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DISTRIBUTIVE MODULES

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ABSTRACT. Let R be a commutative ring with identity. An R-module M is said to be distributive if the lattice of submodules of M is distributive. We characterize such modules and study their properties.

Introduction. Throughout this paper all rings will be commutative with identities, all modules will be unitary modules, and "R" will always denote a ring. Let R be a ring and M an R-module. Then M is said to be distributive if the following condition is satisfied:

$X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$, for all submodules X, Y, Z of M. A ring R is said to be arithmetical if R considered as a module over itself is distributive.

In the last two decades, considerable research has been done on rings with a distributive lattice of ideals [4, 6, and the references therein].

In this paper we study the structure and properties of distributive modules. Our results are motivated in large part by the paper [7] of W. Stephenson.

In section 1, we show that if a ring R has a finitely generated faithful and distributive module M, then R is arithmetical and M is projective of rank one (Proposition 1.3).

In section 2, we give some characterizations of distributive modules in terms of the order ideals of submodules and the homomorphisms of factor modules of submodules (Theorem 2.2 and its Corollary, Proposition 2.3 and its Corollary).

In section 3, we study the properties of distributive modules over Noetherian rings. In particular we show that over a Noetherian arithmetical ring R a finitely generated distributive R-module M is of the form $\bigoplus_{i=1}^{n} M/P_i^{v_i}M$, where $P_1, P_2, \ldots, P_n$ are prime ideals of R and $v_1, v_2, \ldots, v_n$ are positive integers (Theorem 3.4). Then in Proposition 3.5, we show that over a Dedekind domain, a finitely generated torsion distributive module is cyclic.

1. **Some properties of distributive modules**. We begin by recalling the following two well known lemmas.

LEMMA 1.1. [7] *Let R be a local ring and let M be an R-module. Then M is a distributive R-module if, and only if, the set of submodules of M is linearly ordered.*

LEMMA 1.2. [3] *Let R be a ring and M an R-module.*

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(i) *Let S be a multiplicatively closed subset of R. If M is a distributive R-module, then $S^{-1}M$ is a distributive $S^{-1}R$-module.*
(ii) *M is a distributive R-module, if and only if, M_P is a distributive R_P-module for all maximal ideals P of R.*

PROPOSITION 1.3. *Let R be a ring and let M be a finitely generated distributive R-module with $Ann_R(M) = 0$. Then*

(i) *$M_P \simeq R_P$, for all maximal ideals P of R*
(ii) *R is an arithmetical ring;*
(iii) *M is a rank one projective R-module.*

PROOF. (i) $(\mathrm{Ann}_R(M))_P = \mathrm{Ann}_{R_P}(M_P) = 0$ (since M is finitely generated). Since M is a distributive R-module, M_P is a distributive R_P-module, for all maximal ideals P of R (Lemma 1.2). But then for each maximal ideal P, M_P as an R_P-module has a linearly ordered lattice of submodules (Lemma 1.1). That is, for each maximal ideal P, M_P is a finitely generated R_P-module and has a linearly ordered lattice of submodules. Therefore it follows that M_P is a cyclic R_P-module. Since $\mathrm{Ann}_{R_P}(M_P) = 0$, for each maximal ideal P, therefore we have $M_P \simeq R_P$, for all maximal ideals P of R.
(ii) By (i) we have $M_P \simeq R_P$, for all maximal ideals P of R. Hence R_P has a linearly ordered lattice of ideals, for each maximal ideal P of R. Therefore by (ii) of Lemma 1.2, R is arithmetical.
(iii) Since M is finitely generated and $M_P \simeq R_P$, for all maximal ideals P of R, it follows that M is a projective of rank one R-module [5, Theorem 3.3.7].

2. **Direct sum of distributive modules**. In this section we consider the direct sum of a family of distributive modules. First we have the following:

LEMMA 2.1. *Let R be a ring and let $M \oplus N$ be a direct sum of two R-modules. Then the following statements are equivalent.*

(i) *Every submodule of $M \oplus N$ is of the form $A \oplus B$, for some submodule A of M and some submodule B of N.*
(ii) $\mathrm{Ann}(x) + \mathrm{Ann}(y) = R$, *for all $x \in M$ and all $y \in N$.*
(iii) $\mathrm{Hom}_R(A_1/A_2, B_1/B_2) = 0$, *for all submodules $A_2 \subseteq A_1 \subseteq M$ and $B_2 \subseteq B_1 \subseteq N$.*

PROOF. (i) $\rightarrow$ (ii). Let W be any submodule of $M \oplus N$. Then $W = A \oplus B$, for some submodule A of M and some submodule B of N. Consider the following:

$$W \cap M = (A \oplus B) \cap M = A \oplus 0$$

$$W \cap N = (A \oplus B) \cap N = 0 \oplus B.$$

From this, we have $W = A \oplus B = (A \oplus 0) + (0 \oplus B) = (W \cap M) + (W \cap N)$. Take any x in M and y in N and look at the submodule $R(x + y)$ of $M \oplus N$. By the above argument, we have $R(x + y) = (R(x + y) \cap M) + (R(x + y) \cap N)$. Hence $x + y = u + v$, for some $u \in R(x + y) \cap M$ and some $v \in R(x + y) \cap N$. We want to show in fact that $x = u$ and $y = v$. To see this, we observe the following:

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$$(x + y) - (x + v) = (u + v) - (x + v),$$

which implies that $0 + (y - v) = (u - x) + 0$. That is, $(y - v) = 0$ and $(u - x) = 0$. Therefore we have $y = v$ and $u = x$. But, we have $u \in R(x + y) \cap M$ and $v \in R(x + y) \cap N$. Therefore $u = r(x + y) \in M$ and $v = s(x + y) \in N$, for some $r, s \in R$. That is $x = r(x + y)$ which implies that $(1 - r)x = ry \in M \cap N = 0$. Therefore $1 - r \in \text{Ann}(x)$ and $r \in \text{Ann}(y)$. Therefore $1 = (1 - r) + r \in \text{Ann}(x) + \text{Ann}(y)$ and hence $\text{Ann}(x) + \text{Ann}(y) = R$.

(ii) $\rightarrow$ (i). Take any submodule U of $M \oplus N$. Then for any $u \in U$, we have $u = m + n$, for some $m \in M$ and some $n \in N$. Since $\text{Ann}(m) + \text{Ann}(n) = R$, $1 = a + b$ for some $a \in \text{Ann}(m)$ and some $b \in \text{Ann}(n)$. Thus, we have $bu = b(m + n) = bm + bn = bm \in M \cap U$ and $au = a(m + n) = an \in N \cap U$. We also have $m = (a + b)m = bm$ and $n = (a + b)n = an$. Therefore $u = m + n = bm + an = bu + au$. That is $u = bu + au \in (M \cap U) \oplus (N \cap U)$. Since u is taken arbitrarily in U, therefore we have $U = (U \cap M) \oplus (U \cap N)$.

(ii) $\rightarrow$ (iii). Suppose not. Then there are submodules $X_2 \subseteq X_1$ of M and submodules $Y_2 \subseteq Y_1$ of N such that $\text{Hom}_R(X_1/X_2, Y_1/Y_2) \neq 0$. That is, we have a non-zero R-homomorphism $f: X_1/X_2 \rightarrow Y_1/Y_2$. Hence there is a non-zero element $x_1 + X_2$ in X_1/X_2 such that $f(x_1 + X_2) \neq 0$ in Y_1/Y_2, say $f(x_1 + X_2) = y_1 + Y_2$ for some $y_1 \in Y_1 \backslash Y_2$. Now we have $\text{Ann}(x_1) \subseteq (X_2 : x_1) = \text{Ann}(x_1 + X_2)$ and $\text{Ann}(y_1) \subseteq (Y_2 : y_1) = \text{Ann}(y_1 + Y_2)$. But $\text{Ann}(x_1) + \text{Ann}(y_1) = R$. Therefore we have $\text{Ann}(x_1 + X_2) + \text{Ann}(y_1 + Y_2) = R$. If now $r \in \text{Ann}(x_1 + X_2)$, then we have $rf(x_1 + X_2) = f(r(x_1 + X_2)) = 0$ in Y_1/Y_2 (because f is a homomorphism). That is, we have $\text{Ann}(x_1 + X_2) \subseteq \text{Ann}(f(x_1 + X_2)) = \text{Ann}(y_1 + Y_2)$. Therefore it follows that $\text{Ann}(y_1 + Y_2) = \text{Ann}(f(x_1 + X_2)) = R$, which implies that $f(x_1 + X_2) = 0$ in Y_1/Y_2. This is a contradiction to the fact that $f(x_1 + X_2) \neq 0$ in Y_1/Y_2. Therefore we must have $\text{Hom}_R(A_1/A_2, B_1/B_2) = 0$, for all $A_2 \subseteq A_1 \subseteq M$ and $B_2 \subseteq B_1 \subseteq N$.

(iii) $\rightarrow$ (ii). If not, then there are elements m in M and n in N such that $\text{Ann}(m) + \text{Ann}(n) \neq R$. Since $\text{Ann}(m) \subseteq \text{Ann}(m) + \text{Ann}(n)$ and $\text{Ann}(n) \subseteq \text{Ann}(m) + \text{Ann}(n)$, therefore there are well defined R-homomorphisms

$$f_1: Rm \cong R/\text{Ann}(m) \rightarrow R/(\text{Ann}(m) + \text{Ann}(n))$$

$$f_2: Rn \cong R/\text{Ann}(n) \rightarrow R/(\text{Ann}(m) + \text{Ann}(n)).$$

Since $Rm/\text{Ker} f_1 \cong R/(\text{Ann}(m) + \text{Ann}(n))$ and $Rn/\text{Ker} f_2 \cong R/(\text{Ann}(m) + \text{Ann}(n))$ and $\text{Ann}(n) + \text{Ann}(m) \neq R$, therefore it follows that there is a non-zero isomorphism from $Rm/\text{Ker} f_1$ into $Rn/\text{Ker} f_2$. This is again a contradiction. Therefore we have to have $\text{Ann}(m) + \text{Ann}(n) = R$, for all $m \in M$ and all $n \in N$.

THEOREM 2.2. *Let R be a ring and $M_1 \oplus M_2$ be a direct sum of two R-modules. Then the following statements are equivalent:*

(i) *$M_1 \oplus M_2$ is distributive*

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(ii) *M_1 and M_2 are distributive and* $\mathrm{Ann}(m_1) + \mathrm{Ann}(m_2) = R$, *for all* $m_1 \in M_1$ *and all* $m_2 \in M_2$.
(iii) *M_1 and M_2 are distributive and* $\mathrm{Hom}_R(A_1/A_2, B_1/B_2) = 0$, *for all submodules* $A_2 \subseteq A_1 \subseteq M_1$ *and* $B_2 \subseteq B_1 \subseteq M_2$.

PROOF. (i) $\leftrightarrow$ (iii) By [7, Proposition 1.3]. (ii) $\leftrightarrow$ (iii) By Lemma 2.1.

COROLLARY. *Let R be a ring and let M_1 and M_2 be two finitely generated R-modules. Then the following statements are equivalent.*

(i) *$M_1 \oplus M_2$ is distributive.*
(ii) *M_1 and M_2 are distributive and* $\mathrm{Ann}(M_1) + \mathrm{Ann}(M_2) = R$.

PROOF. By Theorem 2.2, it remains to prove that $M_1 \oplus M_2$ is distributive implies that $\mathrm{Ann}_R(M_1) + \mathrm{Ann}_R(M_2) = R$. Let P be any prime ideal of R. Since M_1 and M_2 are finitely generated, therefore $(\mathrm{Ann}_R(M_1))_P + (\mathrm{Ann}_R(M_2))_P = \mathrm{Ann}_{R_P}(M_{1_P}) + \mathrm{Ann}_{R_P}(M_{2_P})$ [1, Proposition 3.14]. If either $M_{1_P} = 0$ or $M_{2_P} = 0$, then $(\mathrm{Ann}_R(M_1))_P + (\mathrm{Ann}_R(M_2))_P = R_P$. Suppose that $M_{1_P} \neq 0$ and $M_{2_P} \neq 0$. Then we have that $M_{1_P} \oplus M_{2_P}$ is a non-zero distributive R_P-module. But then by Lemma 1.1, we have either $M_{1_P} \subseteq M_{2_P}$ or $M_{2_P} \subseteq M_{1_P}$. This is a contradiction. Hence the Corollary.

Let R be a ring and M an R-module. A submodule X of M is said to be fully invariant if $f(X) \subseteq X$, for all $f \in \mathrm{Hom}_R(M, M)$. It is well known that every submodule of a finitely generated distributive module is fully invariant [7, Proposition 4.3]. Here we prove that under certain conditions the converse is also true.

PROPOSITION 2.3. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of distributive R-modules. If every submodule of M is fully invariant, then M is distributive.*

PROOF. Take W to be any submodule of $M = \bigoplus_{i \in I} M_i$ and denote by Π_i the projections, Π_i: $\bigoplus_{i \in I} M_i \to M_i$. Since every submodule of M is fully invariant, therefore we have $\Pi_i(W) \subseteq W$ and (of course) $\Pi_i(W) \subseteq M_i$, for all $i \in I$. So if $i \neq j$ in I, we have $\Pi_i(W) \cap \Pi_j(W) = 0$. If now w is any element of W, we have $w = \Sigma \Pi_i(w)$. Therefore it follows that $W = \oplus \Pi_i(W)$. Obviously $\Pi_i(W) \subseteq W \cap M_i$. Therefore we necessarily have $\Pi_i(W) = W \cap M_i$. That is, $W = \oplus (W \cap M_i)$. Hence the result by [7, Proposition 1.3].

COROLLARY. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of finitely generated distributive R-modules. Then M is distributive if, and only if, every submodule of M is fully invariant.*

PROOF. Suppose that M is distributive. Since M is a direct sum of finitely generated R-modules, therefore M is a finitely generated distributive R-module. Therefore every submodule of M is fully invariant by the remark made before Proposition 2.3.

The converse follows from Proposition 2.3.

PROPOSITION 2.4. *Let R be an integral domain and M a torsion-free distributive R-module. Then M is indecomposable (that is, $M = M_1 \oplus M_2$, for R-modules M_1 and*

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M_2 implies that either $M = M_1$ and $M_2 = 0$ or *$M = M_2$ and $M_1 = 0$).*

PROOF. Suppose that $M_1 \neq 0$, $M_2 \neq 0$ and $M = M_1 \oplus M_2$. Since M is distributive, we have $\mathrm{Ann}(m_1) + \mathrm{Ann}(m_2) = R$, for all m_1 in M_1 and all m_2 in M_2 (Theorem 2.2). But M is a torsion-free R-module. Therefore we have $\mathrm{Ann}(m_1) = 0$, $\mathrm{Ann}(m_2) = 0$, for all $0 \neq m_1 \in M_1$ and all $0 \neq m_2 \in M_2$. That is $\mathrm{Ann}(m_1) + \mathrm{Ann}(m_2) = 0$, whenever $m_1 \neq 0$ in M_1 and $m_2 \neq 0$ in M_2. This is a contradiction. Therefore the result.

3. **Distributive modules over Noetherian rings**. We recall the following well known facts.

LEMMA 3.1. (i) *In an arithmetical ring R any two primary ideals are either comparable or else are comaximal.*
(ii) *In a Noetherian arithmetical ring R, any primary ideal is a power of its radical and every ideal is a product of powers of prime ideals.*

LEMMA 3.2. *Let R be a ring and let M be a finitely generated R-module. If M is a distributive R-module, then for any submodule X of M we have $X = (X: M)M$.*

PROOF. First take R to be a local ring and M a finitely generated distributive R-module. Then (as shown in the proof of part (i) of Proposition 1.3), M is cyclic. That is $M = Rm$, for some $m \in M$. If now X is any submodule of $M = Rm$, then for any $x \in X$, we have $x = rm$ for some $r \in R$, which implies that $r \in (X: M) = (X: m)$ and $X = (X: M)M$.

Consider now the global case. That is, we have R is any ring and M is a finitely generated distributive R-module. If X is any submodule of M and P is any prime ideal of R, then we have

$$
\begin{aligned}
X_P &= (X_P: M_P)M_P && \text{by the local case above} \\
&= (X: M)_P M_P && \text{since } M \text{ is finitely generated.}
\end{aligned}
$$

That is, we have $X_P = ((X: M)M)_P$, for all maximal ideals P of R. Therefore we have $X = (X: M)M$.

PROPOSITION 3.3. *Let R be a Noetherian local arithmetical ring and M a finitely generated R-module. Then M is distributive if, and only if, the set of submodules of M consists of $M \supseteq PM \supseteq P^2M \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} P^nM = 0$, where P is the maximal ideal of R.*

PROOF. Let X be any non-zero submodule of M. Then by the above Lemma, we have $X = (X: M)M$. Since R is a Noetherian local arithmetical ring, $(X: M) = P^k$, for some positive integer k. Therefore it follows that the set of submodules of M consists of $\{M, PM, P^2M, \ldots, 0\}$.

The converse is obvious.

THEOREM 3.4. *Let R be a Noetherian arithmetical ring and let M be a finitely generated distributive R-module. Then there exists a set of prime ideals*

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$\{P_1, P_2, \ldots, P_n\}$ *of* R *and a set of positive integers* $\{v_1, v_2, \ldots, v_n\}$ *such that*

$$M \simeq M/P_1^{v_1}M \oplus M/P_2^{v_2}M \oplus \cdots \oplus M/P_n^{v_n}M.$$

PROOF. Let $0 = N_1 \cap N_2 \cap \cdots \cap N_n$ be a normal primary decomposition of the zero submodule of M. Since M is Noetherian distributive, therefore by Lemma 3.2, we have $N_i = (N_i : M)M$, for all $i = 1, 2, \ldots, n$. Since N_i's are primary submodules of M, therefore $(N_i : M)$'s are primary ideals of R, $(1 < i < n)$.

Put $Q_i = (N_i : M)$. Since the decomposition is a reduced one, therefore for each pair $i \neq j$ in $(\{1, 2, \ldots, n\}$ we have $Q_i + Q_j = R$. [Otherwise $Q_i, Q_j \subseteq \mathcal{M}$, for some maximal ideal $\mathcal{M}$ of R, which implies that either $Q_i \subseteq Q_j$ or $Q_j \subseteq Q_i$ (Lemma 3.1). This implies that either

$$\bigcap_{\substack{k=1\\k\neq j}}^{n} N_k \subseteq N_j \quad \text{or} \quad \bigcap_{\substack{k=1\\k\neq i}}^{n} N_k \subseteq N_i.$$

(This is because if $Q_i \subseteq Q_j$, then $N_i = Q_iM \subseteq Q_jM = N_j$ and hence $\bigcap_{\substack{k=1\\k\neq j}}^{n} N_k \subseteq N_j$.) But this is a contradiction to the fact that the decomposition is taken to be normal].

We now put $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. If for each $i = 1, 2, \ldots, n$, $\text{rad}(Q_i) = P_i$, then by Lemma 3.1, we have $Q_i = P_i^{v_i}$ for some positive integer v_i. That is, we have $Q = Q_1 \cap Q_2 \cdots \cap Q_n = Q_1Q_2 \ldots Q_n = P_1^{v_1}P_2^{v_2} \ldots P_n^{v_n}$. Then we have $R/Q \simeq R/P_1^{v_1} \oplus R/P_2^{v_2} \oplus \cdots \oplus R/P_n^{v_n}$, under the mapping induced by

$$R \to \bigoplus_{i=1}^{n} R/P_i^{v_i}$$

$$x \to x + P_i^{v_i}$$

Tensoring both sides by M over R, we get

$$(R/Q) \bigotimes_R M \simeq \left(\bigoplus_{i=1}^{n} R/P_i^{v_i}\right) \bigotimes_R M,$$

which implies $M/QM \simeq \bigoplus_{i=1}^{n} M/P_i^{v_i}M$.

Since $QM = Q_1M \cap Q_2M \cap \cdots \cap Q_nM = 0$, therefore we have $M \simeq \bigoplus_{i=1}^{n} M/P_i^{v_i}M$.

PROPOSITION 3.5. *Over a Dedekind domain, a finitely generated torsion distributive module is cyclic.*

PROOF. Let R be a Dedekind domain and let M be a finitely generated torsion distributive R-module. Then M is of the form $M = R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$, for some ideals $I_1, I_2, \ldots, I_n$ of R, where $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ [2, Theorem 4, p. 412]. Now using the Corollary to Theorem 2.2 and induction on n, we have $\text{Ann}(R/I_1) + \bigcap_{i=2}^{n} \text{Ann}(R/I_i) = R$. Since $I_1 \supseteq \bigcap_{i=2}^{n} \text{Ann}(R/I_i) = I_n$, it follows that $I_1 = R$. Continuing in this way we get $M = R/I$, for some ideal I of R.

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