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CONVOLUTION PROPERTIES OF A CLASS OF BOUNDED ANALYTIC FUNCTIONS

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Let $\mathbf{A}$ be the class of functions $f(z)$ which are analytic in the unit disk $\mathbf{U}$ with $f(0) = f'(0) - 1 = 0$. A subclass $\mathbf{S}(\lambda, M)$ $(\lambda \geqslant 0, M > 0)$ of $\mathbf{A}$ is introduced. The object of the present paper is to prove some interesting convolution properties of functions $f(z)$ belonging to the class $\mathbf{S}(\lambda, M)$. Also a certain integral operator J for $f(z)$ in the class $\mathbf{A}$ is considered.

1. INTRODUCTION AND LEMMAS

Let $\mathbf{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the unit disk $\mathbf{U} = \{z\colon |z| < 1\}$. We denote by $\mathbf{S}^*(\rho)$ and $\mathbf{K}(\rho)$ the subclasses of $\mathbf{A}$ whose members are starlike and convex of order ρ $(0 \leqslant \rho < 1)$.

For a function $f(z) \in \mathbf{A}$, we say that $f(z)$ is in the class $\mathbf{S}(\lambda, M)$ if and only if it satisfies the condition

$$|f'(z) + \lambda z f''(z) - 1| < M \qquad (z \in \mathbf{U})$$

for some λ $(\lambda \geqslant 0)$ and M $(M > 0)$.

In the present paper, we prove some convolution properties of functions $f(z)$ belonging to the class $\mathbf{S}(\lambda, M)$. Some inclusion relations between $\mathbf{S}(\lambda, M)$ and other subclasses of $\mathbf{A}$ are obtained. We also obtain some new sufficient conditions for $f(z) \in \mathbf{S}^*(\rho)$. Finally, we discuss a class of certain integral operators on $\mathbf{A}$.

We need the following lemmas to derive our results.

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Lemma 1. *Let $\lambda \geqslant 0$ and $M > 0$. If $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$ and satisfies*

$$|p(z) + \lambda z p'(z) - 1| < M \qquad (z \in \mathbf{U}), \tag{1}$$

then we have

$$|p(z) - 1| < \frac{M}{1+\lambda} \qquad (z \in \mathbf{U}), \tag{2}$$

$$\left|\frac{1}{z}\int_0^z p(t)dt - 1\right| < \frac{M}{2(1+\lambda)} \qquad (z \in \mathbf{U}), \tag{3}$$

and

$$\left|\frac{1}{z}\int_0^z p(t)dt - p(z)\right| < \frac{(3+2\lambda)M}{2(1+\lambda)(1+2\lambda)} \qquad (z \in \mathbf{U}). \tag{4}$$

Inequalities in (2) and (3) cannot be improved.

Proof: Let us define the function $p(z)$ by

$$p(z) = 1 + \frac{M}{1+\lambda} w(z), \tag{5}$$

where $w(z)$ is analytic in $\mathbf{U}$ with $w(0) = 0$. We wish to show that $|w(z)| < 1$ for all $z \in \mathbf{U}$. If this is not true, then there exists a point $z_0 \in \mathbf{U}$ satisfying

$$\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by Jack's Lemma [**1**], we can write

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geqslant 1$. It follows that

$$z_0 p'(z_0) = \frac{kM}{1+\lambda} w(z_0)$$

and

$$|p(z_0) + \lambda z_0 p'(z_0) - 1| = \frac{1+\lambda k}{1+\lambda} M \geqslant M.$$

This contradicts the condition (1), and hence we conclude that $|w(z)| < 1$ for all $z \in \mathbf{U}$. Therefore, by using (5), we know that (2) holds true.

In view of Schwarz' Lemma and (2), we have

$$|p(z) - 1| \leqslant \frac{M}{1+\lambda}|z| \qquad (z \in \mathbf{U}),$$

and hence

$$\left|\int_0^z p(t)dt - z\right| = \left|\int_0^z (p(t)-1)dt\right| \leqslant \int_0^{|z|} \frac{M}{1+\lambda} t\,dt = \frac{M}{2(1+\lambda)}|z|^2.$$

This implies that

$$\left|\frac{1}{z}\int_0^z p(t)dt - 1\right| < \frac{M}{2(1+\lambda)} \qquad (z \in \mathrm{U}).$$

Further, let

$$p(z) - \frac{1}{z}\int_0^z p(t)dt = \frac{(3+2\lambda)M}{2(1+\lambda)(1+2\lambda)} w(z), \tag{6}$$

where $w(z)$ is analytic in U with $w(0) = 0$. We can prove that $|w(z)| < 1$ for all $z \in \mathrm{U}$. In fact, if this is not true, then using the same way as in the above there exists a point $z_0 \in \mathrm{U}$ $(z_0 \neq 0)$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = k w(z_0)$, where $k \geqslant 1$. From (1) and (6), we obtain

$$\begin{aligned} &|p(z_0) + \lambda z_0 p'(z_0) - 1| \\ &\quad = \left|\frac{1}{z_0}\int_0^{z_0} p(t)dt + \frac{(3+2\lambda)Mw(z_0)}{2(1+\lambda)(1+2\lambda)} + \frac{(3+2\lambda)\lambda(k+1)}{2(1+\lambda)(1+2\lambda)} Mw(z_0) - 1\right| \\ &\quad < M, \end{aligned}$$

that is,

$$\left|\frac{1}{z_0}\int_0^{z_0} p(t)dt - 1 + \frac{3+2\lambda}{2(1+\lambda)(1+2\lambda)}(1+\lambda+k\lambda)Mw(z_0)\right| < M.$$

Hence, we have

$$\left|\frac{1}{z_0}\int_0^{z_0} p(t)dt - 1\right| > \frac{3+2\lambda}{2(1+\lambda)}M - M = \frac{M}{2(1+\lambda)}.$$

This contradicts (3) and hence $|w(z)| < 1$ for all $z \in \mathrm{U}$. This follows (4) with (6).

Since the function $p_0(z) = 1 + (M/(1+\lambda))z$ satisfies the condition (1), we see that the inequalities in (2) and (3) cannot be improved. Thus we complete the proof of Lemma 1.

Let $A_n = (a_{ij})_{nn}$ denote the real symmetrical matrix of order n. Jian Huaiyu has showed that $|A_n| \geqslant 0$, if A_n satisfies the conditions:

(i) $a_{ij} \geqslant a_{i\,j+1} \geqslant 0 \qquad (i = 1, 2, 3, \ldots, n; i \leqslant j \leqslant n-1)$,

(ii) $a_{i+1\,i+1} \geqslant a_{ii} \qquad (i = 1, 2, 3, \ldots, n-1)$,

(iii) $a_{ij} \geqslant a_{i-1\,j} \qquad (i = 1, 2, 3, \ldots, n; i \leqslant j \leqslant n)$,

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and

$$\text{(iv)} \quad a_{ij} - a_{i\,j+1} \geqslant a_{i-1\,j} - a_{i-1\,j+1} \quad (i = 1, 2, 3, \ldots, n; i \leqslant j \leqslant n-1).$$

In fact, the case $a_{11} = 0$ is trivial. If $a_{11} > 0$, then we have

$$|A_n| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a'_{22} & \ldots & a'_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a'_{n2} & \ldots & a'_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a'_{22} & \ldots & a'_{2n} \\ \cdots & \cdots & \cdots \\ a'_{n2} & \ldots & a'_{nn} \end{vmatrix},$$

where

$$a'_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \qquad (i, j = 1, 2, 3, \ldots, n).$$

By the hypothesis, we see that

$$\begin{pmatrix} a'_{22} & \ldots & a'_{2n} \\ \cdots & \cdots & \cdots \\ a'_{n2} & \ldots & a'_{nn} \end{pmatrix}$$

is a real symmetrical matrix of order $n-1$ and satisfies the conditions (i) - (iv). Hence we can prove that $|A_n| \geqslant 0$ by mathematical induction. □

LEMMA 2. *Let $b_0 > 0$, $b_n \geqslant 0$, and $b_{n-1} - b_n \geqslant b_n - b_{n+1} \geqslant 0$, $n = 1, 2, 3, \ldots$. If*

$$p(z) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n z^n,$$

then $Re(p(z)) > 0\,(z \in \mathrm{U})$.

PROOF: We can write

$$p(z) = \frac{b_0}{2}\{1 + \sum_{n=1}^{\infty} c_n z^n\}$$

with $c_n = 2b_n/b_0$ $(n = 1, 2, 3, \ldots)$. Adopting the convention that $c_0 = 2$, $c_{-n} = c_n$ $(n \geqslant 1)$, we have that

$$A_{m+1} = \begin{pmatrix} c_0 & c_1 & c_2 & \ldots & c_m \\ c_1 & c_0 & c_1 & \ldots & c_{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_i & c_{i-1} & c_{i-2} & \ldots & c_{i-m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_m & c_{m-1} & c_{m-2} & \ldots & c_0 \end{pmatrix} \qquad (i = 0, 1, 2, \ldots, m)$$

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is a real symmetrical matrix of order $m+1$, and satisfies the conditions (i) - (iv). Hence we can prove that A_{m+1} is a semi-positive definite matrix by the mathematical induction.

Since, for $m = 1, 2, 3, \ldots$ and $\lambda_k \in \mathbf{C}\,(0 \leqslant k \leqslant m)$, we have

$$R_m = \sum_{k=0}^{m} \sum_{q=0}^{m} C_{k-q} \lambda_k \overline{\lambda}_q = \lambda' A_{m+1} \lambda,$$

$$\lambda = \begin{pmatrix} \overline{\lambda}_0 \\ \overline{\lambda}_1 \\ \vdots \\ \overline{\lambda}_m \end{pmatrix}$$

and hence $R_m \geqslant 0$; this implies that

$$\mathrm{Re}\{1 + \sum_{n=1}^{\infty} c_n z^n\} > 0 \qquad (z \in \mathbf{U}),$$

so Lemma 2 is completed. □

EXAMPLE. If $\lambda \geqslant 0$ and

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{1 - \lambda + n\lambda} z^n,$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{4\lambda^2 + 3\lambda + 1}{2(1+\lambda)(1+2\lambda)} \qquad (z \in \mathbf{U}). \tag{7}$$

PROOF: Let $b_0 = (1+3\lambda)/((1+\lambda)(1+2\lambda))$ and $b_n = 1/(1+n\lambda)$, $n = 1, 2, 3, \ldots$. Clearly, the sequence $\{b_n\}_0^\infty$ satisfies the conditions in Lemma 2, and hence

$$\mathrm{Re}\left\{\frac{1+3\lambda}{2(1+\lambda)(1+2\lambda)} + \sum_{n=1}^{\infty} \frac{1}{1+n\lambda} z^n\right\} > 0 \qquad (z \in \mathbf{U}). \tag{8}$$

The conclusion follows from (8) at once.

2. The class $\mathrm{S}(\lambda, M)$

Let a function $f(z)$ be in the class $\mathrm{S}(\lambda, M)$. Setting $p(z) = f'(z)$ in Lemma 1, by (2) and (3), we obtain

$$|f'(z) - 1| < \frac{M}{1+\lambda} \qquad (z \in \mathbf{U}) \tag{9}$$

and

$$\left|\frac{f(z)}{z} - 1\right| < \frac{M}{2(1+\lambda)} \qquad (z \in \mathbf{U}), \tag{10}$$

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respectively. From (10), we see that $S(\lambda, M)$ is a class of bounded analytic functions in U. If $M \leqslant 1+\lambda$, by (9), $S(\lambda, M) \subset C$, the usual class of close-to-convex functions in U. From (9), we also obtain

PROPOSITION 1. *Let $0 \leqslant \lambda_2 \leqslant \lambda_1$ and $\lambda_1 > 0$. Then*

$$S(\lambda_1, M) \subset S(\lambda_2, M).$$

THEOREM 2. *Let $f(z) \in S(\lambda, M)$ and $g(z) \in A$ with $\operatorname{Re}\{g(z)/z\} > 1/2$ $(z \in U)$; then $h(z) = (f * g)(z) \in S(\lambda, M)$, where $(f * g)(z)$ denotes the convolution (or Hadamard product) of functions $f(z)$ and $g(z)$.*

PROOF: According to Herglotz Theorem, we have

$$\frac{g(z)}{z} = \int_T \frac{1}{1 - z\tau}\, d\mu(\tau),$$

where μ is a probability measure on the unit circle T. Since

$$h'(z) + \lambda z h''(z) - 1 = (f'(z) + \lambda z f''(z) - 1) * \frac{g(z)}{z},$$

we obtain

$$h'(z) + \lambda z h''(z) - 1 = \int_T (f'(\tau z) + \lambda \tau z f''(\tau z) - 1)\, d\mu(\tau).$$

Moreover, we have

$$|h'(z) + \lambda z h''(z) - 1| < \int_T M\, d\mu(\tau) = M,$$

which shows $h(z) \in S(\lambda, M)$. □

COROLLARY 1. *Let $f(z) \in S(\lambda, M)$, $g(z) \in S(\lambda, M)$ and $M \leqslant 1+\lambda$. Then $h(z) = (f * g)(z) \in S(\lambda, M)$, that is, $S(\lambda, M)$ is closed for the convolution (or Hadamard product) when $M \leqslant 1+\lambda$.*

PROOF: By means of (10), we have $\operatorname{Re}\{f(z)/z\} > 1/2$ $(z \in U)$, and hence the conclusion immediately follows from Theorem 2. □

Next, we derive

THEOREM 3. *Let $f(z) \in S(\lambda, M)$, $g(z) \in S(\lambda, M)$, and $h(z) = (f * g)(z)$.*

(i) *If $M \leqslant 1+\lambda$, then $h(z) \in S^*(0)$ and satisfies*

$$\left| \frac{z h'(z)}{h(z)} - 1 \right| < 1 \qquad (z \in U).$$

(ii) *If either $\lambda \geqslant 1/3$ with $M \leqslant (1+\lambda)/\sqrt{2}$, or $0 < \lambda \leqslant 1/3$ with $M \leqslant \sqrt{2\lambda(1+\lambda)}$, then $h(z) \in K(0)$.*

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PROOF: Defining the functions $f(z)$ and $g(z)$ by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, respectively, we have

$$h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

(i) From (9), we obtain that

$$\iint_{\mathbf{U}} |f'(z) - 1|^2 \, dx dy = \pi \sum_{n=2}^{\infty} n |a_n|^2 < \pi \left(\frac{M}{1+\lambda} \right)^2. \tag{11}$$

Hence

$$\sum_{n=2}^{\infty} n |a_n|^2 < 1. \tag{12}$$

Similarly, we have

$$\sum_{n=2}^{\infty} n |b_n|^2 < 1. \tag{13}$$

By means of the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=2}^{\infty} n |a_n b_n| \leqslant \left(\sum_{n=2}^{\infty} n |a_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} n |b_n|^2 \right)^{1/2} < 1. \tag{14}$$

Therefore, we know that $h(z) \in \mathbf{S}^*(0)$, so $h(z)/z \neq 0$ $(z \in \mathbf{U})$. It follows from (14) that

$$\sum_{n=2}^{\infty} n |a_n b_n| |z|^{n-1} < 1 \qquad (z \in \mathbf{U}),$$

or

$$\sum_{n=2}^{\infty} (n-1) |a_n b_n| |z|^{n-1} < 1 - \sum_{n=2}^{\infty} |a_n b_n| |z|^{n-1} \qquad (z \in \mathbf{U}).$$

This implies that

$$\left| \sum_{n=2}^{\infty} (n-1) a_n b_n z^{n-1} \right| < \left| 1 + \sum_{n=2}^{\infty} a_n b_n z^{n-1} \right| \qquad (z \in \mathbf{U}),$$

that is, that

$$\left|h'(z) - \frac{h(z)}{z}\right| < \left|\frac{h(z)}{z}\right| \qquad (z \in \mathbf{U}).$$

Consequently, we obtain that

$$\left|\frac{zh'(z)}{h(z)} - 1\right| < 1 \qquad (z \in \mathbf{U}).$$

(ii) Since $f(z) \in \mathbf{S}(\lambda, M)$, we have

$$\left|\sum_{n=2}^{\infty} n(1-\lambda+n\lambda)a_n z^{n-1}\right| < M \qquad (z \in \mathbf{U}),$$

and hence

$$\text{(15)} \qquad \iint_{\mathbf{U}} \left|\sum_{n=2}^{\infty} n(1-\lambda+n\lambda)a_n z^{n-1}\right|^2 dx\,dy = \pi \sum_{n=2}^{\infty} n(1-\lambda+n\lambda)^2 |a_n|^2 < \pi M^2.$$

Since $\lambda \geqslant 1/3$ with $M \leqslant (1+\lambda)/\sqrt{2}$, or $0 < \lambda \leqslant 1/3$ with $M \leqslant \sqrt{2\lambda(1+\lambda)}$, we can prove that $(1-\lambda+n\lambda)^2 \geqslant nM^2$ for every $n \geqslant 2$, and hence by (15) we have

$$\sum_{n=2}^{\infty} n^2 |a_n|^2 < 1.$$

Similarly

$$\sum_{n=2}^{\infty} n^2 |b_n|^2 < 1.$$

Therefore, we see that

$$\sum_{n=2}^{\infty} n^2 |a_n b_n| \leqslant \left(\sum_{n=2}^{\infty} n^2 |a_n|^2\right)^{1/2} \left(\sum_{n=2}^{\infty} n^2 |b_n|^2\right)^{1/2} < 1.$$

This implies that $h(z)$ belongs to the class $\mathbf{K}(0)$. ∎

From the proof of (i) in Theorem 3, we have

Corollary 2. *If*

$$F(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

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is in the class $\mathbf{A}$ *with*

$$\sum_{n=2}^{\infty} n\,|c_n| \leqslant 1,$$

then

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < 1 \qquad (z \in \mathbf{U}).$$

Letting $\lambda = M - 1 = 0$ in (i) of Theorem 3, we have

COROLLARY 3. *Let* $f(z) \in \mathbf{A}$ *and* $g(z) \in \mathbf{A}$ *with* $|f'(z) - 1| < 1$ $(z \in \mathbf{U})$ *and* $|g'(z) - 1| < 1$ $(z \in \mathbf{U})$. *Then* $h(z) = (f * g)(z) \in \mathbf{S}^*(0)$ *and*

$$\left|\frac{zh'(z)}{h(z)} - 1\right| < 1 \qquad (z \in \mathbf{U}).$$

THEOREM 4. *Let* $f(z) \in \mathbf{S}(\lambda, M)$.

(i) *If* $M \leqslant 2(1+\lambda)/\sqrt{5}$, *then* $f(z) \in \mathbf{S}^*(0)$.

(ii) *If* $M \leqslant (1+2\lambda)/2$, *then* $|zf'(z)/f(z) - 1| < 1$ $(z \in \mathbf{U})$.

(iii) *If* $M \leqslant 2(1+\lambda)(1+2\lambda)/(5+6\lambda)$, *then* $f(z) \in \mathbf{S}^*(1/2)$.

PROOF: (i) Since $M/(1+\lambda) \leqslant 2/\sqrt{5} < 1$, in view of (9), we obtain $\operatorname{Re}\{f'(z)\} > 0$ $(z \in \mathbf{U})$, and

$$|\arg f'(z)| < \sin^{-1}\left(\frac{M}{1+\lambda}\right) \leqslant \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) < \frac{\pi}{2} \quad (z \in \mathbf{U}).$$

By (10), we have $\operatorname{Re}\{f(z)/z\} > 0$ $(z \in \mathbf{U})$, and

$$\left|\arg \frac{f(z)}{z}\right| < \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) < \frac{\pi}{2} \qquad (z \in \mathbf{U}).$$

Noting that

$$\sin\left(\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) = 1,$$

we have

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqslant |\arg f'(z)| + \left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2} \qquad (z \in \mathbf{U}),$$

which implies $f(z) \in \mathbf{S}^*(0)$.

(ii) Setting $p(z) = f'(z)$ in Lemma 1, we have by (4)

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{3+2\lambda}{4(1+\lambda)} \qquad (z \in \mathbf{U}).$$

Since (10) gives

$$\left|\frac{f(z)}{z}\right| > 1 - \frac{1+2\lambda}{4(1+\lambda)} = \frac{3+2\lambda}{4(1+\lambda)} \qquad (z \in \mathbf{U}),$$

we have

$$\left|f'(z) - \frac{f(z)}{z}\right| < \left|\frac{f(z)}{z}\right| \qquad (z \in \mathbf{U}),$$

which proves

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathbf{U}).$$

(iii) It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbf{U}).$$

Since (10) leads to

$$\left|\frac{f(z)}{z}\right| > \frac{4(1+\lambda)}{5+6\lambda} \qquad (z \in \mathbf{U}),$$

we see that $f(z)/z \neq 0$ $(z \in \mathbf{U})$. Therefore, we only need to show that

$$\left|f'(z) - \frac{f(z)}{z}\right| < |f'(z)| \qquad (z \in \mathbf{U}). \tag{16}$$

With the aid of (4) and (9), we obtain

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{3+2\lambda}{5+6\lambda} \qquad (z \in \mathbf{U})$$

and

$$|f'(z)| > \frac{3+2\lambda}{5+6\lambda} \qquad (z \in \mathbf{U}).$$

Thus we prove the inequality (16). □

REMARK. Taking $\lambda = 0$ and $M = 1$ in (i) of Theorem 4, we obtain Theorem 2 and Theorem 3 by Mocanu [2]. Further, letting $M = 1$ in (ii) of Theorem 4, we obtain the main result by Mocanu [2], that is, Theorem 4.

Making $\lambda = 0$ in (iii) of Theorem 4, we have

COROLLARY 4. $\mathrm{S}(0, 2/5) \subset \mathrm{S}^*(1/2)$.

By Corollary 4, we see

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COROLLARY 5. $S(1, 2/5) \subset K(1/2)$.

Next, in view of (9), we derive

THEOREM 5. *If* $zf'(z) \in S(\lambda, M)$, *then* $f(z) \in S(1, M/(1+\lambda))$. *Conversely, if* $f(z) \in S(\lambda, M)$, *then* $zf'(z) \in S(0, 2M/\lambda(1+\lambda))$ *when* $0 < \lambda \leqslant 1$, *and* $zf'(z) \in S(0, 2M/(1+\lambda))$ *when* $\lambda \geqslant 1$.

THEOREM 6. *Let*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

belong to the class $S(\lambda, M)$. *Then, for every* $n \geqslant 1$, *the* n*th partial sum* $f_n(z)$ *of* $f(z)$ *satisfies*

(i)
$$\left|\frac{f_n(z)}{z} - 1\right| < \frac{M}{1+\lambda} \qquad (z \in U)$$

and

(ii)
$$|f_n'(z) - 1| < M \qquad (z \in U),$$

where $\lambda \geqslant 1$.

PROOF: We define the function $g(z)$ by $g(z) = \log(1/(1-z))$. Then, we have $g(z) \in K(1/2)$, and $\mathrm{Re}\{g_n(z)/z\} > 1/2$ $(z \in U)$ by Singh [**3**, Theorem 2], where $g_n(z)$ denotes the nth partial sum of $g(z)$.

(i) Since $f(z) \in S(\lambda, M)$, by (9) and the equality

$$\frac{f_n(z)}{z} - 1 = (f'(z) - 1) * \frac{g_n(z)}{z} \qquad (z \in U),$$

in the same method as Theorem 2, we obtain

$$\left|\frac{f_n(z)}{z} - 1\right| < \frac{M}{1+\lambda} \qquad (z \in U).$$

(ii) By Proposition 1, we see that $f(z) \in S(1, M)$, and hence

$$\left|1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1} - 1\right| < M \qquad (z \in U),$$

Since

$$f_n'(z) - 1 = \left(1 + \sum_{k=2}^{\infty} k^2 a_k z^{k-1} - 1\right) * \frac{g_n(z)}{z} \qquad (z \in U),$$

by the same way as the part (i), we obtain

$$|f_n'(z) - 1| < M \qquad (z \in U)$$

for all $\lambda \geqslant 1$. ∎

COROLLARY 6. *If* $f(z) \in S(\lambda, 1)$, *then* $f_n(z) \in C$ *for all* $\lambda \geqslant 1$ *and for every* $n \geqslant 1$.

3. Integral operators

We now discuss integral operators

$$g(z) = J(f)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt \qquad (\gamma > -1) \tag{17}$$

for $f(z) \in \mathbf{A}$. Writing $\gamma = 1/\lambda - 1$ $(\lambda > 0)$, we see that

$$f(z) = (1-\lambda)g(z) + \lambda z g'(z) \tag{18}$$

and

$$f'(z) = g'(z) + \lambda z g''(z). \tag{19}$$

Clearly, if $\lambda > 0$ and $g(z) \in \mathrm{S}(\lambda, M)$, then we observe that $f(z)$ defined by (17) is in the class $\mathrm{S}(0, M)$. Conversely, we have

Theorem 7. *The integral operator J defined by (17) satisfies*

$$J\colon \mathrm{S}(1/(1+\gamma), M) \longrightarrow \mathrm{S}(1/(1+\gamma), (1+\gamma)M/(2+\gamma)).$$

Proof: Setting $\lambda = 1/(1+\gamma)$ and $p(z) = g'(z) + \lambda z g''(z)$, we see from (19) that

$$f'(z) + \lambda z f''(z) - 1 = p(z) + \lambda z p'(z) - 1.$$

Suppose that $f(z) \in \mathrm{S}(\lambda, M) = \mathrm{S}(1/(1+\gamma), M)$. Then it follows from (2) that

$$|p(z) - 1| < \frac{M}{1+\lambda} \qquad (z \in \mathrm{U}),$$

and hence $g(z) \in \mathrm{S}(\lambda, M/(1+\lambda))$. This completes the proof of Theorem 7. ∎

Theorem 8. *Let $M \leqslant 1+\lambda$, $-1 < \gamma = 1/\lambda - 1 \leqslant 0$, and $\lambda \geqslant 1$. If $f(z) \in \mathrm{S}(\lambda, M)$ and $g(z)$ is defined by (17), then $(g * h)(z) \in \mathrm{K}(0)$ for every $h(z) \in \mathrm{S}(\lambda, M)$.*

Proof: Defining the functions $g(z)$ and $h(z)$ by

$$g(z) = z + \sum_{n=2}^\infty a_n z^n \text{ and } h(z) = z + \sum_{n=2}^\infty b_n z^n,$$

(18) leads to

$$f(z) = z + \sum_{n=2}^\infty (1 - \lambda + n\lambda) a_n z^n \in \mathrm{S}(\lambda, M).$$

Therefore, from (12), we have

$$\sum_{n=2}^{\infty} n(1-\lambda+n\lambda)^2 |a_n|^2 < \left(\frac{M}{1+\lambda}\right)^2 < 1.$$

Noting that $\lambda \geqslant 1$, we have

$$\sum_{n=2}^{\infty} n^3 |a_n|^2 < 1.$$

Further, by (12) we obtain

$$\sum_{n=2}^{\infty} n |b_n|^2 < 1.$$

Consequently, we know that

$$\sum_{n=2}^{\infty} n^2 |a_n b_n| < 1,$$

which implies that $(g * h)(z) \in \mathbf{K}(0)$. □

Next, we prove

THEOREM 9. *If $f(z) \in \mathbf{A}$ satisfies $Re\{f(z)/z\} > \rho$ $(\rho < 1; z \in \mathbf{U})$, then the function $g(z)$ defined by (17) satisfies*

$$Re\left\{\frac{g(z)}{z}\right\} > \begin{cases} \dfrac{2+4\rho+5\rho\gamma+\rho\gamma^2}{(2+\gamma)(3+\gamma)} & (-1<\gamma\leqslant 0) \\ \dfrac{1+2\rho+2\rho\gamma}{3+2\rho} & (\gamma>0), \end{cases},$$

for $z \in \mathbf{U}$.

PROOF: Letting

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and $\gamma = 1/\lambda - 1$ $(\lambda > 0)$, (18) gives

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} = \operatorname{Re}\left\{1 + \sum_{n=2}^{\infty}(1-\lambda+n\lambda)a_n z^{n-1}\right\} > \rho \qquad (z \in \mathbf{U}).$$

Hence we have

$$\operatorname{Re}\left\{1 + \frac{1}{2(1-\rho)}\sum_{n=2}^{\infty}(1-\lambda+n\lambda)a_n z^{n-1}\right\} > \frac{1}{2} \qquad (z \in \mathbf{U}). \tag{20}$$

Note that

(21)
$$\frac{g(z)}{z}=\left\{1+\frac{1}{2(1-\rho)}\sum_{n=2}^{\infty}(1-\lambda+n\lambda)a_n z^{n-1}\right\}*\left\{1+2(1-\rho)\sum_{n=2}^{\infty}\frac{z^{n-1}}{1-\lambda+n\lambda}\right\}.$$

Thus (7) leads to

$$\text{Re}\left\{1+2(1-\rho)\sum_{n=2}^{\infty}\frac{z^{n-1}}{1-\lambda+n\lambda}\right\}>\frac{2\lambda^2+(1-3\lambda)\rho}{(1+\lambda)(1+2\lambda)}\qquad(z\in\mathbf{U}). \tag{22}$$

Combining (20), (21) and (22), in a similar way to Theorem 2, we obtain

$$\text{Re}\left\{\frac{g(z)}{z}\right\}>\frac{2\lambda^2+(1+3\lambda)\rho}{(1+\lambda)(1+2\lambda)}\qquad(z\in\mathbf{U})$$

for all $\lambda>0$, that is,

$$\text{Re}\left\{\frac{g(z)}{z}\right\}>\frac{2+4\rho+5\rho\gamma+\rho\gamma^2}{(2+\gamma)(3+\gamma)}\qquad(z\in\mathbf{U})$$

for all $\gamma>-1$. But for $\gamma>0$, that is, for $0<\lambda<1$, we have

$$\frac{2+4\rho+5\rho\gamma+\rho\gamma^2}{(2+\gamma)(3+\gamma)}<\frac{1+2\rho+2\rho\gamma}{3+2\gamma}.$$

Applying Jack's Lemma [1], we can prove

$$\text{Re}\left\{\frac{g(z)}{z}\right\}>\frac{1+2\rho+2\rho\gamma}{3+2\gamma}\qquad(z\in\mathbf{U}) \tag{23}$$

for $\gamma>0$. □

REMARK. The above inequality (23) was recently proved by Owa and Nunokawa [4] when $0\leqslant\rho<1$ and $\gamma>-1$.

With the help of the proof of Theorem 9, we have

THEOREM 10. *If $f(z)\in\mathbf{A}$ satisfies $Re\{f'(z)\}>\rho$ $(\rho<1; z\in\mathbf{U})$, then the function $g(z)$ defined by (17) satisfies*

$$Re\{g'(z)\}>\begin{cases}\dfrac{2+4\rho+5\rho\gamma+\rho\gamma^2}{(2+\gamma)(3+\gamma)} & (-1<\gamma\leqslant 0)\\[2ex] \dfrac{1+2\rho+2\rho\gamma}{3+2\gamma} & (\gamma>0)\end{cases}$$

for $z\in\mathbf{U}$.

REMARK. The second inequality in Theorem 10 was proved by Owa and Nunokawa [4] when $0\leqslant\rho<1$ and $\gamma>-1$.

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