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# ULTRADISCRETIZATION OF THE TZITZEICA EQUATION

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**Abstract.** The trilinear form of the discrete Tzitzeica equation by Schief is found to be a discrete Toda molecule equation with a special boundary condition. Based on this fact, a higher order discrete Tzitzeica equation and an ultradiscrete Tzitzeica equation are obtained.

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**1. Introduction.** Tzitzeica [**1**, **2**] obtained the equation

$$(\log h)_{xy} = h - h^{-2} \tag{1}$$

as the compatibility condition of the Gauss equation

$$\begin{aligned} r_{xx} &= (h_x r_x + \lambda r_y)/h, \\ r_{xy} &= hr, \\ r_{yy} &= (h_y r_y + \lambda^{-1} r_x)/h, \end{aligned}$$

in connection with an *affine sphere*.

Kaptsov and Shan'ko [**3**] transformed the Tzitzeica equation, using the dependent variable transformation

$$v = 1 - 2(\log\tau)_{xy},$$

into

$$(1 - 2(\log\tau)_{xy})^2\{(\log(\tau^2 - 2\tau\tau_{xy} + 2\tau_x\tau_y))_{xy} - 1\} + 1 = 0, \tag{2}$$

whose numerator is a trilinear equation. They have obtained N-soliton solutions to the trilinear equation.

On the other hand, Schief [**4**, **5**] has obtained an integrable discrete version of the Tzitzeica equation as the compatibility condition of the discrete Gauss equation,

$$\begin{aligned} \mathbf{r}_{11} - \mathbf{r}_1 &= \alpha(\mathbf{r}_1 - \mathbf{r}) + \beta(\mathbf{r}_{12} - \mathbf{r}_1), \\ \mathbf{r}_{12} + \mathbf{r} &= H(\mathbf{r}_1 + \mathbf{r}_2), \\ \mathbf{r}_{22} - \mathbf{r}_2 &= \gamma(\mathbf{r}_2 - \mathbf{r}) + \delta(\mathbf{r}_{12} - \mathbf{r}_2). \end{aligned}$$

Schief has transformed the discrete Tzitzeica equation into the trilinear form

$$\begin{vmatrix} \tau(m,n) & \tau(m,n+1) & \tau(m,n+2) \\ \tau(m+1,n) & \tau(m+1,n+1) & \tau(m+1,n+2) \\ \tau(m+2,n) & \tau(m+2,n+1) & \tau(m+2,n+2) \end{vmatrix} = ab\,\tau(m+1,n+1)^3, \quad a, b \text{ being parameters} \tag{3}$$

which is reduced to the following trilinear equation in the continuous limit

$$\begin{vmatrix} \tau & \tau_x & \tau_{xx} \\ \tau_y & \tau_{xy} & \tau_{xxy} \\ \tau_{yy} & \tau_{xyy} & \tau_{xxyy} \end{vmatrix} = c\,\tau^3, \quad \text{c being a parameter.} \tag{4}$$

We shall transform the Tzitzeica equation (1) into the bilinear form and find a relation between the trilinear equations (4) and (2). Let $h = G/F$, then we have

$$\frac{\partial^2}{\partial x \partial y} \log h = \frac{G_{xy}G - G_x G_y}{G^2} - \frac{F_{xy}F - F_x F_y}{F^2}.$$

Here we introduce the bilinear operators $D_x$ and $D_y$ operating on an ordered pair of $F$ and $G$

$$D_x^n F \cdot G = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n F(x)G(x')\Big|_{x'=x}$$

$$D_x^n D_y^m F \cdot G = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m F(x,y)G(x',y')\Big|_{x'=x, y'=y}$$

which gives

$$D_x D_y F \cdot F = 2(F_{xy}F - F_x F_y)$$

$$D_x D_y G \cdot G = 2(G_{xy}G - G_x G_y).$$

Accordingly the Tzitzeica equation (1) is transformed into

$$\frac{D_x D_y G \cdot G - 2\alpha G^2 + 2F^2}{2G^2} - \frac{D_x D_y F \cdot F - 2\alpha F^2 + 2GF}{2F^2} = 0,$$

where $\alpha$ is an arbitrary parameter. Accordingly the Tzitzeica equation (1) is decoupled into the bilinear equations

$$D_x D_y G \cdot G = 2(\alpha G^2 - F^2), \tag{5}$$

$$D_x D_y F \cdot F = 2(\alpha F^2 - GF). \tag{6}$$

Let us introduce a new dependent variable $\tau_1$ and express $F$ and $G$ as follows

$$F = \tau_1^2, \tag{7}$$

$$G = \alpha \tau_1^2 - D_x D_y \tau_1 \cdot \tau_1. \tag{8}$$

Equation (6) is satisfied by this choice of $F$ and $G$, because of the identity

$$D_x D_y \tau_1^2 \cdot \tau_1^2 = 2\tau_1^2 D_x D_y \tau_1 \cdot \tau_1.$$

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Let $G = \tau_2$, then equations (8) and (5) are written as

$$D_x D_y \tau_1 \cdot \tau_1 = \alpha \tau_1^2 - \tau_2, \tag{9}$$

$$D_x D_y \tau_2 \cdot \tau_2 = 2\left(\alpha \tau_2^2 - \tau_1^4\right). \tag{10}$$

Equations (9) and (10) are reduced to the trilinear equations (2) and (4) for $\alpha = 1$ and $\alpha = 0$, respectively, by eliminating $\tau_2$.

**2. Toda molecule equation.** Equations (9) and (10) for $\alpha = 0$ remind us of the Toda molecule equation expressed with the bilinear form

$$D_x D_y \tau_n \cdot \tau_n = 2\tau_{n+1}\tau_{n-1}, \tag{11}$$

for $n = 1, 2, \ldots, N$. For $N = 2$ we have

$$D_x D_y \tau_1 \cdot \tau_1 = 2\tau_2 \tau_0, \tag{12}$$

$$D_x D_y \tau_2 \cdot \tau_2 = 2\tau_3 \tau_1, \tag{13}$$

which become equations (9) and (10) for $\alpha = 0$ by choosing the boundary condition

$$\tau_0 = -1/2, \quad \tau_3 = -\tau_1^3. \tag{14}$$

The bilinear equation (11) is transformed into the ordinary form [**6**]

$$\frac{\partial^2}{\partial x \partial y} \log V_n = V_{n+1} - 2V_n + V_{n-1}, \tag{15}$$

for $n = 1, 2, \ldots, N$, through the dependent variable transformation

$$V_n = \frac{\partial^2}{\partial x \partial y} \log \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{16}$$

Accordingly equations (12) and (13) with the boundary condition (14) are transformed into

$$\frac{\partial^2}{\partial x \partial y} \log V_1 = V_2 - 2V_1, \tag{17}$$

$$\frac{\partial^2}{\partial x \partial y} \log V_2 = 4V_1 - 2V_2. \tag{18}$$

Adding twice equation (17) to equation (18) we obtain

$$\frac{\partial^2}{\partial x \partial y} \log\left(V_1^2 V_2\right) = 0, \tag{19}$$

which is consistent with

$$V_2 = -\frac{1}{4} V_1^{-2}, \tag{20}$$

given by equations (14) and (16).

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Substituting equation (20) into equation (17) we obtain

$$\frac{\partial^2}{\partial x \partial y} \log V_1 = -\frac{1}{4} V_1^{-2} - 2V_1. \tag{21}$$

Let

$$V_1 = -\frac{h}{2}, \tag{22}$$

then equation (21) becomes the Tzitzeica equation

$$\frac{\partial^2}{\partial x \partial y} \log h = h - h^{-2}. \tag{23}$$

Hence we have shown that the Tzitzeica equation is equivalent to the Toda molecule equation with the special boundary condition. The Toda molecule equation is discretized in [7]. Hence the discrete Tzitzeica equation could be obtained by using the discrete Toda molecule equation with a special boundary condition.

The discrete Toda molecule equation [7] is expressed with the bilinear form as follows

$$\begin{aligned} &\tau_s(m+1, n+1)\tau_s(m, n) - \tau_s(m+1, n)\tau_s(m, n+1) \\ &\quad = q\tau_{s+1}(m, n)\tau_{s-1}(m+1, n+1), \end{aligned} \tag{24}$$

for $s = 1, 2, 3, \ldots, N$, $q$ being a constant, with the boundary conditions

$$\tau_0(m, n) = 1, \tag{25}$$

$$\tau_{N+1}(m, n) = f(m)g(n). \tag{26}$$

We replace the boundary condition (26) by

$$\tau_{N+1}(m, n) = F(\tau), \tag{27}$$

in order to obtain the discrete Tzitzeica equation, where $F(\tau)$ is a function of $\tau_j$ for $j = 1, 2, \ldots, N$.

For $N = 2$, equation (24) gives

$$\tau_1(m+1, n+1)\tau_1(m, n) - \tau_1(m+1, n)\tau_1(m, n+1) = q\tau_2(m, n), \tag{28}$$

$$\begin{aligned} &\tau_2(m+1, n+1)\tau_2(m, n) - \tau_2(m+1, n)\tau_2(m, n+1) \\ &\quad = q\tau_3(m, n)\tau_1(m+1, n+1). \end{aligned} \tag{29}$$

Eliminating $\tau_2(m, n)$ from these equations we obtain $\tau_3(m, n)$ expressed in terms of $\tau_1(m, n)$

$$\tau_3(m, n) = q^{-3} \begin{vmatrix} \tau_1(m, n) & \tau_1(m, n+1) & \tau_1(m, n+2) \\ \tau_1(m+1, n) & \tau_1(m+1, n+1) & \tau_1(m+1, n+2) \\ \tau_1(m+2, n) & \tau_1(m+2, n+1) & \tau_1(m+2, n+2) \end{vmatrix}. \tag{30}$$

Hence equation (30) with the boundary condition

$$\tau_3(m, n) = ab\tau(m+1, n+1)^3/q^3 \tag{31}$$

is the discrete Tzitzeica equation (4) obtained by Schief.

**3. A Higher order discrete Tzitzeica equation.** The discrete Toda molecule equation (24) suggests that we may extend the discrete Tzitzeica equation (4) to a higher order one. We have for $N = 3$ and for $\tau_0 = 1$

$$\begin{aligned}
&\tau_1(m+1,n+1)\tau_1(m,n) - \tau_1(m+1,n)\tau_1(m,n+1) = q\tau_2(m,n),\\
&\tau_2(m+1,n+1)\tau_2(m,n) - \tau_2(m+1,n)\tau_2(m,n+1) = q\tau_3(m,n)\tau_1(m+1,n+1),\\
&\tau_3(m+1,n+1)\tau_3(m,n) - \tau_3(m+1,n)\tau_3(m,n+1) = q\tau_4(m,n)\tau_2(m+1,n+1).
\end{aligned}$$

Eliminating $\tau_2$ and $\tau_3$ from these equations we obtain

$$\tau_4(m,n) = q^{-6}\begin{vmatrix} \tau_1(m,n) & \tau_1(m,n+1) & \dots & \tau_1(m,n+3) \\ \tau_1(m+1,n) & \tau_1(m+1,n+1) & \dots & \tau_1(m+1,n+3) \\ \cdots & \cdots & \cdots & \cdots \\ \tau_1(m+3,n) & \tau_1(m+3,n+1) & \dots & \tau_1(m+3,n+3) \end{vmatrix}. \tag{32}$$

The *r.h.s.* is a homogeneous polynomial function of $\tau_1$ of degree 4 and is invariant under the following gauge transformation:

$$\tau_1(m,n) \to \tau_1(m,n)\exp(a_0 m + b_0 n), \qquad a_0, b_0 \text{ being constant.}$$

Taking the gauge invariance into account, we propose the following equation

$$\begin{vmatrix} \tau(m,n) & \tau(m,n+1) & \tau(m,n+2) & \tau(m,n+3) \\ \tau(m+1,n) & \tau(m+1,n+1) & \tau(m+1,n+2) & \tau(m+1,n+3) \\ \tau(m+2,n) & \tau(m+2,n+1) & \tau(m+2,n+2) & \tau(m+2,n+3) \\ \tau(m+3,n) & \tau(m+3,n+1) & \tau(m+3,n+2) & \tau(m+3,n+3) \end{vmatrix} = ab\begin{vmatrix} \tau(m+1,n+1) & \tau(m+1,n+2) \\ \tau(m+2,n+1) & \tau(m+2,n+2) \end{vmatrix}^2,$$

as an integrable higher order Tzitzeica equation. The integrability of the equation has not been proved yet. However we have a test of identifying integrable discrete systems proposed by Hietarinta and Viallet [**8**, **9**] which analyses the complexity (algebraic entropy) of the map using the growth of the degree of its solution. Numerical calculations of the algebraic entropy of the equation indicate the system's integrability.

**4. Ultradiscretization of the discrete Tzitzeica equation.** In 1990 one of authors (D.T) and Satsuma reported on a simple cellular automaton which shows soliton-like behavior [**10**]. It is called "soliton cellular automaton". Tokihiro and coworkers have found that the cellular automaton models are obtained as a special limit, known as the ultradiscrete limit, of the integrable equations [**11**]. Typically, the limiting procedure simply replaces the summation by the Max operator,

$$\lim_{\varepsilon\to+0} \varepsilon \log(\exp(A/\varepsilon) + \exp(B/\varepsilon)) = \max(A, B),$$

and the product by the summation,

$$\lim_{\varepsilon\to+0} \varepsilon \log(\exp(A/\varepsilon) \times \exp(B/\varepsilon)) = A + B.$$

Let

$$V_s(m,n) = \frac{\tau_{s+1}(m,n)\tau_{s-1}(m+1,n+1)}{\tau_s(m+1,n)\tau_s(m,n+1)}, \tag{33}$$

then the bilinear form

$$\tau_s(m+1,n+1)\tau_s(m,n) - \tau_s(m+1,n)\tau_s(m,n+1) \\ = q\tau_{s+1}(m,n)\tau_{s-1}(m+1,n+1), \quad \text{for } s = 1,2,\ldots,N,$$

is rewritten as

$$1 + qV_s(m,n) = \frac{\tau_s(m+1,n+1)\tau_s(m,n)}{\tau_s(m+1,n)\tau_s(m,n+1)}, \qquad \text{for } s = 1,2,\ldots,N. \tag{34}$$

Equations (33) and (34) give

$$\frac{V_s(m+1,n+1)V_s(m,n)}{V_s(m+1,n)V_s(m,n+1)} = \frac{[1+qV_{s+1}(m,n)][1+qV_{s-1}(m+1,n+1)]}{[1+qV_s(m+1,n)][1+qV_s(m,n+1)]}, \\ \text{for } s = 1,2,\ldots,N. \tag{35}$$

The boundary condition $\tau_3(m,n) = ab\tau_1^3(m+1,n+1)$ is transformed, using the expression

$$1 + qV_s(m,n) = \frac{\tau_s(m+1,n+1)\tau_s(m,n)}{\tau_s(m+1,n)\tau_s(m,n+1)},$$

into

$$1 + qV_3(m,n) = \frac{\tau_3(m+1,n+1)\tau_3(m,n)}{\tau_3(m+1,n)\tau_3(m,n+1)} = [1 + qV_1(m+1,n+1)]^3.$$

Accordingly we have the discrete Tzitzeica equation in the ordinary form,

$$\frac{V_1(m+1,n+1)V_1(m,n)}{V_1(m+1,n)V_1(m,n+1)} = \frac{[1+qV_2(m,n)]}{[1+qV_1(m+1,n)][1+qV_1(m,n+1)]},$$
$$\frac{V_2(m+1,n+1)V_2(m,n)}{V_2(m+1,n)V_2(m,n+1)} = \frac{[1+qV_1(m+1,n+1)]^4}{[1+qV_2(m+1,n)][1+qV_2(m,n+1)]}.$$

Following the limiting procedure we put

$$V_s(j,k) = \exp(x_s(j,k)/\epsilon), \quad q = \exp(-c/\epsilon), \quad \text{for } s = 1,2,3.$$

The discrete Tzitzeica is transformed into the following max-plus equations in the small limit of $\epsilon$,

$$\begin{aligned} x_1(m+1,n+1) &= x_1(m+1,n) + x_1(m,n+1) - x_1(m,n) \\ &\quad + \max(0, x_2(m,n) - c) - \max(0, x_1(m+1,n) - c) \\ &\quad - \max(0, x_1(m,n+1) - c), \\ x_2(m+1,n+1) &= x_2(m+1,n) + x_2(m,n+1) - x_2(m,n) \\ &\quad + 4\max(0, x_1(m+1,n+1) - c) - \max(0, x_2(m+1,n) - c) \\ &\quad - \max(0, x_2(m,n+1) - c). \end{aligned}$$

We remark that the discrete sine-Gordon equation is obtained in connection with a discrete geometry [**12**] as well as the discrete Tzitzeica equation. However it is very difficult to ultradiscretize the former equation in contrast to the latter.

**5. Periodic Boundary Conditions.** We transform the coordinates $m, n$ into new coordinates $j, k$:

$$m = \frac{1}{2}(j + k), \qquad n = \frac{1}{2}(j - k),$$

in order to obtain equations with the periodic boundary condition. We express functions of $m, n$ in the new coordinates

$$V_s(m, n) = V_s(j, k), \quad V_s(m + 1, n) = V_s(j + 1, k + 1),$$
$$V_s(m, n + 1) = V_s(j + 1, k - 1), \quad V_s(m + 1, n + 1) = V_s(j + 2, k).$$

Then, the discrete Tzitzeica equation is transformed into

$$\frac{V_1(j + 2, k)V_1(j, k)}{V_1(j + 1, k + 1)V_1(j + 1, k - 1)} = \frac{[1 + qV_2(j, k)]}{[1 + qV_1(j + 1, k + 1)][1 + qV_1(j + 1, k - 1)]}, \tag{36}$$

$$\frac{V_2(j + 2, k)V_2(j, k)}{V_2(j + 1, k + 1)V_2(j + 1, k - 1)} = \frac{[1 + qV_1(j + 2, k)]^4}{[1 + qV_2(j + 1, k + 1)][1 + qV_2(j + 1, k - 1)]}. \tag{37}$$

It is generally accepted that the system's integrability is not destroyed by imposing a periodic boundary condition on the system. We use the periodic boundary condition $V_s(j, k) = V_s(j, k + N)$ in order to transform the partial difference equations (36) and (37) into a coupled form of ordinary difference equations. The algebraic entropy of the higher order discrete Tzitzeica equation (33) is calculated by imposing the periodic boundary condition $V_s(j, k) = V_s(j, k + N)$ with periods $N = 1, 3, 5, \ldots$.

The discrete Tzitzeica equation is reduced, for $N = 1$ (uniform in $k$-direction), to

$$\frac{V_1(j + 2)V_1(j)}{V_1^2(j + 1)} = \frac{[1 + qV_2(j)]}{[1 + qV_1(j + 1)]^2}, \tag{38}$$

$$\frac{V_2(j + 2)V_2(j)}{V_2^2(j + 1)} = \frac{[1 + qV_1(j + 2)]^4}{[1 + qV_2(j + 1)]^2}. \tag{39}$$

Equations (38) and (39) give a relation

$$\left[\frac{V_1(j + 3)V_1(j + 1)}{V_1^2(j + 2)}\right]^2 \frac{V_2(j + 2)V_2(j)}{V_2^2(j + 1)} = 1,$$

hence

$$V_2(j) = c_0 c_1^j V_1^{-2}(j + 1), \tag{40}$$

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where $c_0$ and $c_1$ are integration constants. Substituting the relation into equation (38) we obtain

$$\frac{V_1(j+2)V_1(j)}{V_1^2(j+1)} = \frac{[1+qc_0c_1^jV_1^{-2}(j+1)]}{[1+qV_1(j+1)]^2}. \tag{41}$$

Equation (41) is a non-autonomous equation if $c_1 \neq 1$, while the Tzitzeica equation is an autonomous equation. Hence we choose $c_1 = 1$. Putting $V_1(j+1) = x_n$ and $c_0 = q = c$ we obtain a one-dimensional discrete Tzitzeica equation in the following form,

$$x_{n+1}x_{n-1} = c^2\frac{(1+x_n^2/c^2)}{(1+cx_n)^2}. \tag{42}$$

Let $x_n = e^{X_n/\epsilon}$, $c = e^{C/\epsilon}$, then the one-dimesional discrete Tzitzeica equation (42) is transformed into the following max-plus equations in the small limit of $\epsilon$,

$$X_{n+1} = 2C + 2\max(0, X_n - C) - 2\max(0, X_n + C) - X_{n-1}. \tag{43}$$

**6. Periods of the ultradiscrete Tzitzeica equation.** Equation (43) has the following conserved quantity,

$$H = \max(X_{n-1} - X_n - 2C,\ -X_{n-1} + X_n - 2C, -X_{n-1} - X_n,\ X_{n-1} + X_n), \tag{44}$$

where $H$ is constant for $n$. This quantity $H$ means that equation (43) is integrable. Moreover, a solution from any initial data has the following remarkable features;

(a) If $|(H-2C)/4C|$ is rational, the solution is periodic with a finite period. Otherwise, the period becomes infinite.

(b) The finite period of a solution is determined only by $C$ and $H$.

(c) Therefore, if $C$ is fixed, finite periods of all solutions with the same $H$ are the same.

If we consider equation (43) as a mapping from $(X_{n-1}, X_n)$ to $(X_n, X_{n+1})$, we can show the above features by a geometrical analysis of solution orbits in a phase plane. We omit the details of the analysis since it needs a long space and only show how a period is determined by $C$ and $H$. Consider the following relation,

$$\frac{q}{p} = \left|\frac{H-2C}{4C}\right|, \tag{45}$$

where $p$ and $q$ are positive integers and relatively prime. Then the period is expressed by $p$ and $q$ as follows,

| | | |
|---|---|---|
| $C > 0$ | $H = 0$ | 2 |
| | $0 < H < 2C$ | $3p - 4q$ |
| | $H = 2C$ | 3 |
| | $2C < H$ | $3p + 4q$ |
| $C = 0$ | $0 \leq H$ | 4 |
| $C < 0$ | $H = 2\|C\|$ | 1 |
| | $2\|C\| < H$ | $-3p + 4q$ |

Note that $0 \leq H$ if $0 \leq C$ and $2|C| \leq H$ if $C < 0$, which can be shown from equation (44).

For example, if $C = 3$, $X_0 = 1$ and $X_1 = 3$, then $H = 4$ and $|(H - 2C)/4C| = 1/6$. Consequently we have $q = 1$ and $p = 6$, and the period becomes $3p - 4q = 14$. Indeed, the solution becomes

$$X_n:\ 1,\ 3,\ -7,\ 3,\ 1,\ -5,\ 5,\ -1,\ -3,\ 7,\ -3,\ -1,\ 5,\ -5,\ 1,\ 3,\ \ldots.$$

If we take the same $C$ and $X_0 = X_1 = 2$, the same $H$, $p$ and $q$ are obtained. The solution is

$$X_n:\ 2,\ 2,\ -6,\ 4,\ 0,\ -4,\ 6,\ -2,\ -2,\ 6,\ -4,\ 0,\ 4,\ -6,\ 2,\ 2,\ \ldots,$$

and the period is also 14.

**7. Concluding remarks.** We have ultradiscretized the discrete Tzitzeica equation by Schief. The discrete Tzitzeica equation is connected to the discrete affine geometry. So it would be of strong interest to find a geometry connected to the ultradiscrete Tzitzeica equation.

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