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EMBEDDING A SEMIGROUP OF TRANSFORMATIONS

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Let X be an arbitrary set and θ a transformation of X. One may use θ to induce an associative operation in $\mathscr{T}_X$, the set of all mappings of X to itself as follows:

$$\alpha * \beta = \alpha\theta\beta \qquad (\alpha, \beta \in \mathscr{T}_X).$$

We denote the resulting semigroup by $(\mathscr{T}_X; \theta)$ Magill (1967) introduced this structure and it has been studied by Sullivan and by myself.

Sullivan asks when $(\mathscr{T}_X; \theta)$ can be embedded in $(\mathscr{T}_X, \circ)$, the full transformation semigroup under composition. He shows that if X is finite this can be done if and only if θ is a permutation of X and that any embedding (an isomorphism perforce) is of the form

$$\alpha \to g^{-1}\theta\alpha g \qquad (\alpha \in \mathscr{T}_X)$$

where g is a permutation of X. The infinite case is left open. The purpose of this note is to prove the following.

THEOREM 1. *If X is infinite then any $(\mathscr{T}_X, \theta)$ may be embedded in $(\mathscr{T}_X, \circ)$.*

PROOF. Let $X = X_E \cup X_0$ where X_0 and X_0 are disjoint and of the same cardinality, necessarily that of X. Select bijections g and h such that

$$h: X \to X_E \text{ and } g: X \to X_0.$$

For α in $\mathscr{T}_X$ we define $\alpha\phi$ as follows

$$\begin{aligned} x\alpha\phi &= xh^{-1}\alpha g \qquad (x \in X_E) \\ &= xg^{-1}\theta\alpha g \quad (x \in X_0). \end{aligned}$$

It is clear that the first part of the definition guarantees that $\alpha \to \alpha\phi$ is one to one. Observe that $\alpha\phi: X \to X_g = X_0$. It follows that for α and β in $\mathscr{T}_X$ we have that $\alpha\phi\beta\phi = \alpha\phi g^{-1}\theta\beta g$. Thus if x is in X_E,

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$$x\alpha\phi\beta\phi = xh^{-1}\alpha g \cdot g^{-1}\theta\beta g = xh^{-1}\alpha\theta\beta g = x(\alpha * \beta)\phi$$

while if x is in X_0,

$$x\alpha\phi\beta\phi = xg^{-1}\theta\alpha g . g^{-1}\theta\beta g = xg^{-1}\theta\alpha\theta\beta g = (x * \beta)\phi .$$

It follows that $(\alpha * \beta)\phi = \alpha\phi\beta\phi$, as required.

The classification of the embeddings of $(\mathscr{T}_X; \theta)$ in $(\mathscr{T}_X, \circ)$ is extremely difficult. Partial results have been obtained. We shall describe our most pleasant result in this direction.

We call a transformation semigroup $S \subseteq \mathscr{T}_X$ irreducible if the set

$$xS = \{x\alpha; \alpha \in S\}$$

coincides with X for each x in X. Further, we shall say than an embedding ϕ of $(\mathscr{T}_X; \theta)$ in $(\mathscr{T}_X, 0)$ is irreducible if $\mathscr{T}_X\phi$ is irreducible.

THEOREM 2. *Any irreducible embedding of* $(\mathscr{T}_X; \theta)$ *in* $(\mathscr{T}_X, \circ)$ *is of the form*

$$\alpha \to g^{-1}\theta\alpha g \qquad (\alpha \in \mathscr{T}_X)$$

for some fixed permutation g *of* X.

PROOF. Assume ϕ is such an embedding and let $\kappa = \kappa_x$ denote the constant function in $\mathscr{T}_x$ with range x. We choose y in the range of $\kappa\phi$ and consider $y(\kappa\phi)^{-1}$. If z is any member of this latter set then for any α in $\mathscr{T}_X$

$$(z\alpha\phi)\kappa\phi = z(\alpha\kappa)\phi = z\kappa\phi = y .$$

This shows that $y(\kappa\phi)^{-1}$ is invariant under $\mathscr{T}_X\phi$ and hence, by irreducibility, coincides with X. This shows that ϕ maps constants to constants from which follows

$$\kappa_x\phi = \kappa_{xg} \qquad (x \in X)$$

where g is an injective transformation of X. But then for each α

$$\kappa_{x\theta\alpha g} = (\kappa_x * \alpha)\phi = \kappa_x\phi\alpha\phi$$

which implies $\theta\alpha g = g\alpha\phi$. Thus $Xg\alpha\phi \subseteq Xg$, and this is contrary to irreducibility unless g is onto. In this case g permutes X and $\alpha\phi = g^{-1}\theta\alpha g$, as required.

It is clear that ϕ above is an embedding if and only if θ is onto X. Hence we have the following:

COROLLARY. *It is possible to irreducibly embed* $(\mathscr{T}_X; \theta)$ *in* $(\mathscr{T}_X, \circ)$ *if and only if* θ *is onto.*

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