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Γ-CONVERGENCE OF INHOMOGENEOUS FUNCTIONALS IN ORLICZ–SOBOLEV SPACES

MARIAN BOCEA[1] AND MIHAI MIHĂILESCU[2,3]

[1] *Department of Mathematics and Statistics, Loyola University Chicago, 1032 West Sheridan Road, Chicago, IL 60660, USA* (mbocea@luc.edu)
[2] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania* (mmihailes@yahoo.com)
[3] *'Simion Stoilow' Institute of Mathematics of the Romanian Academy, PO Box 1-764, 014700 Bucharest, Romania*

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Abstract The asymptotic behaviour of inhomogeneous power-law type functionals is undertaken via De Giorgi's Γ-convergence. Our results generalize recent work dealing with the asymptotic behaviour of power-law functionals acting on fields belonging to variable exponent Lebesgue and Sobolev spaces to the Orlicz–Sobolev setting.

Keywords: Γ-convergence; Orlicz–Sobolev spaces; inhomogeneous power-law functionals

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1. Introduction

Motivated by applications to dielectric breakdown and polycrystal plasticity, the study of the asymptotic behaviour of power-law functionals has been the subject of increased interest during the last decade. In the context of (first-failure) dielectric breakdown for composite materials made of two isotropic phases, Garroni *et al*. [**14**] have introduced natural variational principles that provide a rigorous justification, via De Giorgi's Γ-convergence [**10**, **11**], of the classical dielectric breakdown model as a limiting case of power-law models, and suggest a new non-degenerate variational principle in L^∞ that can be used to efficiently characterize the effective yield set. The Γ-convergence results in [**14**] have been extended [**4**, **5**] to the framework of $\mathcal{A}$-quasiconvexity, which allows for more general linear partial differential equation constraints on the underlying fields. The analysis in [**4**] leads to variational characterizations of the yield set in the framework of electrical resistivity, where one works with divergence-free underlying fields, while [**5**] deals with applications to models of antiplane shear and plane stress plasticity. These results have recently been extended to the setting of power-law functionals defined on variable exponent Sobolev spaces for curl-free fields (see [**3**]) as well as for the general case of fields that are subject to constant rank differential constraints [**6**].

The main goal of this paper is to study whether Γ-convergence results in the spirit of Garroni *et al*. [**14**] can be obtained in the very general framework of Orlicz–Sobolev spaces. To this aim, we analyse the asymptotic behaviour of several classes of inhomogeneous functionals involving energy densities that are defined either in terms of Young functions Φ_n (see Theorem 3.2 in § 3 of the paper) or in terms of the Orlicz norms associated with these functions (Theorems 3.3 and 3.4 in § 3). In the particular case where the Young functions are of power-law-type $\Phi_n(t) = t^n/n$, Theorem 3.2 allows us to recover the traditional dielectric breakdown model discussed in [**14**] (see also [**16**]), while the Γ-limit obtained in Theorem 3.3 suggests an alternative derivation of the new variational principle proposed by Garroni *et al*. as a limit of variational principles involving the Orlicz norm. Our next Γ-convergence result, Theorem 3.4, shows that in the setting of conductivity, even if one considers nonlinear materials whose elastic energies are characterized by a combination of the two previous types of energies (which type of energy is activated depends on the magnitude of the Orlicz norm of the electric field), the predicted limiting material behaviour coincides again with the one in [**14**].

Overall, our results indicate that widely accepted models of dielectric breakdown can be justified as limiting cases of much more flexible models. They should be viewed in the context of the recent literature (see, for example, [**4**, **6**] and references therein) concerned with the derivation and analysis of a variety of models in the more general setting of polycrystal plasticity starting from appropriate power-law and related models. Furthermore, understanding the asymptotic behaviour of inhomogeneous functionals satisfying non-standard growth conditions is also of independent interest, and we expect that our results will find further applications in other areas, such as electrorheological fluids or image processing, where these types of functionals play a key role in accurately describing the underlying physical phenomena.

The layout of the paper is as follows: in § 2 we briefly describe some basic properties of Orlicz and Orlicz–Sobolev spaces; our Γ-convergence results, Theorems 3.2, 3.3 and 3.4, are stated in § 3 of the paper; the remaining sections are devoted to the proofs of our main results, together with a number of remarks and auxiliary results.

2. Orlicz–Sobolev spaces

In this section we provide a brief review of the basic properties of Orlicz and Orlicz–Sobolev spaces. For more details we refer the reader to the books [**1**, **2**, **19**, **20**] and to the papers [**7**, **8**, **13**, **15**].

Assume that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and define

$$\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s, \qquad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\,\mathrm{d}s. \tag{2.1}$$

Note that $\Phi(0) = 0$, Φ is convex and $\lim_{t\to\infty} \Phi(t) = +\infty$, which makes Φ a *Young function*. Moreover, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \Phi(t)/t = 0$, and $\lim_{t\to\infty} \Phi(t)/t = +\infty$, Φ is an N*-function* (see [**1**] or [**2**] for more details). $\Phi^\star$ is called the *complementary* function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s)\colon s \geqslant 0\} \quad \text{for all } t \geqslant 0.$$

In addition, $\Phi^\star$ is also an N-function and *Young's inequality*

$$st \leqslant \Phi(s) + \Phi^\star(t) \quad \text{for all } s, t \geqslant 0$$

holds. In what follows, we assume that

$$1 < \varphi^- \leqslant \frac{t\varphi(t)}{\Phi(t)} \leqslant \varphi^+ < \infty \quad \text{for all } t > 0, \tag{2.2}$$

where

$$\varphi^- := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^+ := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

We indicate below several examples of functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ that are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and for which (2.2) holds. For more details, the reader is referred to [**7**, Examples 1–3, p. 243].

(1) $\varphi(t) = |t|^{p-2}t$ with $p > 1$. It can be shown that $\varphi^- = \varphi^+ = p$.

(2) $\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t$ with $p, r > 1$. In this case $\varphi^- = p$ and $\varphi^+ = p + r$.

(3) $\varphi(t) = |t|^{p-2}t/\log(1 + |t|)$ if $t \neq 0$, $\varphi(0) = 0$, with $p > 2$. In this case it turns out that $\varphi^- = p - 1$ and $\varphi^+ = p$.

With φ, Φ and $\Phi^\star$ as above, the Orlicz space $L^\Phi(\Omega)$ is the space of measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\Phi} := \sup\left\{\int_\Omega uv \,\mathrm{d}x\colon \int_\Omega \Phi^\star(|v|)\,\mathrm{d}x \leqslant 1\right\} < \infty. \tag{2.3}$$

Endowed with the so-called *Orlicz norm*, given by (2.3), $L^\Phi(\Omega)$ is a Banach space. The *Luxemburg norm*, defined by

$$\|u\|_\Phi := \inf\left\{\mu > 0\colon \int_\Omega \Phi\left(\frac{u(x)}{\mu}\right)\mathrm{d}x \leqslant 1\right\}, \tag{2.4}$$

is equivalent to the Orlicz norm on $L^\Phi(\Omega)$.

In the context of Orlicz spaces, Hölder's inequality reads as follows (see [**20**, Inequality 4, p. 79]):

$$\int_\Omega uv \,\mathrm{d}x \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^{\Phi^\star}} \quad \text{for all } u \in L^\Phi(\Omega) \text{ and } v \in L^{\Phi^\star}(\Omega).$$

The Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ is defined by

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega)\colon \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\ i = 1, \dots, N\right\}$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

We note that if φ^- and φ^+ are defined as above, our hypothesis (2.2) implies that Φ satisfies the Δ_2-*condition*:

$$\Phi(2t) \leqslant K\Phi(t) \quad \forall t \geqslant 0, \tag{2.5}$$

where K is a positive constant (see [**18**, Proposition 2.3]). On the other hand (see, for example, [**12**, Lemma 2.1] or [**18**, Proposition 2.1]), we have

$$\|u\|_\Phi^{\varphi^+} \leqslant \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x \leqslant \|u\|_\Phi^{\varphi^-} \quad \forall u \in L^\Phi(\Omega),\ \|u\|_\Phi < 1, \tag{2.6}$$

and

$$\|u\|_\Phi^{\varphi^-} \leqslant \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x \leqslant \|u\|_\Phi^{\varphi^+} \quad \forall u \in L^\Phi(\Omega),\ \|u\|_\Phi > 1. \tag{2.7}$$

Finally, we assume that Φ is such that

$$\text{the map } [0,\infty) \ni t \to \Phi(\sqrt{t}) \text{ is convex.} \tag{2.8}$$

We note that this, together with (2.5), implies that the Orlicz space $L^\Phi(\Omega)$ is a uniformly convex (and hence reflexive) Banach space (see [**18**, Proposition 2.2]).

Remark 2.1. Let $p > 1$ and define $\varphi(t) = |t|^{p-2}t$, $t \in \mathbb{R}$. As we already mentioned in example (1), it can be shown that in this case we have $\varphi^- = \varphi^+ = p$ and the corresponding Orlicz space $L^\Phi(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, while the Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ becomes the Sobolev space $W^{1,p}(\Omega)$.

Finally, we note that under assumption (2.2) we have (see, for example, [**12**, Lemma A.2]):

$$\beta(\rho)\Phi(t) \leqslant \Phi(\rho t) \leqslant \gamma(\rho)\Phi(t) \quad \forall t > 0,\ \rho > 0, \tag{2.9}$$

where

$$\beta(\rho) := \begin{cases} \rho^{\varphi^+} & \text{if } \rho \in (0,1], \\ \rho^{\varphi^-} & \text{if } \rho \in (1,\infty), \end{cases}$$

$$\gamma(\rho) := \begin{cases} \rho^{\varphi^-} & \text{if } \rho \in (0,1], \\ \rho^{\varphi^+} & \text{if } \rho \in (1,\infty). \end{cases}$$

3. Main results

We start by recalling the definition of Γ-convergence (introduced in [**10**, **11**]) in metric spaces. The reader is referred to [**9**] for a comprehensive introduction to the subject.

Definition 3.1. Let X be a metric space. A sequence $\{F_n\}$ of functionals $F_n \colon X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(X)$-converge to $F_\infty \colon X \to \bar{\mathbb{R}}$, and we write $\Gamma(X) - \lim_{n\to\infty} F_n = F_\infty$, if the following hold:

(i) for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have

$$F_\infty(u) \leqslant \liminf_{n\to\infty} F_n(u_n);$$

(ii) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$ (called a *recovery sequence*) such that $u_n \to u$ in X and

$$F_\infty(u) \geqslant \limsup_{n\to\infty} F_n(u_n).$$

Let $\Omega \subset \mathbb{R}^N$ be an open set of finite Lebesgue measure, $|\Omega| < +\infty$, with sufficiently smooth boundary. To simplify the presentation we will assume in what follows that $|\Omega| = 1$. However, this additional assumption is only imposed so that unnecessary complications in the proofs can be avoided; our results still hold, with straightforward modifications, in the general case.

Let Φ_n be defined as in (2.1), and assume that for each $n \in \mathbb{N}$, (2.2) and (2.8) hold (with φ and Φ replaced with φ_n and Φ_n, respectively). Moreover, we will assume that φ_n satisfies the following conditions:

$$\varphi_n^- \to \infty \quad \text{as} \quad n \to \infty \tag{3.1}$$

and

$$\text{there exists a real constant } \beta > 1 \text{ such that } \varphi_n^+ \leqslant \beta\varphi_n^- \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Let $\lambda \in L^\infty(\Omega)$ be such that $0 < a \leqslant \lambda(x) \leqslant b$, where a and b are two positive real numbers. For each $n \in \mathbb{N}$, consider the functionals $I_n, J_n \colon L^1(\Omega) \to [0, +\infty]$ defined by

$$I_n(u) = \begin{cases} \displaystyle\int_\Omega \frac{1}{\varphi_n(1)} \Phi_n(|\lambda(x)\nabla u(x)|)\,\mathrm{d}x & \text{if } u \in W^{1,\Phi_n}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$J_n(u) = \begin{cases} \|\lambda \nabla u\|_{\Psi_n} & \text{if } u \in W^{1,\Psi_n}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\Psi_n(t) := \Phi_n(t)/\Phi_n(1)$.

Theorem 3.2. *Assume that the sequence $\{\varphi_n\}$ satisfies (3.1) and (3.2). Define $I_\infty \colon L^1(\Omega) \to [0, +\infty]$ by*

$$I_\infty(u) = \begin{cases} 0 & \textit{if } |\lambda(x)\nabla u(x)| \leqslant 1 \textit{ for almost every (a.e.) } x \in \Omega, \\ +\infty & \textit{otherwise.} \end{cases}$$

Then the following hold.

(i) *For every $u \in L^1(\Omega)$, and $\{u_n\} \subset L^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$, we have*

$$I_\infty(u) \leqslant \liminf_{n\to\infty} I_n(u_n). \tag{3.3}$$

(ii) *For every* $u \in L^1(\Omega)$ *there exists a sequence* $\{u_n\} \subset L^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $L^1(\Omega)$, *and*

$$\limsup_{n\to\infty} I_n(u_n) \leqslant I_\infty(u). \tag{3.4}$$

In particular, $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} I_n = I_\infty$.

In the case where the sequence $\{\varphi_n\}$ satisfies (3.1), and with (3.2) replaced by the stronger requirement

$$\lim_{n\to\infty} \varphi_n^+/\varphi_n^- = 1, \tag{3.5}$$

two other Γ-convergence results can be established. The first one concerns the sequence $\{J_n\}$ defined above, while the second one involves a rescaled version of the sequence $\{I_n\}$ considered in Theorem 3.2. Precisely, we will prove in § 5 of the paper the following theorem.

Theorem 3.3. *Assume that the sequence* $\{\varphi_n\}$ *satisfies (3.1) and (3.5). Define* $J_\infty\colon L^1(\Omega) \to [0, +\infty]$ *by*

$$J_\infty(u) = \begin{cases} \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} & \text{if } u \in W^{1,\infty}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Then the following hold.

(i) *For every* $u \in L^1(\Omega)$, *and* $\{u_n\} \subset L^1(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^1(\Omega)$, *we have*

$$J_\infty(u) \leqslant \liminf_{n\to\infty} J_n(u_n).$$

(ii) *For every* $u \in L^1(\Omega)$, *there exists a sequence* $\{u_n\} \subset L^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $L^1(\Omega)$, *and*

$$\limsup_{n\to\infty} J_n(u_n) \leqslant J_\infty(u).$$

In particular, $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} J_n = J_\infty$.

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, define $\Psi_n(t) := \Phi_n(t)/\Phi_n(1)$ and consider the sequence $\{K_n\}$ of functionals $K_n\colon L^1(\Omega) \to [0, +\infty]$ given by

$$K_n(u) = \begin{cases} \left(\displaystyle\int_\Omega \Psi_n(|\lambda(x)\nabla u(x)|)\,\mathrm{d}x\right)^{1/\varphi_n^+} & \text{if } u \in W^{1,\Psi_n}(\Omega) \text{ with } \|\lambda\nabla u\|_{\Psi_n} \leqslant 1, \\ \|\lambda\nabla u\|_{\Psi_n} & \text{if } u \in W^{1,\Psi_n}(\Omega) \text{ with } \|\lambda\nabla u\|_{\Psi_n} > 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.4. *Assume that the sequence* $\{\varphi_n\}$ *satisfies (3.1) and (3.5) and let* J_∞ *be defined by (3.6). Then the following hold.*

(i) *For every $u \in L^1(\Omega)$, and $\{u_n\} \subset L^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$, we have*

$$J_\infty(u) \leqslant \liminf_{n\to\infty} K_n(u_n).$$

(ii) *For every $u \in L^1(\Omega)$ there exists a sequence $\{u_n\} \subset L^1(\Omega)$ such that $u_n \to u$ strongly in $L^1(\Omega)$, and*

$$\limsup_{n\to\infty} K_n(u_n) \leqslant J_\infty(u).$$

In particular, $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} K_n = J_\infty$.

Remark 3.5. We recall that a sequence $\{F_n\}$ of functionals $F_n\colon L^1(\Omega) \to \bar{\mathbb{R}}$ is said to be *equicoercive* with respect to the strong topology of $L^1(\Omega)$ if, whenever $\{u_n\} \subset L^1(\Omega)$ is a sequence with bounded energy, i.e. such that $\sup_{n\in\mathbb{N}} F_n(u_n) < \infty$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, and $u \in L^1(\Omega)$ such that $u_{n_k} \to u$ strongly in $L^1(\Omega)$. Although in part (i) of Theorems 3.2–3.4 we are able to prove the Γ-liminf inequalities under the less restrictive assumption that $u_n \rightharpoonup u$ weakly (rather than strongly) in $L^1(\Omega)$, our Γ-convergence results are then explicitly stated as holding with respect to the strong topology of $L^1(\Omega)$. This topology is natural to consider here because the sequences of functionals $\{I_n\}$, $\{J_n\}$ and $\{K_n\}$ in the statements of Theorems 3.2–3.4 are in fact equicoercive with respect to it. Indeed, as it can be seen from the proofs of these theorems presented in §§ 4 and 5 of the paper, in each case, the sequences with bounded energy turn out to be uniformly bounded in any Sobolev space $W^{1,q}(\Omega)$ with $q > 1$. The equicoercivity property then follows from the Rellich–Kondrachov theorem (precisely, from fact that the embedding of $W^{1,q}(\Omega)$ into $L^1(\Omega)$ is compact).

4. Proof of Theorem 3.2

We start by verifying (3.4). If $I_\infty(u) = \infty$, the inequality clearly holds for any sequence $u_n \to u$ strongly in $L^1(\Omega)$. On the other hand, if $I_\infty(u) < +\infty$ we must have $I_\infty(u) = 0$ and, consequently, $|\lambda(x)\nabla u(x)| \leqslant 1$ for a.e. $x \in \Omega$. For each $n \in \mathbb{N}$ let $u_n := u$ and note that we have

$$\begin{aligned}\limsup_{n\to\infty} I_n(u_n) &= \limsup_{n\to\infty} \int_\Omega \frac{\Phi_n(|\lambda(x)\nabla u(x)|)}{\varphi_n(1)}\,\mathrm{d}x \\ &\leqslant \limsup_{n\to\infty} \frac{|\Omega|\Phi_n(1)}{\varphi_n(1)} \\ &\leqslant \limsup_{n\to\infty} \frac{|\Omega|}{\varphi_n^-} \\ &= 0 \\ &= I_\infty(u),\end{aligned}$$

where we have used hypotheses (2.2) and (3.1). Thus, the constant sequence $\{u_n\} = \{u\}$ is a recovery sequence for the Γ-limit.

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To prove (3.3) we may assume, without loss of generality, that $u_n \in W^{1,\Phi_n}(\Omega)$ and

$$\liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} I_n(u_n) < \infty. \tag{4.1}$$

Note that since $\Phi_n(t)$ dominates $t^{\varphi_n^-}$ near infinity for each $n \in \mathbb{N}$, we have $W^{1,\Phi_n}(\Omega) \subset W^{1,\varphi_n^-}(\Omega)$, and thus $u_n \in W^{1,\varphi_n^-}(\Omega)$ (see [**17**, Lemma 2]).

Let $x \in \Omega$ be a Lebesgue point for $\lambda \nabla u \in L^1(\Omega)$. For any ball $B(x,r) \subset \Omega$ and $n \in \mathbb{N}$ sufficiently large we have, in view of Hölder's inequality,

$$\int_{B(x,r)} |\lambda(y)\nabla u_n(y)|\,\mathrm{d}y \leqslant \|\lambda\nabla u_n\|_{L^{\varphi_n^-}} \|\chi_{B(x,r)}\|_{L^{(\varphi_n^-)'}}, \tag{4.2}$$

where $(\varphi_n^-)' := (\varphi_n^- - 1)/\varphi_n^-$. We also have

$$\|\chi_{B(x,r)}\|_{L^{(\varphi_n^-)'}} = |B(x,r)|^{(\varphi_n^- -1)/\varphi_n^-}. \tag{4.3}$$

Before proceeding further, we note that (2.9) implies that

$$\Phi_n(1)\left\{\begin{matrix} t^{\varphi_n^-} & \text{if } t \geqslant 1 \\ t^{\varphi_n^+} & \text{if } t \in (0,1) \end{matrix}\right\} \leqslant \Phi_n(t) \leqslant \Phi_n(1)\left\{\begin{matrix} t^{\varphi_n^+} & \text{if } t \geqslant 1 \\ t^{\varphi_n^-} & \text{if } t \in (0,1) \end{matrix}\right\} \tag{4.4}$$

for all $n \in \mathbb{N}$. Consider the sets

$$\Omega_n^+ = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| \geqslant 1\} \quad \text{and} \quad \Omega_n^- = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| < 1\}.$$

In view of (2.2), (3.2) and (4.4), we have

$$\begin{aligned}
\int_\Omega |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x &= \int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x + \int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x \\
&\leqslant 1 + \int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x \\
&\leqslant 1 + \frac{\varphi_n(1)}{\Phi_n(1)} \int_\Omega \frac{\Phi_n(|\lambda(x)\nabla u_n(x)|)}{\varphi_n(1)}\,\mathrm{d}x \\
&\leqslant 1 + \varphi_n^+ I_n(u_n) \\
&\leqslant 1 + \beta\varphi_n^- I_n(u_n).
\end{aligned}$$

Thus,

$$\|\lambda\nabla u_n\|_{L^{\varphi_n^-}} \leqslant [1 + \beta\varphi_n^- I_n(u_n)]^{1/\varphi_n^-}. \tag{4.5}$$

Combining (4.2), (4.3) and (4.5), we obtain

$$\int_{B(x,r)} |\lambda(y)\nabla u_n(y)|\,\mathrm{d}y \leqslant |B(x,r)|^{(\varphi_n^- -1)/\varphi_n^-} [1 + \beta\varphi_n^- I_n(u_n)]^{1/\varphi_n^-},$$

which, in view of (3.1) and (4.1), implies that

$$\limsup_{n\to\infty} \int_{B(x,r)} |\lambda(y)\nabla u_n(y)|\,\mathrm{d}y \leqslant |B(x,r)|. \tag{4.6}$$

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Let $q \geqslant 1$ be an arbitrary real number. By (3.1), $q < \varphi_n^-$ for sufficiently large $n \in \mathbb{N}$. Using Hölder's inequality we have

$$\begin{aligned}\int_\Omega |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x &\leqslant \biggl(\int_\Omega |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x\biggr)^{q/\varphi_n^-} |\Omega|^{(\varphi_n^- - q)/\varphi_n^-} \\ &= \biggl(\int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x + \int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x\biggr)^{q/\varphi_n^-} \\ &\leqslant \biggl(1 + \int_\Omega \frac{\Phi_n(|\lambda(x)\nabla u_n(x)|)}{\Phi_n(1)}\,\mathrm{d}x\biggr)^{q/\varphi_n^-} \\ &\leqslant (1 + \beta\varphi_n^- I_n(u_n))^{q/\varphi_n^-}.\end{aligned}$$

Thus, the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ for any $q \geqslant 1$. Since $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$ we deduce, in view of the Poincaré–Wirtinger inequality, that $\{u_n\}$ is bounded in $L^q(\Omega)$. It follows that $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$, and thus we may extract a subsequence (not relabelled) such that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$. Well-known lower semicontinuity results now give

$$\int_{B(x,r)} |\lambda(y)\nabla u(y)|\,\mathrm{d}y \leqslant \liminf_{n\to\infty} \int_{B(x,r)} |\lambda(y)\nabla u_n(y)|\,\mathrm{d}y,$$

which implies, in view of (4.6), that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |\lambda(y)\nabla u(y)|\,\mathrm{d}y \leqslant 1.$$

Since almost every $x \in \Omega$ is a Lebesgue point for $\lambda\nabla u$, passing to the limit $r \to 0^+$ in the above inequality yields $|\lambda(x)\nabla u(x)| \leqslant 1$ for a.e. $x \in \Omega$. It follows that $I_\infty(u) = 0$ and this implies that the inequality (3.3) holds. This concludes the proof of Theorem 3.2.

5. Proofs of Theorems 3.3 and 3.4

We begin this section by establishing several auxiliary results that will be needed later. The following two lemmas generalize to the Orlicz space setting the classical result which asserts that if $\Omega \subset \mathbb{R}^N$ has finite Lebesgue measure and if $u \in L^\infty(\Omega)$, then

$$\lim_{q\to\infty} \|u\|_{L^q(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{5.1}$$

Lemma 5.1. *Let $\{\varphi_n\}$ be a sequence of odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ such that $\varphi_n^+ \to \infty$ as $n \to \infty$. Then, if $u \in L^\infty(\Omega) \setminus \{0\}$, we have*

$$\lim_{n\to\infty} \biggl[\frac{1}{|\Omega|\Phi_n(1)}\int_\Omega \Phi_n\biggl(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\biggr)\,\mathrm{d}x\biggr]^{1/\varphi_n^+} = 1. \tag{5.2}$$

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Proof. Define $v \in L^\infty(\Omega)$ by

$$v(x) := \frac{u(x)}{\|u\|_{L^\infty(\Omega)}}.$$

We have $|v(x)| \leqslant 1$ for a.e. $x \in \Omega$, and using the fact that $t\varphi_n(t)/\Phi_n(t) \leqslant \varphi_n^+$ for all $t > 0$ and $n \in \mathbb{N}$, we deduce that for every fixed $n \in \mathbb{N}$ the function $(0,\infty) \ni t \to t^{\varphi_n^+}/\Phi_n(t)$ is increasing on $(0,\infty)$. Thus, for every $t \in (0,1)$ and $n \in \mathbb{N}$, we have

$$\frac{t^{\varphi_n^+}}{\Phi_n(t)} \leqslant \frac{1}{\Phi_n(1)}.$$

Since $|v(x)| \leqslant 1$ for a.e. $x \in \Omega$ we find

$$\frac{|v(x)|^{\varphi_n^+}}{\Phi_n(|v(x)|)} \leqslant \frac{1}{\Phi_n(1)} \quad \text{for a.e. } x \in \Omega.$$

It follows that

$$\int_\Omega |v(x)|^{\varphi_n^+}\,\mathrm{d}x \leqslant \frac{1}{\Phi_n(1)} \int_\Omega \Phi_n(|v(x)|)\,\mathrm{d}x \leqslant \frac{1}{\Phi_n(1)} \Phi_n(1)|\Omega| = |\Omega|,$$

which gives

$$\left(\frac{1}{|\Omega|}\int_\Omega |v(x)|^{\varphi_n^+}\,\mathrm{d}x\right)^{1/\varphi_n^+} \leqslant \left[\frac{1}{|\Omega|\Phi_n(1)}\int_\Omega \Phi_n\left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right)\mathrm{d}x\right]^{1/\varphi_n^+} \leqslant 1$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, and taking into account (5.1), we conclude that (5.2) holds. □

Remark 5.2. In particular, (5.2) holds when our hypothesis (3.1) is satisfied. Also, note that (5.1) follows from (5.2); indeed, given an arbitrary sequence $\{q_n\}$ of real numbers such that $q_n \to \infty$, it suffices to consider $\Phi_n(t) := t^{q_n}$ in (5.2).

Lemma 5.3. *Let $u \in L^\infty(\Omega)$ and let $\{\varphi_n\}$ be a sequence of odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ such that (3.1) and (3.2) hold. Then*

$$\lim_{n\to\infty} \|u\|_{\Psi_n} = \|u\|_{L^\infty(\Omega)}, \tag{5.3}$$

where $\Psi_n(t) := \Phi_n(t)/\Phi_n(1)$.

Remark 5.4. Elementary computations show that with $\psi_n = \Psi_n'$, we have $\psi_n^- = \varphi_n^-$ and $\psi_n^+ = \varphi_n^+$ for all $n \in \mathbb{N}$.

Proof. We may assume, without loss of generality, that $u \not\equiv 0$. We will show that

$$\lim_{n\to\infty} \left\|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right\|_{\Psi_n} = 1. \tag{5.4}$$

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First note that, since $|u(x)| \leqslant \|u\|_{L^\infty(\Omega)}$ for a.e. $x \in \Omega$, we have

$$\frac{1}{\Psi_n(1)} \int_\Omega \Psi_n\left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right) \mathrm{d}x \leqslant 1 \quad \forall n \in \mathbb{N}.$$

Taking into account the fact that $\Psi_n(1) = 1$ for each $n \in \mathbb{N}$, the definition of the Luxemburg norm gives

$$\left\|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right\|_{\Psi_n} \leqslant 1 \quad \forall n \in \mathbb{N}. \tag{5.5}$$

In what follows we will work with the Luxemburg norm (2.4) of the Orlicz space $L^\Phi(\Omega)$ rather than the equivalent Orlicz norm (2.3).

In view of (2.6) and (5.5) we obtain that

$$\left\|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right\|_{\Psi_n}^{\varphi_n^-} \geqslant \int_\Omega \Psi_n\left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right) \mathrm{d}x,$$

and hence

$$\left\|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right\|_{\Psi_n} \geqslant \left[\left(\int_\Omega \Psi_n\left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right) \mathrm{d}x\right)^{1/\varphi_n^+}\right]^{\varphi_n^+/\varphi_n^-} \quad \forall n \in \mathbb{N}. \tag{5.6}$$

By Lemma 5.1, the sequence $\{a_n\}$ with

$$a_n := \left(\int_\Omega \Psi_n\left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}}\right) \mathrm{d}x\right)^{1/\varphi_n^+}$$

converges to 1 as $n \to \infty$. On the other hand, (3.2) implies that the sequence $\{\varphi_n^+/\varphi_n^-\}$ is bounded. Consequently,

$$\lim_{n\to\infty} a_n^{\varphi_n^+/\varphi_n^-} = \lim_{n\to\infty} \exp\left(\frac{\varphi_n^+}{\varphi_n^-} \ln(a_n)\right) = 1.$$

Passing to the limit as $n \to \infty$ in (5.6) gives

$$\liminf_{n\to\infty} \left\|\frac{u}{\|u\|_{L^\infty(\Omega)}}\right\|_{\Psi_n} \geqslant 1.$$

Hence, taking into account (5.5), we deduce that (5.4) holds. □

Remark 5.5. We point out that in the case when the additional assumption $|\Omega| = 1$ is removed (that is, we just assume that $|\Omega| < +\infty$), the conclusion of Lemma 5.3 (see (5.3)) should read

$$\lim_{n\to\infty} \|u\|_{|\Omega|^{-1}\Psi_n} = \|u\|_{L^\infty(\Omega)}. \tag{5.7}$$

Clearly, (5.7) implies (5.3). We also note that, given an arbitrary sequence of real numbers $q_n \to \infty$, by taking $\Phi_n(t) := t^{q_n}$ in (5.7) we again recover (5.1).

Lemma 5.6. *Let $\Omega_1 \subset \Omega_2$ be two open sets and $u \in L^{\Phi}(\Omega_2)$. Then $u \in L^{\Phi}(\Omega_1)$ and*

$$\|u\|_{\Phi,\Omega_1} \leqslant \|u\|_{\Phi,\Omega_2},$$

where $\|u\|_{\Phi,\Omega_i}$ stands for the Luxemburg norm of u in $L^{\Phi}(\Omega_i)$, $i \in \{1,2\}$.

Proof. Assume that $u \not\equiv 0$ in Ω_2. Clearly,

$$\int_{\Omega_1} \Phi(|u|)\,\mathrm{d}x \leqslant \int_{\Omega_2} \Phi(|u|)\,\mathrm{d}x,$$

and thus $u \in L^{\Phi}(\Omega_1)$. The previous inequality can be rewritten as

$$\int_{\Omega_2} \Phi(|u|\chi_{\Omega_1})\,\mathrm{d}x \leqslant \int_{\Omega_2} \Phi(|u|)\,\mathrm{d}x,$$

which holds for all $u \in L^{\Phi}(\Omega_2)$. Combining this with the definition of the Luxemburg norm in $L^{\Phi}(\Omega_2)$, we deduce that

$$\int_{\Omega_2} \Phi\biggl(\frac{|u|}{\|u\|_{\Phi,\Omega_2}}\chi_{\Omega_1}\biggr)\,\mathrm{d}x \leqslant \int_{\Omega_2} \Phi\biggl(\frac{|u|}{\|u\|_{\Phi,\Omega_2}}\biggr)\,\mathrm{d}x \leqslant 1.$$

Hence,

$$\int_{\Omega_1} \Phi\biggl(\frac{|u|}{\|u\|_{\Phi,\Omega_2}}\biggr)\,\mathrm{d}x \leqslant 1$$

and the definition of the Luxemburg norm in $L^{\Phi}(\Omega_1)$ now yields $\|u\|_{\Phi,\Omega_1} \leqslant \|u\|_{\Phi,\Omega_2}$. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. We begin by establishing the existence of a recovery sequence for the Γ-limit. Let $u \in L^1(\Omega)$ be such that $J_\infty(u) < +\infty$. Thus, $u \in W^{1,\infty}(\Omega)$ and $J_\infty(u) = \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}$. For $n \in \mathbb{N}$ define $u_n := u$. We have $u_n \in W^{1,\Psi_n}(\Omega)$, and using Lemma 5.3 we obtain

$$\limsup_{n\to\infty} J_n(u_n) = \limsup_{n\to\infty} \|\lambda\nabla u\|_{\Psi_n} = \|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}.$$

We deduce that the constant sequence $\{u_n\} = \{u\}$ is a recovery sequence for the Γ-limit.

It remains to show that for any $u \in L^1(\Omega)$ we have

$$J_\infty(u) \leqslant \liminf_{n\to\infty} J_n(u_n) \tag{5.8}$$

whenever $\{u_n\} \subset L^1(\Omega)$ is such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$.

We may assume, without loss of generality, that $u_n \in W^{1,\Psi_n}(\Omega)$ and, after eventually extracting a subsequence (not relabelled),

$$\liminf_{n\to\infty} J_n(u_n) = \lim_{n\to\infty} J_n(u_n) < \infty. \tag{5.9}$$

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Let $q \geqslant 1$ be arbitrary. By (3.1), $q < \varphi_n^-$ for sufficiently large $n \in \mathbb{N}$. For $n \in \mathbb{N}$ consider the sets

$$\Omega_n^+ = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| \geqslant 1\} \quad \text{and} \quad \Omega_n^- = \{x \in \Omega;\ |\lambda(x)\nabla u_n(x)| < 1\}.$$

We have

$$\int_\Omega |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x = \int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x + \int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x.$$

Hölder's inequality yields

$$\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x \leqslant |\Omega_n^+|^{(\varphi_n^- - q)/\varphi_n^-} \biggl(\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \,\mathrm{d}x \biggr)^{q/\varphi_n^-} \tag{5.10}$$

and

$$\int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^q \,\mathrm{d}x \leqslant |\Omega_n^-|^{(\varphi_n^+ - q)/\varphi_n^+} \biggl(\int_{\Omega_n^-} |\lambda(x)\nabla u_n(x)|^{\varphi_n^+} \,\mathrm{d}x \biggr)^{q/\varphi_n^+}. \tag{5.11}$$

For $n \in \mathbb{N}$ and $x \in \Omega_n^+$ we have, in view of (2.9), that

$$|\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \leqslant \Psi_n(|\lambda(x)\nabla u_n(x)|). \tag{5.12}$$

Given $n \in \mathbb{N}$, we have two alternatives: either $|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+} \leqslant 1$ or $|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+} > 1$. We will assume first that we are in the case where $|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+} \leqslant 1$. Taking into account (2.9), (5.12) and the definition of the Luxemburg norm in $L^{\Psi_n}(\Omega_n^+)$, we obtain that

$$\int_{\Omega_n^+} \biggl| \frac{\lambda(x)\nabla u_n(x)}{|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+}} \biggr|^{\varphi_n^-} \mathrm{d}x \leqslant \int_{\Omega_n^+} \Psi_n\biggl(\frac{\lambda(x)\nabla u_n(x)}{|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+}} \biggr) \leqslant 1,$$

which further yields

$$\biggl(\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \,\mathrm{d}x \biggr)^{q/\varphi_n^-} \leqslant |||\lambda\nabla u_n|||^q_{\Psi_n,\Omega_n^+}. \tag{5.13}$$

Next, assume that $|||\lambda\nabla u_n|||_{\Psi_n,\Omega_n^+} > 1$. Integrating (5.12) over Ω_n^+ and using (2.7), we get

$$\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \,\mathrm{d}x \leqslant \int_{\Omega_n^+} \Psi_n(|\lambda(x)\nabla u_n(x)|) \,\mathrm{d}x \leqslant |||\lambda\nabla u_n|||^{\varphi_n^+}_{\Psi_n,\Omega_n^+}.$$

Thus,

$$\biggl(\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \,\mathrm{d}x \biggr)^{q/\varphi_n^-} \leqslant |||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega_n^+}. \tag{5.14}$$

From (5.13) and (5.14) it follows that

$$\biggl(\int_{\Omega_n^+} |\lambda(x)\nabla u_n(x)|^{\varphi_n^-} \,\mathrm{d}x \biggr)^{q/\varphi_n^-} \leqslant \max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega_n^+}, |||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega_n^+}\},$$

and hence, in view of Lemma 5.6,

$$\left(\int_{\Omega_n^+}|\lambda(x)\nabla u_n(x)|^{\varphi_n^-}\,\mathrm{d}x\right)^{q/\varphi_n^-}\leqslant\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega}\}$$

for all $n\in\mathbb{N}$. A similar argument implies that we have

$$\left(\int_{\Omega_n^-}|\lambda(x)\nabla u_n(x)|^{\varphi_n^+}\,\mathrm{d}x\right)^{q/\varphi_n^+}\leqslant\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^-/\varphi_n^+}_{\Psi_n,\Omega}\}\quad\forall n\in\mathbb{N}.$$

The last two inequalities, combined with (5.10) and (5.11), respectively, imply that

$$\begin{aligned}\left(\int_{\Omega}|\lambda(x)\nabla u_n(x)|^q\,\mathrm{d}x\right)^{1/q}&\\ &\leqslant\left(|\Omega_n^+|^{(\varphi_n^--q)/\varphi_n^-}+|\Omega_n^-|^{(\varphi_n^+-q)/\varphi_n^+}\right)^{1/q}\\ &\quad\times\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^-/\varphi_n^+}_{\Psi_n,\Omega}\}^{1/q}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\left(\int_{\Omega}|\lambda(x)\nabla u_n(x)|^q\,\mathrm{d}x\right)^{1/q}&\\ &\leqslant\left(|\Omega_n^+|^{(\varphi_n^--q)/\varphi_n^-}+(1-|\Omega_n^+|)^{(\varphi_n^+-q)/\varphi_n^+}\right)^{1/q}\\ &\quad\times\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^-/\varphi_n^+}_{\Psi_n,\Omega}\}^{1/q}.\end{aligned}$$

Using the fact that $(\varphi_n^- - q)/\varphi_n^-\leqslant(\varphi_n^+ - q)/\varphi_n^+$ for each $n\in\mathbb{N}$, the above inequality implies that

$$\begin{aligned}|||\lambda\nabla u_n|||_{L^q(\Omega)}&\leqslant\left(|\Omega_n^+|^{(\varphi_n^--q)/\varphi_n^-}+(1-|\Omega_n^+|)^{(\varphi_n^--q)/\varphi_n^-}\right)^{1/q}\\ &\quad\times\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^-/\varphi_n^+}_{\Psi_n,\Omega}\}^{1/q}.\end{aligned}$$

Since $x^\theta+(1-x)^\theta\leqslant 2^{1-\theta}$ for all $x,\theta\in(0,1)$, it follows that

$$\begin{aligned}|||\lambda\nabla u_n|||_{L^q(\Omega)}&\leqslant 2^{(1-(\varphi_n^--q)/\varphi_n^-)/q}\\ &\quad\times\max\{|||\lambda\nabla u_n|||^q_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^+/\varphi_n^-}_{\Psi_n,\Omega},|||\lambda\nabla u_n|||^{q\varphi_n^-/\varphi_n^+}_{\Psi_n,\Omega}\}^{1/q}.\end{aligned}\tag{5.15}$$

Hence, by (5.9) and (3.5), we obtain that the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$ for any $q\geqslant 1$. Since $u_n\rightharpoonup u$ weakly in $L^1(\Omega)$ we deduce, after eventually extracting a subsequence (not relabelled), that $u_n\rightharpoonup u$ weakly in $W^{1,q}(\Omega)$. The weak lower semicontinuity of the norm implies that

$$\|\lambda\nabla u\|_{L^q(\Omega;\mathbb{R}^N)}\leqslant\liminf_{n\to\infty}\|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)}.\tag{5.16}$$

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Passing to the limit as $n \to \infty$ in (5.15) and taking into account (5.9) and our hypothesis (3.5), we obtain

$$\limsup_{n\to\infty} \|\lambda \nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \leqslant \limsup_{n\to\infty} \|\lambda \nabla u_n\|_{\Psi_n} = \lim_{n\to\infty} J_n(u_n). \tag{5.17}$$

Combining (5.16) and (5.17) we find that, for any $q > 1$, we have

$$\|\lambda \nabla u\|_{L^q(\Omega;\mathbb{R}^N)} \leqslant \lim_{n\to\infty} J_n(u_n). \tag{5.18}$$

Using a localization argument similar to the one used in the proof of Theorem 3.2, it can be shown that $\nabla u \in L^\infty(\Omega;\mathbb{R}^N)$. Thus, letting $q \to \infty$ in (5.18) and taking into account (5.1), we deduce that

$$J_\infty(u) \leqslant \lim_{n\to\infty} J_n(u_n) = \liminf_{n\to\infty} J_n(u_n).$$

Hence, (5.8) holds, which concludes the proof of Theorem 3.3. □

Before we can prove Theorem 3.4 we need to establish an auxiliary result which is a slight refinement of Lemma 5.1.

Lemma 5.7. *Let $\{\varphi_n\}$ be a sequence of odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ such that (3.1) and (3.5) are satisfied, and let $u \in L^\infty(\Omega)$ be such that $\|u\|_{L^\infty(\Omega)} \leqslant 1$. Then*

$$\lim_{n\to\infty} \left(\int_\Omega \Psi_n(|u(x)|)\,\mathrm{d}x \right)^{1/\varphi_n^+} = \|u\|_{L^\infty(\Omega)},$$

where, for $n \in \mathbb{N}$ and $t > 0$, $\Psi_n(t) := \Phi_n(t)/\Phi_n(1)$.

Proof. Using the fact that $|u(x)| \leqslant \|u\|_{L^\infty(\Omega)} \leqslant 1$ for a.e. $x \in \Omega$, together with (4.4), we obtain that

$$\begin{aligned} \left(\int_\Omega |u(x)|^{\varphi_n^+}\,\mathrm{d}x \right)^{1/\varphi_n^+} &\leqslant \left(\int_\Omega \Psi_n(|u(x)|)\,\mathrm{d}x \right)^{1/\varphi_n^+} \\ &\leqslant \left[\left(\int_\Omega |u(x)|^{\varphi_n^-}\,\mathrm{d}x \right)^{1/\varphi_n^-} \right]^{\varphi_n^-/\varphi_n^+} \end{aligned}$$

for every $n \in \mathbb{N}$. Taking into account (3.1), (3.5) and (5.1), the conclusion now follows by letting $n \to \infty$ in the above estimates. □

Proof of Theorem 3.4. Let $u \in L^1(\Omega)$ be arbitrary. To prove the existence of a recovery sequence, we only need to consider the non-trivial case where $J_\infty(u) < \infty$. Thus, $u \in W^{1,\infty}(\Omega)$ and $J_\infty(u) = \|\lambda \nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)}$. Define $u_n \in W^{1,\Psi_n}(\Omega)$ by $u_n := u$ for $n \in \mathbb{N}$. We have two possibilities: either $\|\lambda \nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} \leqslant 1$ or $\|\lambda \nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} > 1$. In the first case, applying (5.5) to $\lambda|\nabla u| \in L^\infty(\Omega)$ gives $\|\lambda \nabla u\|_{\Psi_n} \leqslant 1$, which implies that

$$K_n(u_n) = K_n(u) = \left(\int_\Omega \Psi_n(|\lambda(x) \nabla u(x)|)\,\mathrm{d}x \right)^{1/\varphi_n^+}.$$

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On the other hand, if $\|\lambda\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} > 1$, we have, by Lemma 5.3, that $\|\lambda\nabla u\|_{\Psi_n} > 1$ for $n \in \mathbb{N}$ sufficiently large. Thus, in this case, $K_n(u_n) = K_n(u) = \|\lambda\nabla u\|_{\Psi_n}$. In view of Lemmas 5.3 and 5.7 we conclude that

$$\lim_{n\to\infty} K_n(u_n) = J_\infty(u),$$

and hence $\{u_n\} = \{u\}$ is again a recovery sequence for the Γ-limit.

It remains to prove that

$$J_\infty(u) \leqslant \liminf_{n\to\infty} K_n(u_n) \tag{5.19}$$

whenever $\{u_n\} \subset L^1(\Omega)$ and $u \in L^1(\Omega)$ are such that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$. Extracting a subsequence if necessary, we may assume, without loss of generality, that $u_n \in W^{1,\Psi_n}(\Omega)$ and

$$\liminf_{n\to\infty} K_n(u_n) = \lim_{n\to\infty} K_n(u_n) < \infty.$$

Let $q \geqslant 1$ be arbitrary. In view of (3.1), $q < \varphi_n^- \leqslant \varphi_n^+$ for sufficiently large $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we have either $\|\lambda\nabla u_n\|_{\Psi_n} \leqslant 1$ or $\|\lambda\nabla u_n\|_{\Psi_n} > 1$. Revisiting the proof of Theorem 3.3 we recall that, in either case, (5.15) gives

$$\begin{aligned}\|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \leqslant{}& 2^{(1-(\varphi_n^- - q)/\varphi_n^-)/q} \\ &\times \max\{\|\|\lambda\nabla u_n\|\|_{\Psi_n}^q, \|\|\lambda\nabla u_n\|\|_{\Psi_n}^{q\varphi_n^+/\varphi_n^-}, \|\|\lambda\nabla u_n\|\|_{\Psi_n}^{q\varphi_n^-/\varphi_n^+}\}^{1/q}.\end{aligned} \tag{5.20}$$

In addition, when $\|\lambda\nabla u_n\|_{\Psi_n} \leqslant 1$, the above inequality and (2.6) give

$$\begin{aligned}\|\lambda\nabla u_n\|_{L^q(\Omega;\mathbb{R}^N)} \leqslant{}& 2^{(1-(\varphi_n^- - q)/\varphi_n^-)/q} \\ &\times \max\{T_n(\lambda\nabla u_n)^q, T_n(\lambda\nabla u_n)^{q\varphi_n^+/\varphi_n^-}, T_n(\lambda\nabla u_n)^{q\varphi_n^-/\varphi_n^+}\}^{1/q},\end{aligned} \tag{5.21}$$

where

$$T_n(v) := \left(\int_\Omega \Psi_n(|v|)\,\mathrm{d}x\right)^{1/\varphi_n^+}.$$

Thus, by (5.20), (5.21) and (3.5), we deduce that the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^N)$. To conclude that (5.19) holds, one may now proceed along the lines of the last part of the proof of Theorem 3.3. □

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