb{N}\}$ and $A := M^{\mathbb{N}} \setminus \{\hat{3}\}$. Let $x : \mathbf{A} \to \mathbf{M}$ be a homomorphism. Assume that $i, j, k, l \in \mathbb{N}$ are distinct with $x(3^1_i) = x(3^1_j)$ and $x(3^1_k) = x(3^1_l)$. We aim to prove that $x(3^1_i) = x(3^1_k)$.

Since we have assumed that $\delta_m \subseteq r^{\mathbf{M}}$, we have $(3^1_i, 1^{22}_{ik}), (3^1_j, 2^{11}_{ik}), (1^{22}_{ik}, 2^{11}_{ik}) \in r^{\mathbf{A}}$. Hence,

$$(x(3^1_i), x(1^{22}_{ik})), (x(3^1_j), x(2^{11}_{ik})), (x(1^{22}_{ik}), x(2^{11}_{ik})) \in r^{\mathbf{M}}. \tag{3.1}$$

Suppose that $m = 3$ and let $X := \{x(3^1_i), x(1^{22}_{ik}), x(2^{11}_{ik})\}$. Then by (3.1) and the fact that $x(3^1_i) = x(3^1_j)$, it follows that $\mathbf{X} \cong \mathbf{K}_3$. Similarly, if $Y := \{x(3^1_k), x(1^{22}_{ik}), x(2^{11}_{ik})\}$, then $\mathbf{Y} \cong \mathbf{K}_3$. If $x(3^1_i) \neq x(3^1_k)$, then the subgraph formed by $X \cup Y$ is an isomorphic copy of $\mathbf{K}_{m+1}$ or $\mathbf{K}^*_{m+1}$, which is a contradiction. Therefore, $x(3^1_i) = x(3^1_k)$, as required.

Now suppose that $m \geq 4$. For all $s \in \{4, \ldots, m\}$ and all $t \in \{4, \ldots, m\} \setminus \{s\}$, we have

$$\begin{aligned} &(3^1_i, \hat{s}), (1^{22}_{ik}, \hat{s}), (2^{11}_{ik}, \hat{s}), (\hat{s}, \hat{t}) \in r^{\mathbf{A}} \\ &\implies (x(3^1_i), x(\hat{s})), (x(1^{22}_{ik}), x(\hat{s})), (x(2^{11}_{ik}), x(\hat{s})), (x(\hat{s}), x(\hat{t})) \in r^{\mathbf{M}}. \end{aligned} \tag{3.2}$$

Let $X := \{x(3^1_i), x(1^{22}_{ik}), x(2^{11}_{ik}), x(\hat{4}), \ldots, x(\hat{m})\}$. Then $\mathbf{X} \cong \mathbf{K}_m$ by (3.1) and (3.2). Similarly, if we let $Y := \{x(3^1_k), x(1^{22}_{ik}), x(2^{11}_{ik}), x(\hat{4}), \ldots, x(\hat{m})\}$, then $\mathbf{Y} \cong \mathbf{K}_m$. If $x(3^1_i) \neq x(3^1_k)$, then the subgraph formed by $X \cup Y$ is an isomorphic copy of $\mathbf{K}_{m+1}$ or $\mathbf{K}^*_{m+1}$, which is a contradiction. Thus, $x(3^1_i) = x(3^1_k)$.

We have shown that $\ker(x{\restriction}_{A_0})$ has a unique block of size greater than 1. The element given by part (ii) of Lemma 2.3 (nondualizability) is $g = \hat{3} \notin A$, so $\mathbf{M}$ is nondualizable. $\square$

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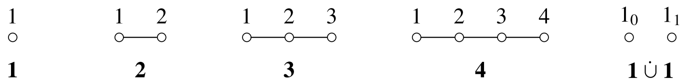

FIGURE 2. Caicedo's list of bipartite graphs.

Using Lemmas 3.6 and 3.7 (with $\mathbf{M} = \mathbf{K}_n$ and $m = n$), and Theorem 2.1 (independence of the generator), we have the following corollary.

COROLLARY 3.8. *For all $n \geq 3$, there is no natural duality for the class of all freely n-colourable graphs.*

4. Bipartite graphs

We will prove that every finite bipartite graph is dualizable. Let **1**, **2**, **3**, **4** be the simple graphs depicted in Figure 2. It is shown by Caicedo [3] that, if **M** is a finite bipartite graph, then

$$\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M}'),$$

for some $\mathbf{M}' \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1}\,\dot\cup\,\mathbf{1}\}$. By Theorem 2.1 (independence of the generator), it therefore suffices to show that the graphs **1**, **2**, **3**, **4** and $\mathbf{1}\,\dot\cup\,\mathbf{1}$ are dualizable.

It is trivial that **1** is dualizable. Banaschewski's duality for sets [1] (see also [4]) tells us that $\mathbf{1}\,\dot\cup\,\mathbf{1}$ is dualizable. The fact that **4** is dualizable follows from Theorem 3.4, since $\mathbf{4} \cong \mathbf{\Gamma}_2$ so $\mathbb{ISP}(\mathbf{4})$ is the class of all bipartite graphs. It remains to prove that **2** and **3** are dualizable.

It is shown in [16] that if $\mathbf{M} = \langle M; \mathrm{graph}(f)\rangle$, for some finite set M and some unary operation f on M, then **M** is dualizable. Let $'$ be the unary operation on $\{1, 2\}$ with $1' = 2$ and $2' = 1$. Since $\mathbf{2} = \langle\{1, 2\}; \mathrm{graph}(')\rangle$, it follows that **2** is dualizable. We will give a direct proof of this fact.

LEMMA 4.1. *The graph **2** is dualized by the alter ego $\underset{\sim}{\mathbf{2}} = \langle\{1, 2\}; R_3, \mathcal{T}\rangle$, where R_3 consists of all ternary relations on $\{1, 2\}$.*

PROOF. We will use Theorem 2.2 (duality compactness). Let $\mathbf{A} \leq \mathbf{2}^n$, for some $n \in \mathbb{N}$, and let $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{2}}$ be a morphism. We want to show that α is an evaluation map.

Let $\varphi : \mathbf{A} \to \mathbf{2}$ be a homomorphism. We can extend the unary operation $'$ pointwise to $\{1, 2\}^n$. Let

$$B_1 := \{a \in A \mid a' \in A \text{ and } \varphi(a) = \alpha(\varphi)\}, \qquad B_2 := \{a \in A \mid a' \in A \text{ and } \varphi(a) = \alpha(\varphi)'\},$$
$$C_1 := \{a \in A \mid a' \notin A \text{ and } \varphi(a) = \alpha(\varphi)\}, \qquad C_2 := \{a \in A \mid a' \notin A \text{ and } \varphi(a) = \alpha(\varphi)'\}.$$

Every map ψ from $B_1 \cup C_1 \cup C_2$ to $\{1, 2\}$ extends to a homomorphism ψ^+ from **A** to **2** with $\psi^+(a) = \psi(a')'$, for $a \in B_2$.

Consider the unary relation $\varphi(A)$ on $\{1, 2\}$. Clearly, $\varphi \in \varphi(A)$ in $\mathrm{D}(\mathbf{A})$, so $\alpha(\varphi) \in \varphi(A)$ since α preserves all relations on $\{1, 2\}$ of arity at most 3. Thus, we can enumerate the elements of $B_1 \cup C_1$ as $a_1, \ldots, a_m$, for some $m \in \mathbb{N}$. For each $i \in \{0, \ldots, m\}$, let $\psi_i : \mathbf{A} \to \mathbf{2}$ be the unique homomorphism determined by $\psi_i(C_2) = \{\alpha(\varphi)'\}$ and

$$\psi_i(a_j) = \begin{cases} \alpha(\varphi)' & \text{if } j \leq i, \\ \alpha(\varphi) & \text{if } j > i. \end{cases}$$

Define the binary relation $r := \{1, 2\}^2 \setminus \{(\alpha(\varphi), \alpha(\varphi))\}$. Then $(\varphi, \psi_m) \in r$ in $\mathrm{D}(\mathbf{A})$, so $\alpha(\psi_m) = \alpha(\varphi)'$. Let $k := \min\{i \in \{1, \ldots, m\} \mid \alpha(\psi_i) = \alpha(\varphi)'\}$. If $k \geq 2$ then $\alpha(\psi_{k-1}) = \alpha(\varphi)$ by the definition of k as the minimum. If $k = 1$ then $\psi_{k-1} = \varphi$ so $\alpha(\psi_{k-1}) = \alpha(\varphi)$. We will show that $\alpha(\xi) = \xi(a_k)$, for all homomorphisms $\xi : \mathbf{A} \to \mathbf{2}$.

Let $\xi : \mathbf{A} \to \mathbf{2}$. Define a ternary relation s on $\{1, 2\}$ by

$$s := \{(x, y, y) \mid x, y \in \{1, 2\}\} \cup \{(\xi(a_k), \alpha(\varphi), \alpha(\varphi)')\} \cup \{(\xi(a_k)', \alpha(\varphi)', \alpha(\varphi))\}.$$

It then follows that

$$\begin{aligned} (\xi, \psi_{k-1}, \psi_k) \in s \quad &\text{in } \mathrm{D}(\mathbf{A}), \\ \Longrightarrow \alpha(\xi) = \xi(a_k)&, \end{aligned}$$

since α preserves all ternary relations on $\{1, 2\}$. We have therefore shown that α is given by evaluation at a_k. By the duality compactness theorem, we can conclude that $\mathbf{2}$ is dualizable. □

It remains to prove that $\mathbf{3}$ is dualizable. We will use Lemma 2.7 and Theorem 2.6.

LEMMA 4.2. *The graph* $\mathbf{3}$ *is dualizable.*

PROOF. Observe that $N_{\mathbf{3}}(1) = \{2\} = N_{\mathbf{3}}(3)$. By Lemma 2.7, the elements 1 and 3 are strong idempotents of $\mathbf{3}$ and are gst-elements for 2. Let $\psi : \mathbf{3} \to \mathbf{3}$ be the homomorphism defined by $\psi(1) = 2 = \psi(3)$ and $\psi(2) = 1$. Then $\psi \in \mathrm{End}(\mathbf{3})$ so $1 \in \{f(2) \mid f \in \mathrm{End}(\mathbf{3})\}$. By Theorem 2.6, the graph $\mathbf{3}$ is dualizable. □

COROLLARY 4.3. *Every finite bipartite graph is dualizable.*

5. Graphs containing odd cycles

We will give a complete characterization of dualizable 3-colourable graphs containing a 3-cycle. Within this class, we will show that dualizability is dependent upon the existence of a certain subgraph. We will also show that there exist arbitrarily long finite chains of 3-colourable graphs, under the subgraph order, that are alternately dualizable and nondualizable. This is similar to the situation for algebras; it is known that there exist infinite chains of unary algebras, under the subalgebra order, that are alternately dualizable and nondualizable [20].

Let $\mathbf{M} = \langle M; \sim \rangle$ be a simple graph and let $n \in \mathbb{N}$. By a *cycle* of length n in $\mathbf{M}$ we mean a collection of n distinct elements $a_1, \ldots, a_n \in M$ with

$$a_1 \sim a_2 \sim \cdots \sim a_n \sim a_1.$$

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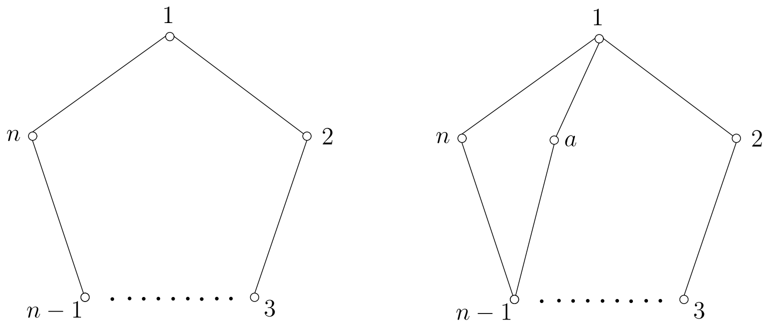

FIGURE 3. The cycle graph $\mathbf{O}_n$ and the graph $\mathbf{O}'_n$.

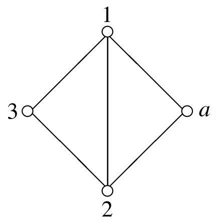

FIGURE 4. The graph $\mathbf{O}'_3$.

If $n \in \mathbb{N}$ with $n \geq 3$, then we let $\mathbf{O}_n = \langle\{1, \ldots, n\}; \theta_n\rangle$ be the n-element cycle graph, as given in Figure 3. Let $\mathbf{O}'_n = \langle\{1, \ldots, n, a\}; \theta'_n\rangle$ be obtained from $\mathbf{O}_n$ by adding an extra element a, an edge between 1 and a, and an edge between $n-1$ and a (see Figure 3). Thus,

$$\theta'_n = \theta_n \cup \{(1, a), (a, 1), (n-1, a), (a, n-1)\}.$$

The graph $\mathbf{O}'_3$ will be particularly important in our characterization of dualizability (see Figure 4).

Let $\mathfrak{C}$ be a class of relational structures. We say that a finite structure $\mathbf{M} \in \mathfrak{C}$ is *inherently dualizable* within $\mathfrak{C}$ if every finite member of $\mathfrak{C}$ that has $\mathbf{M}$ as a substructure is dualizable.

LEMMA 5.1. *The graph $\mathbf{O}'_3$ is inherently dualizable within the class of 3-colourable graphs.*

PROOF. Let $\mathbf{M}$ be a finite 3-colourable graph with $\mathbf{O}'_3 \leq \mathbf{M}$. We will prove that, for all $z \in M$, there exists a homomorphism $\psi_z : \mathbf{M} \to \mathbf{O}'_3$ with $\psi_z^{-1}(a) = \{z\}$.

Let $z \in M$. Since $\mathbf{M}$ is 3-colourable, there exists a homomorphism from $\mathbf{M}$ to $\mathbf{O}_3$. Therefore, there exists a homomorphism $\varphi_z : \mathbf{M} \to \mathbf{O}_3$ with $\varphi_z(z) = 3$. Define $\psi_z : M \to O'_3$ by $\psi_z{\restriction}_{M\setminus\{z\}} = \varphi_z{\restriction}_{M\setminus\{z\}}$ and $\psi_z(z) = a$. Since $N_{\mathbf{O}'_3}(a) = N_{\mathbf{O}'_3}(3)$, the map ψ_z is a homomorphism. We now observe that $\psi_z^{-1}(a) = \{z\}$. In particular, $\psi_a^{-1}(a) = \{a\}$.

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Applying Lemma 2.7 with $\mathbf{A} = \mathbf{O}'_3$, $x = a$ and $y = 3$, we see that a is a strong idempotent of $\mathbf{M}$ and a is a gst-element for each $z \in M \setminus \{a\}$. The map ψ_z is an endomorphism of $\mathbf{M}$ with $\psi_z(z) = a$, for each $z \in M \setminus \{a\}$. Thus, $\mathbf{M}$ is dualizable by Theorem 2.6. □

THEOREM 5.2. *Let $\mathbf{M}$ be a finite 3-colourable graph containing a cycle of length 3. Then $\mathbf{M}$ is dualizable if and only if $\mathbf{O}'_3$ embeds into $\mathbf{M}$.*

PROOF. Assume that $\mathbf{M}$ contains a subgraph isomorphic to $\mathbf{O}'_3$. Then $\mathbf{M}$ is dualizable by Lemma 5.1. Assume that $\mathbf{M}$ does not contain a subgraph isomorphic to $\mathbf{O}'_3$. We observe that $\mathbf{O}_3 = \mathbf{K}_3$ and $\mathbf{O}'_3 \cong \mathbf{K}^*_4$. Since $\mathbf{M}$ contains a cycle of length 3, we know that $\mathbf{K}_3$ embeds into $\mathbf{M}$. Also, $\mathbf{K}_4$ does not embed into $\mathbf{M}$ since $\mathbf{M}$ is 3-colourable. By assumption, $\mathbf{K}^*_4$ does not embed into $\mathbf{M}$. So $\mathbf{M}$ is nondualizable by Lemma 3.7. □

Note that any graph that is not bipartite must contain an odd cycle.

LEMMA 5.3. *Assume that $\mathbf{M} = \langle M; \sim\rangle$ is a finite 3-colourable graph that is not bipartite and does not contain a cycle of length 3. Let p be the length of the shortest odd cycle in $\mathbf{M}$, so $p \geq 5$. Assume that if $\mathbf{X}, \mathbf{Y} \leq \mathbf{M}$ with $\mathbf{X} \cong \mathbf{O}_p \cong \mathbf{Y}$, then $X = Y$ or $X \cap Y = \varnothing$. Then $\mathbf{M}$ is nondualizable.*

PROOF. We will use Lemma 2.3 (nondualizability) and the notation introduced in the proof of Lemma 3.7.

We can assume that $\mathbf{O}_p \leq \mathbf{M}$. Let $A_0 := \{1^3_i \mid i \in \mathbb{N}\}$ and let $A := O_p^{\mathbb{N}} \setminus \{\hat{1}\}$. Let $x : \mathbf{A} \to \mathbf{M}$ be a homomorphism. Assume that $i, j, k, l \in \mathbb{N}$ are distinct with $x(1^3_i) = x(1^3_j)$ and $x(1^3_k) = x(1^3_l)$. Suppose that $x(1^3_i) \neq x(1^3_k)$. We will arrive at a contradiction by using the following claims, which are straightforward to prove.

Claim 1. If $\mathbf{B} \leq \mathbf{A}$ with $\mathbf{B} \cong \mathbf{O}_p$, then $x(\mathbf{B}) \cong \mathbf{O}_p$.

Claim 2. Suppose that $\mathbf{C} \leq \mathbf{M}$ with $\mathbf{C} \cong \mathbf{O}_p$. If $x \sim y \sim z$ and $x, z \in C$ are distinct, then $y \in C$.

Let

$$B_1 := \{1^3_i, \hat{2}, 3^1_i, \ldots, p^4_i\}, \quad B_2 := \{1^3_j, \hat{2}, 3^1_j, \ldots, p^4_j\},$$
$$B_3 := \{1^3_k, \hat{2}, 3^1_k, \ldots, p^4_k\}, \quad B_4 := \{1^3_l, \hat{2}, 3^1_l, \ldots, p^4_l\}.$$

For each $m \in \{1, 2, 3, 4\}$, let $\mathbf{B}_m$ be the substructure of $\mathbf{A}$ formed by the set B_m and let $x(\mathbf{B}_m)$ be the substructure of $\mathbf{M}$ formed by the set $x(B_m)$. Then using Claim 1 and the fact that $x(\hat{2}) \in x(B_1) \cap \cdots \cap x(B_4)$,

$$\begin{aligned} &\mathbf{B}_1 \cong \mathbf{B}_2 \cong \mathbf{B}_3 \cong \mathbf{B}_4 \cong \mathbf{O}_p \\ \Longrightarrow\ & x(\mathbf{B}_1) \cong x(\mathbf{B}_2) \cong x(\mathbf{B}_3) \cong x(\mathbf{B}_4) \cong \mathbf{O}_p \\ \Longrightarrow\ & x(\mathbf{B}_1) = x(\mathbf{B}_2) = x(\mathbf{B}_3) = x(\mathbf{B}_4). \end{aligned}$$

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We can assume, without loss of generality, that $x(\mathbf{B}_m)=\mathbf{O}_p$, for all $m\in\{1, 2, 3, 4\}$. Since $1_i^3 \sim \hat{2} \sim 1_k^3$ and $x(1_i^3) \neq x(1_k^3)$, we can further assume that $x(1_i^3) = 1$, $x(\hat{2}) = 2$ and $x(1_k^3) = 3$. Then since $1_i^3 \sim \hat{2} \sim 3_i^1$, we have $x(3_i^1) = 3$. Similarly, we can show that $x(3_j^1) = 3$ and $x(3_k^1) = x(3_l^1) = 1$.

Now, $3_l^1 \sim 4_{lj}^{22} \sim 3_j^1$ so $1 = x(3_l^1) \sim x(4_{lj}^{22}) \sim x(3_j^1) = 3$. By Claim 2, we have $x(4_{lj}^{22}) \in O_p$ so $x(4_{lj}^{22}) = 2$. We now let $B_5 := \{2_{lj}^{4p}, 3_j^1, 4_{lj}^{22}, \dots, 1_{lj}^{5\,p-1}\}$. Then $\mathbf{B}_5$ is an isomorphic copy of $\mathbf{O}_p$ in $\mathbf{A}$, so $x(\mathbf{B}_5) \cong \mathbf{O}_p$ by Claim 1. Since $x(3_j^1) \in x(B_5) \cap O_p$, we must have $x(B_5) = O_p$. Using the fact that $x(3_j^1) = 3$ and $x(4_{lj}^{22}) = 2$, it follows that $x(2_{lj}^{4p}) = 4$. Observe now that $1_i^3 \sim 2_{ij}^{4p} \sim 3_j^1$ implies that $1 = x(1_i^3) \sim x(2_{ij}^{4p}) \sim x(3_j^1) = 3$. By Claim 2, we have $x(2_{ij}^{4p}) \in O_p$ so $x(2_{ij}^{4p}) = 2$. Using a similar argument and the fact that $2 = x(2_{ij}^{4p}) \sim x(3_{kj}^{1\,p-1}) \sim x(2_{lj}^{4p}) = 4$, we have $x(3_{kj}^{1\,p-1}) = 3$.

Let $B_6 = \{1_k^3, 2_j^p, 3_{kj}^{1\,p-1}, \dots, p_{kj}^{42}\}$. Then $\mathbf{B}_6 \cong \mathbf{O}_p$ so $x(\mathbf{B}_6) \cong \mathbf{O}_p$ by Claim 1. But this contradicts that $x(1_k^3) = 3 = x(3_{kj}^{1\,p-1})$. Hence, we can conclude that $x(1_i^3) = x(1_k^3)$ so part (i) of Lemma 2.3 (nondualizability) holds. The element given by part (ii) of Lemma 2.3 is $g = \hat{1} \notin A$, so $\mathbf{M}$ is nondualizable. $\square$

We now show that we can construct arbitrarily long finite chains of 3-colourable graphs, under the subgraph order, that alternate between being dualizable and nondualizable. We continue to use the notation $\mathbf{O}_n$ to denote the cycle graph with n-elements, where $n \geq 3$. We will now label the elements of $\mathbf{O}_n$ as $\{1_n, \dots, n_n\}$ and label the element of $\mathbf{O}_n'$ added to $\mathbf{O}_n$ as a_n.

LEMMA 5.4. *If j and k are odd with $3 \leq j \leq k$, then $\mathbf{O} := \mathbf{O}_j' \,\dot\cup\, \mathbf{O}_{j+2}' \,\dot\cup \cdots \dot\cup\, \mathbf{O}_k'$ is dualizable.*

PROOF. Let $n \in \{j, j+2, \dots, k\}$. Since $N_{\mathbf{O}}(n_n) = N_{\mathbf{O}}(a_n)$, it follows from Lemma 2.7 that n_n and a_n are strong idempotents of $\mathbf{O}$, and both are gst-elements for the remaining elements in O.

Let $x \in O_n' \setminus \{n_n, a_n\}$. Let $\varphi_x : O \to O$ be defined by $\varphi_x{\restriction_{O_i'}} = \mathrm{id}_{O_i'}$, if $i \neq n$, and $\varphi_x{\restriction_{O_n'}}$ is any endomorphism of $\mathbf{O}_n'$ that sends x to n_n. Then φ_x is an endomorphism of $\mathbf{O}$ sending x to the strong idempotent n_n, so $\mathbf{O}$ is dualizable by Theorem 2.6. $\square$

THEOREM 5.5. *Let $n \in \mathbb{N}$ with $3 \leq n$. There exist 3-colourable graphs $\mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ such that $\mathbf{A}_n \leq \mathbf{A}_{n-1} \leq \cdots \leq \mathbf{A}_3 \leq \mathbf{A}_2$, and $\mathbf{A}_i$ is dualizable if and only if i is even.*

PROOF. Note that, for all odd $i \in \mathbb{N}$, the only odd length cycles contained in $\mathbf{O}_i'$ have length i, and $\mathbf{O}_i$ and $\mathbf{O}_i'$ are 3-colourable. Let $n \in \mathbb{N}$ with $n \geq 3$. We can assume, without loss of generality, that n is odd. Set $\mathbf{A}_n := \mathbf{O}_n$. Let $i \in \mathbb{N}$ with $i < n$. Then we

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define $\mathbf{A}_i$ by

$$\mathbf{A}_i := \begin{cases} \mathbf{O}'_n \mathbin{\dot\cup} \mathbf{O}'_{n-2} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathbf{O}'_{i+2} \mathbin{\dot\cup} \mathbf{O}_i & \text{if } i \text{ is odd,} \\ \mathbf{O}'_n \mathbin{\dot\cup} \mathbf{O}'_{n-2} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathbf{O}'_{i+1} & \text{if } i \text{ is even.} \end{cases}$$

It is clear from this definition that each $\mathbf{A}_i$ is the disjoint union of 3-colourable graphs and is therefore 3-colourable. Also, $\mathbf{A}_{i+1} \le \mathbf{A}_i$, for all $i \in \{2, \ldots, n-1\}$. It remains to prove that $\mathbf{A}_i$ is dualizable if and only if i is even.

Let $i \in \mathbb{N}$ with $2 \le i \le n$. Assume that i is odd. Then

$$\mathbf{A}_i = \mathbf{O}'_n \mathbin{\dot\cup} \mathbf{O}'_{n-2} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathbf{O}'_{i+2} \mathbin{\dot\cup} \mathbf{O}_i$$

if $i < n$, and $\mathbf{A}_i = \mathbf{O}_i$ if $i = n$. The shortest odd length cycle in $\mathbf{A}_i$ has length i, and there is only one such cycle (namely $\mathbf{O}_i$). If $i > 3$ then $\mathbf{A}_i$ is nondualizable by Lemma 5.3. If $i = 3$ then $\mathbf{A}_i$ is nondualizable by Theorem 5.2. Assume instead that i is even. Then $\mathbf{A}_i = \mathbf{O}'_n \mathbin{\dot\cup} \mathbf{O}'_{n-2} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathbf{O}'_{i+1}$ so $\mathbf{A}_i$ is dualizable by Lemma 5.4. □

REMARK 5.6. In the proof of Theorem 5.5, we constructed chains of graphs that alternated between being dualizable and nondualizable by adding vertices and edges. By starting with a bigger underlying set, we can modify this proof to obtain alternating chains of graphs by only adding edges. Clearly, only finite chains are possible here.

6. Graphs and graph algebras

In this section, we will prove that if a finite graph algebra is dualizable, then the corresponding graph is also dualizable.

Let $\mathbf{M} = \langle M; r\rangle$ be a graph (so r is a binary relation that is symmetric but not necessarily anti-reflexive). We let $\mathcal{A}(\mathbf{M}) = \langle \mathcal{A}(M); \cdot\rangle$ be the corresponding *graph algebra*, where $\mathcal{A}(M) = M \cup \{0\}$ and we insist that $0 \notin M$. The operation $\cdot$ is a binary operation defined in the following way: if $u, v \in \mathcal{A}(M)$, then

$$u \cdot v = \begin{cases} u & \text{if } u, v \in M \text{ and } (u, v) \in r, \\ 0 & \text{otherwise.} \end{cases}$$

For example, consider the path $\mathbf{4}$ that we met in Figure 2. The graph algebra $\mathcal{A}(\mathbf{4})$ has universe $\{0, 1, 2, 3, 4\}$ and $\cdot$ is given by the table in Figure 5.

The dualizability of graph algebras has been studied by Davey *et al.* [8] and Lampe *et al.* [18]. A complete characterization of dualizable graph algebras is given in [8] (see Theorem 6.1).

Let $\mathbf{M}$ be a graph. We say that $\mathbf{M}$ is *complete* if every pair of elements of M is joined by an edge and every element has a loop. We say that $\mathbf{M}$ is *bipartite complete* if M can be partitioned into two nonempty sets such that any two elements belonging to different sets are joined by an edge but no elements belonging to the same set are joined by an edge. We say that $\mathbf{M}$ is a *loose vertex* if $\mathbf{M}$ consists of a single element and no edges.

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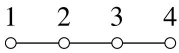

4

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	1	0	0
2	0	2	0	2	0
3	0	0	3	0	3
4	0	0	0	4	0

FIGURE 5. The graph **4** and its graph algebra.

THEOREM 6.1 (See [8, Theorem 1]). *Let* $\mathbf{M}$ *be a finite graph. The graph algebra* $\mathcal{A}(\mathbf{M})$ *is dualizable if and only if each connected component of* $\mathbf{M}$ *is either complete, bipartite complete, or a loose vertex.*

We will use this characterization of dualizability for graph algebras to prove that if $\mathbf{M}$ is a finite graph and the graph algebra $\mathcal{A}(\mathbf{M})$ is dualizable, then $\mathbf{M}$ is also dualizable. (We note that the reverse implication does not hold. To see this, consider the graph **4**. We saw in Section 4 that **4** is dualizable, however, the corresponding graph algebra $\mathcal{A}(\mathbf{4})$ is nondualizable by Theorem 6.1.)

We will use the following theorem, which is the analogue of [9, Theorem 5.1.10] for relational structures. This theorem provides conditions under which the disjoint union of two relational structures is dualizable. Most of the proof of [9, Theorem 5.1.10] still works for relational structures. We will only need to make some minor changes. Note that given a relational type R, we let $\mathbf{1}_R$ denote the complete one-element structure of type R, that is, a one-element structure such that every relation in R is nonempty. (Recall that if $\mathbf{M}$ is a finite relational structure of type R, then $\mathbf{1}_R$ may not be contained in $\mathbb{ISP}(\mathbf{M})$ since we take powers over nonempty index sets.)

THEOREM 6.2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite relational structures of type* R*, such that* $\mathbf{N} \in \mathbb{ISP}(\mathbf{M}) \cup \{\mathbf{1}_R\}$*. If* $\mathbf{M}$ *is dualizable, then* $\mathbf{M} \dot\cup \mathbf{N}$ *is dualizable.*

PROOF. We will assume that $M \cap N = \varnothing$. Let $\boldsymbol{\mathcal{B}} := \mathbb{ISP}(\mathbf{M} \cup \mathbf{N})$. Let $\mathbf{B} \in \boldsymbol{\mathcal{B}}$ and let $\beta : \boldsymbol{\mathcal{B}}(\mathbf{B}, \mathbf{M} \cup \mathbf{N}) \to M \cup N$ be a continuous map that preserves every finitary relation on $M \cup N$. We will prove that β is given by evaluation. It will then follow that $\mathbf{M} \cup \mathbf{N}$ is dualized by the discretely topologized structure on $M \cup N$ that has every finitary relation on $M \cup N$ in its type. We consider three cases.

Case 1. β is constant and $\mathbf{N} \cong \mathbf{1}_R$. It follows from Case 1 of [9, Theorem 5.1.10] that β is an evaluation map.

Case 2. β is constant and $\mathbf{N} \ncong \mathbf{1}_R$. Let m denote the value of β on $\boldsymbol{\mathcal{B}}(\mathbf{B}, \mathbf{M} \cup \mathbf{N})$. We will show that $m \in M$. It then follows from Case 2 of [9, Theorem 5.1.10] that β is an evaluation map.

Since $\mathbf{N} \ncong \mathbf{1}_R$, we know that $\mathbf{N} \in \mathbb{ISP}(\mathbf{M})$. Therefore, there is a homomorphism from $\mathbf{N}$ to $\mathbf{M}$. It follows that there exists some $x \in \boldsymbol{\mathcal{B}}(\mathbf{B}, \mathbf{M} \cup \mathbf{N})$ with $x(B) \subseteq M$. Now, β preserves the unary relation M, so $m = \beta(x) \in M$.

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Case 3. β is not constant. The proof of Case 3 from [9, Theorem 5.1.10] shows that β is given by evaluation.

We can therefore conclude that $\mathbf{M} \cup \mathbf{N}$ is dualizable. $\square$

COROLLARY 6.3. *Let* $\mathbf{M}$ *be a finite relational structure of type* R. *If* $\mathbf{M}$ *is dualizable then* $\mathbf{M} \mathbin{\dot\cup} \mathbf{1}_R$ *is dualizable.*

THEOREM 6.4. *Let* $\mathbf{M}$ *be a finite graph. If the graph algebra* $\mathcal{A}(\mathbf{M})$ *is dualizable, then* $\mathbf{M}$ *is dualizable.*

PROOF. Let $\mathbf{M}$ be a finite graph and assume that $\mathcal{A}(\mathbf{M})$ is dualizable. By Theorem 6.1, each connected component of $\mathbf{M}$ is either complete, bipartite complete, or a loose vertex. If $\mathbf{M}$ contains a complete connected component with at least two elements, then $\mathbf{M}$ is dualizable by [15, Lemma 2.4]. We can therefore assume that each connected component of $\mathbf{M}$ is either a single looped vertex, bipartite complete, or a loose vertex. Let $\mathbf{A}$ be the union of the connected components of $\mathbf{M}$ that are looped vertices and let $\mathbf{B}$ be the union of the connected components of $\mathbf{M}$ that are bipartite complete or loose vertices. Suppose that $B \neq \varnothing$. Then $\mathbf{B}$ is bipartite and therefore dualizable, by Corollary 4.3. It follows from Corollary 6.3 that $\mathbf{M} = \mathbf{A} \cup \mathbf{B}$ is dualizable. Suppose instead that $B = \varnothing$. Then $\mathbf{M}$ is the disjoint union of looped vertices. If $|M| = 1$ then it is trivial that $\mathbf{M}$ is dualizable. If $|M| > 1$ then it follows from Corollary 6.3 that $\mathbf{M}$ is dualizable. $\square$

REMARK 6.5. In fact, if each connected component of $\mathbf{M}$ is either a single looped vertex, bipartite complete, or a loose vertex, then $\mathbb{ISP}(\mathbf{M})$ is one of only eight possible classes.

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