3
Some optics

To obtain insight into the electron scattering process, we appeal to some elementary optics, with which the reader is certainly familiar from an introductory physics course. If one looks through a telescope at a star, or shines a laser through a pinhole, one does not really observe a point of light, but actually a diffraction pattern with a bright disc at the center and a series of concentric rings with diminishing intensity. If the radius of the aperture through which the light passes is a, and the wavelength of the incident light is λ_1, then the angle θ to the first diffraction minimum of the central *Airy disc* is given by

$$a\theta \approx 0.61\lambda_1 \tag{3.1}$$

Here θ is measured from the central ray, starting at the aperture. Now introduce the incident wave number k_1 and "momentum transfer" κ

$$\begin{aligned} k_1 &\equiv \frac{2\pi}{\lambda_1} && ;\ \text{wave number} \\ \kappa &\approx k_1\theta && ;\ \text{momentum transfer} \end{aligned} \tag{3.2}$$

Equation (3.1) can then be rewritten as

$$\kappa a \approx 1.22\,\pi \tag{3.3}$$

This relation has a marvelous consequence. Suppose one shines light from a laser of given wavelength on a pinhole, and projects the resulting diffraction pattern on a screen behind the pinhole. The angle to the first minimum can be determined by making *macroscopic measurements* of the distance of the screen from the aperture and the transverse distance on the screen out to the first minimum. Equation (3.3) then allows one to determine the radius a of the pinhole. *One can measure a radius of arbitrarily small size if only the momentum transfer is large enough!* The

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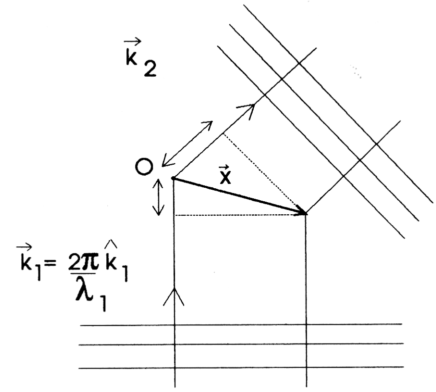

Fig. 3.1. Optical pathlength with respect to central ray in Fraunhofer diffraction.

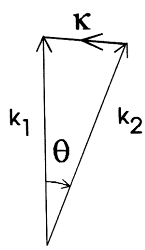

Fig. 3.2. The "momentum transfer" $\boldsymbol{\kappa} = \mathbf{k}_1 - \mathbf{k}_2$.

momentum transfer is inversely proportional to the wavelength. Thus to obtain large momentum transfer, one has to go to short wavelength. One evidently needs a wavelength comparable to the size of the aperture to make this measurement.

Let us extend these simple considerations. In Fraunhofer diffraction one has an incident plane wave and an outgoing plane wave in the direction of observation as illustrated in Fig. 3.1. The optical pathlength of an arbitrary ray with respect to the central ray is evidently given from this figure as

$$\Delta_{\text{opt}} = \frac{2\pi}{\lambda_1}(\hat{\mathbf{k}}_1 \cdot \mathbf{x} - \hat{\mathbf{k}}_2 \cdot \mathbf{x}) = \boldsymbol{\kappa} \cdot \mathbf{x} \tag{3.4}$$

where $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ are unit vectors in the incident and outgoing directions respectively. Here the momentum transfer $\boldsymbol{\kappa}$ is defined by (Fig. 3.2)

$$\boldsymbol{\kappa} = \mathbf{k}_1 - \mathbf{k}_2 \tag{3.5}$$

Since the lengths of the incoming and outgoing wave numbers are identical

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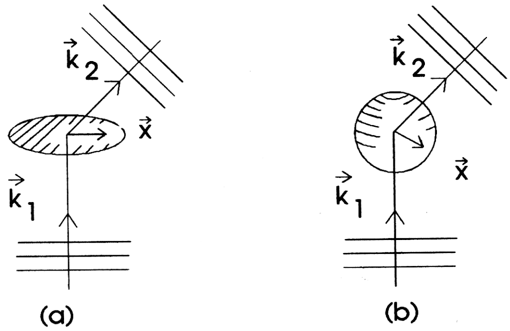

Fig. 3.3. (a) Fraunhofer diffraction of light from a circular aperture; and (b) Electron scattering through the Coulomb interaction from a spherical charge distribution.

$|\mathbf{k}_2| = |\mathbf{k}_1|$, the square of the momentum transfer is given by

$$\begin{aligned}\boldsymbol{\kappa}^2 &= 2k_1^2(1-\cos\theta) \\ &= 4k_1^2\sin^2\frac{\theta}{2}\end{aligned} \tag{3.6}$$

Here θ is the angle between the incident and outgoing wave number vectors (Fig. 3.2). *Huygens Principle* says that each point on a wavefront serves as a new source of outgoing waves. The outgoing waves interfere. To determine the net outgoing wave from a circular aperture one must add the contributions from each little element of the disc weighted by $\exp\{i\Delta_{\rm opt}\}$ as illustrated in Fig. 3.3 (a). The resulting amplitude of the light wave far from the scatterer is thus given by

$$\mathscr{A}_\gamma = \int_{\rm Aperture} d^2x\, e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \tag{3.7}$$

The diffraction pattern evidently measures the two-dimensional Fourier transform of the aperture.

Now consider the scattering of an electron from a spherical charge distribution through the Coulomb interaction. de Broglie and quantum mechanics tell us that there is a wave associated with the electron of wavelength

$$\lambda_1 = \frac{h}{p_1} \qquad ;\ \text{electron} \tag{3.8}$$

Here $h \equiv 2\pi\hbar$ is Planck's constant and p_1 the incident electron momentum. The scattering amplitude from each little element will be proportional to the amount of charge there, or to the charge density $\rho_{\rm ch}(x)$. The resulting

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amplitude of the electron wave far from the target is thus given in direct analogy with the above by (see Fig. 3.3 b)

$$\mathscr{A}_{\rm el} = \int_{\rm Nucleus} d^3x\, \rho_{\rm ch}(x)\, e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} \tag{3.9}$$

The diffraction pattern of the scattered electron measures the three-dimensional Fourier transform of the target charge distribution.

In electron scattering, the incident electron momentum is evidently $\mathbf{p}_1 = \hbar\mathbf{k}_1$ and the momentum transferred from the electron is $\hbar\boldsymbol{\kappa}$; this is the reason for the terminology.

One can see a macroscopic diffraction pattern from arbitrarily small charge distributions if only the momentum transfer is large enough, or equivalently, if the wavelength is small enough. It follows from Eq. (3.8) that to achieve very small wavelengths, one must go to very high electron energies. It is an irony (and an expensive one!) that to look in detail with accuracy and precision at very small objects such as nuclei and nucleons one needs accurate and precise high-energy electron accelerators to produce the incident electron beams and correspondingly large, accurate and precise spectrometers to detect the scattered electrons.

One can put in some numbers. To have an electron with wavelength 1 fm $= 10^{-13}$ cm, a typical nuclear dimension, one needs a relativistic electron of energy[1]

$$\begin{aligned} \lambda &= 1 \text{ fm} \\ \Rightarrow E &= pc = \hbar kc = 1240 \text{ MeV} \end{aligned} \tag{3.10}$$

To obtain some insight, it is useful to evaluate the above amplitudes for the simple cases of a circular disc and unit spherical charge distribution, both of radius a. With the introduction of polar coordinates in the first case, and the use of spherical coordinates in the second, one finds [Fe80]

$$\begin{aligned} \mathscr{A}_\gamma^{\rm disc} &= \int_0^a \rho\, d\rho \int_0^{2\pi} d\phi\, \exp\{i\kappa_\perp \rho\cos\phi\} \\ &= \pi a^2 \left[\frac{2J_1(\kappa_\perp a)}{\kappa_\perp a}\right] \\ \mathscr{A}_{\rm el}^{\rm sphere} &= \int_0^a r^2\, dr \int d\Omega_r\, \exp\{i\kappa r\cos\theta_r\} \\ &= \left(\frac{4\pi a^3}{3}\right)\left[\frac{3j_1(\kappa a)}{\kappa a}\right] \end{aligned} \tag{3.11}$$

[1] Recall $\hbar c = 197.3$ MeV fm. Here c is the speed of light.

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Here $J_n(\alpha)$ and $j_n(\alpha)$ are cylindrical and spherical Bessel functions respectively. The quantities in square brackets in the above expressions are known as *form factors*. It is instructive to make some log plots of the square of these quantities on your PC. The first zero of $J_1(\alpha)$ occurs at $\alpha_{1,1} = 1.22\,\pi$; this is the origin of Eq. (3.3).