CONJUGATES OF DIFFERENTIABLE FLOWS

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The work in this paper is directed at the question: What differentiable flows on $[0,1]$ [1] are conjugates of linear fractional flows on $[0,1]$?

LEMMA. *If h is a homeomorphism of $[0,1]$ onto $[0,1]$ such that for some number $a \in (0,1]$ h has a continuous positive derivative on $(0,a]$ and there is a number r such that* $\lim_{x\to 0} x^r h'(x) > 0$ *then* $r < 1$.

PROOF. Suppose $r \geqq 1$ and $\lim_{x\to 0} x^r h'(x) = A > 0$. Then there is a number $b \in (0,a]$ such that if $x \in (0,b]$ then $x^r h'(x) > A/2$. Since $r \geqq 1$ we have that $x^r h'(x) \leqq (x^r h(x))'$ and hence that

$$\int_0^b (x^r h(x))' dx \geqq \int_0^b x^r h'(x) dx \geqq Ab/2.$$

It then follows that $b^r h(b) \geqq Ab/2$ and hence that $d^{r-1}h(d) \geqq A/2$ if $d \in (0,b]$. This is impossible, hence $r < 1$.

DEFINITION. A differentiable flow F_t on $[0,1]$ is said to be of *type I* if

1. 0 and 1 are the only fixed points of F_t,
2. $F_t'(0) = c^t, F_t'(1) = d^t$ where $c > 1 > d$ and
3. there are homeomorphisms ϕ and Φ from $[0,1)$ and $(0,1]$ respectively onto $[0,\infty)$ each having a continuous nonzero derivative such that $F_t = \phi^{-1}(c^t\phi) = \Phi^{-1}(d^t\Phi)$.

THEOREM. *A necessary and sufficient condition that a differentiable flow F_t on $[0,1]$ be of type I is that there is:*

1. *a homeomorphism h of $[0,1]$ onto $[0,1]$ which has a continuous positive derivative on $(0,1]$ and a number r such that* $\lim_{x\to 0} x^r h'(x) > 0$. *and*
2. *a linear fractional flow L_t on $[0,1]$ such that $F_t = h \circ L_t \circ h^{-1}$.*

PROOF. Suppose F_t is a differentiable flow of type I with $F_t'(0) = c^t, F^t(1) = d^t$ where $c > 1 > d$.

Define a linear fractional flow L_t by

$$L_t(x) = d^{-t}x/[(d^{-t}-1)x+1] \text{ if } x\in[0,1] \text{ and } t\in(-\infty,\infty).$$

Fix $b\in(0,1)$ and define a function h^{-1} by

$$h^{-1}(x) = [1+y(\phi(x))^{-p}]^{-1} \text{ if } 0 \leqq x < 1 \text{ and}$$
$$h^{-1}(1) = 1$$

where $p = -\ln d/\ln c$ and $y = (1-b)/b(\phi(b))^p$.

It is easily verified that h^{-1} is a homeomorphism of $[0,1]$ onto $[0,1]$ which has a positive continuous derivative on $(0,1)$. Also a sequence of straightforward computations establishes that $F_t = h\circ L_t \circ h^{-1}$.

Because $h'\circ h^{-1}(x) = [(\phi(x))^p + y]^2/yp(\phi(x))^{p-1}\phi'(x)$ we have that

$$(h^{-1}(x))^r h'\circ h^{-1}(x) = [\phi(x)+y]^{2-r}/yp\phi'(x)$$

where $r = 1-1/p$. It then follows that

$$\lim_{x\to 0} x^r h'(x) = y^{1-r}/p\phi'(0) > 0.$$

All that remains to establish the result going one way, is that h has a continuous positive derivative on $(0,1]$. This follows from the following observations. Fix $k\in(0,1)$ and recall that

$$\phi^{-1}(c^t\phi(k)) = \Phi^{-1}(d^t\Phi(k)).$$

If $m = \phi(k)$, $n = \Phi(k)$ and $z = \Phi^{-1}(d^t n)$ then $\Phi(z) = nd^t$ and hence $t = \ln[\Phi(\frac{1}{2})/n]/\ln d$. We then have that

$$\phi(z) = mc^{\ln[\Phi(z)/n]/\ln d} = m(\Phi(z)/n)^{\ln c/\ln d}.$$

Therefore $(\phi(z))^p = m'n/\Phi(z)$ and $h^{-1}(z) = [1+y\Phi(z)/m'n]^{-1}$, where $m' = m^p$. Hence h^{-1} has a continuous positive derivative on $(0,1]$.

To establish the remaining half of the theorem suppose h is a homeomorphism of $[0,1]$ onto $[0,1]$ having the required properties, and that L_t is a linear fractional flow on $[0, 1]$ with $L_t'(0) = a^t$ where $a > 1$. Let $F_t = h\circ L_t\circ h^{-1}$. Clearly F_t is a flow on $[0,1]$ which is a differentiable flow on $(0,1]$, also only 0 and 1 are fixed points on F_t.

Since $F_t'(x) = h'\circ L_t\circ h^{-1}(x)\cdot L_t'\circ h^{-1}(x)\cdot h^{-1\prime}(x)$ we have that

$$\begin{aligned}\lim_{x\to 0} F_t'(x) &= \lim_{x\to 0} h'\circ L_t(x)\cdot L_t'(x)/h'(x)\\ &= \lim_{x\to 0}(L_t(x))^r h'\circ L_t(x)\cdot x^r L_t'(x)/(L_t(x))^r x^r h'(x)\\ &= (L_t'(0))^{1-r}\\ &= a^{(1-r)t}.\end{aligned}$$

Note that $F_t'(1) = L_t'(1) = a^{-t}$, thus $a^{1-r} > 1 > a^{-1}$.

We need only to produce the desired homeomorphisms ϕ and Φ to complete the argument.

Define a function θ on $(0,1]$ by

$$\theta(x) = h'(1)/[h' \circ h^{-1}(x)(h^{-1}(x))^2] \text{ for } x \text{ in } (0,1].$$

It is clear that θ is continuous and positive on $(0,1]$.

If $\Phi(x) = \int_x^1 \theta$ then there is a number $B > 0$ such that $\Phi(x) = B[(1/h^{-1}(x)) - 1]$ and hence Φ is a homeomorphism of $(0,1]$ onto $[0,\infty)$ which has a negative continuous derivative on $(0,1]$. Moreover, a sequence of computations yields

$$\begin{aligned}\Phi^{-1}(a^{-t}\Phi(x)) &= h[B/(a^{-t}\Phi(x)+B)]\\ &= h \circ L_t \circ h^{-1}(x)\\ &= F_t(x).\end{aligned}$$

Now define a function ϕ on $[0,1)$ by

$$\phi(x) = (\Phi(x))^{r-1} \text{ if } x \in (0,1)$$

and

$$\phi(0) = 0.$$

Hence ϕ is a homeomorphism of $[0,1)$ onto $[0,\infty)$ which has a positive continuous derivative on $(0,1)$.

Now
$$\begin{aligned}\phi'(x) &= (r-1)(\Phi(x))^{r-2}\Phi'(x)\\ &= (1-r)B^{r-2}\theta(x)[(1/h^{-1}(x)) - 1]^{r-2} \text{ on } (0,1).\end{aligned}$$

Using the definitions of θ and h^{-1} and the above we have that

$$\begin{aligned}\phi'(x) &= [(1-r)B^{r-2}h'(1)/h' \circ h^{-1}(x)(h^{-1}(x))^2][(1/h^{-1}(x)) - 1]^{r-2}\\ &= A[1-h^{-1}(x)]^{r-2}/(h^{-1}(x))^r h' \circ h^{-1}(x)\end{aligned}$$

where $A = (1-r)B^{r-2}h'(1)$.

Hence $\lim_{x\to 0}\phi'(x) > 0$ and therefore ϕ has a positive continuous derivative on $[0,1)$. Also a simple computation shows that

$$F_t = \phi^{-1}(a^{(1-r)t}\phi)$$

which concludes the proof of the theorem.

References

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