

LENS SPACE SURGERIES ALONG CERTAIN 2-COMPONENT LINKS RELATED WITH PARK'S RATIONAL BLOW DOWN, AND REIDEMEISTER-TURAEV TORSION

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Abstract

We study lens space surgeries along two different families of 2-component links, denoted by $A_{m,n}$ and $B_{p,q}$, related with the rational homology 4-ball used in J. Park's (generalized) rational blow down. We determine which coefficient r of the knotted component of the link yields a lens space by Dehn surgery. The link $A_{m,n}$ yields a lens space only by the known surgery with $r = mn$ and unexpectedly with $r = 7$ for $(m, n) = (2, 3)$. On the other hand, $B_{p,q}$ yields a lens space by infinitely many r . Our main tool for the proof are the Reidemeister-Turaev torsions, that is, Reidemeister torsions with combinatorial Euler structures. Our results can be extended to the links whose Alexander polynomials are same as those of $A_{m,n}$ and $B_{p,q}$.

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1. Introduction

For a coprime pair of nonzero positive integers (m, n) , let $A_{m,n}$ be a 2-component link in S^3 in Figure 1, where K_1 is the (m, n) -torus knot $T_{m,n}$ and K_2 is an unknot. The linking number of K_1 and K_2 is $m + n$. Next, for a coprime pair of nonzero integers (p, q) , let $B_{p,q}$ be a 2-component link in S^3 in Figure 2, where K_1 is the closure of the (p, q) -torus braid (the standard p -braid of the (p, q) -torus knot $T_{p,q}$) and K_2 is the braid axis. The linking number of K_1 and K_2 is p . For a μ -component link $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$, by $(L; r_1, r_2, \dots, r_\mu)$, we denote the result of (r_1, r_2, \dots, r_μ) -surgery along L , where $r_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ ($i = 1, 2, \dots, \mu$). A Dehn surgery along a link is called a *lens space*

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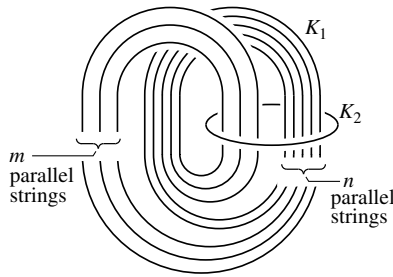


FIGURE 1. $A_{m,n}$ (ex. $(m, n) = (3, 5)$).

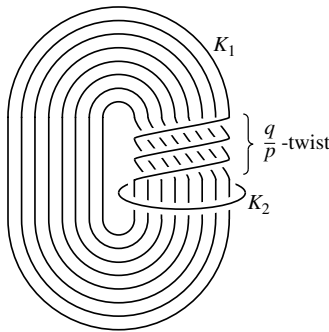


FIGURE 2. $B_{p,q}$ (ex. $(p, q) = (8, 3)$).

surgery if the result is a lens space. We study lens space surgeries along the links $A_{m,n}$ and $B_{p,q}$, fixing the surgery coefficient of K_2 as 0 except in Section 6.4 and Section 7. Our convention on lens spaces is ‘ $L(a, b)$ is the result of $-a/b$ -surgery along the trivial knot’.

The result of $(r, 0)$ -surgery along a 2-component link $L = K_1 \cup K_2$ in S^3 (as ∂B^4) such that K_2 is an unknot and $r \in \mathbb{Z}$ bounds a 4-manifold by attaching a 1-handle along K_2 , and a 2-handle along K_1 with a framing r , to a 0-handle B^4 along S^3 . We denote the 4-manifold by $W^4(L; r, \dot{0})$ following Akbulut [Ak] (see also [GS, Kir]). Then $\pi_1(W^4(L; r, \dot{0})) \cong \mathbb{Z}/|l|\mathbb{Z}$ and $H_1(\partial W^4(L; r, \dot{0}); \mathbb{Z}) \cong \mathbb{Z}/l^2\mathbb{Z}$, where l is the linking number of K_1 and K_2 , and $W^4(L; r, \dot{0})$ is a rational homology 4-ball if and only if $l \neq 0$. We also note that K_1 can be regarded as a knot in $S^1 \times S^2$.

We explain a background of our targets $A_{m,n}$ and $B_{p,q}$. Park [Pa] discussed *generalized rational blow down*, which is an operation on a 4-manifold cutting a certain submanifold $C_{p,q}$ and pasting a 4-manifold $W_{p,q}$ along $\partial C_{p,q} \cong \partial W_{p,q}$. The 4-manifold $W_{p,q}$ is a rational homology 4-ball that is characterized by $\pi_1(W_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$ and $\partial W_{p,q} \cong L(p^2, pq - 1)$ (see [CH, FS]). To the best of the authors’ knowledge, it is unknown whether the diffeomorphism type of $W_{p,q}$ is determined by these properties. A lens space surgery along $B_{p,q}$ whose result is $L(p^2, pq - 1)$ is often

used to describe $W_{p,q}$ (see [Li]). On the other hand, the second author [Yam3] found a lens space surgery along $A_{m,n}$ whose result is $L(p^2, pq - 1)$ and defined an algorithm to determine (m, n) from (p, q) using the Euclidean algorithm. We also remark that the link $A_{m,n}$ appears as a bi-product of Stipsicz–Szabó–Wahl’s construction [SSW, Remark 7.1], see also Endo–Mark–Horn–Morris [EMM]. For a recent work on $W_{p,q}$ constructed from $B_{p,q}$, see Lekili–Maydanskiy [LM]. We compare the links $A_{m,n}$ and $B_{p,q}$ by studying lens space surgeries along them. Our problem is the following question.

PROBLEM 1.1. When is $(A_{m,n}; r, 0)$ ($(B_{p,q}; r, 0)$, respectively) a lens space ?

There are some trivial and overlapping cases: $B_{1,q}$ is the Hopf link and $A_{1,n} = B_{n+1,1}$. The links have some symmetries: $A_{m,n} = A_{n,m}$ and $B_{p,-q}$ is the mirror image of $B_{p,q}$. Throughout the paper, we make the following assumption.

ASSUMPTION. For $A_{m,n}$, $\gcd(m, n) = 1$ and $2 \leq m < n$. For $B_{p,q}$, $\gcd(p, q) = 1$, $p \geq 2$ and $q \geq 1$.

The following theorem asserts that the link $A_{m,n}$ has at least one lens space surgery.

THEOREM 1.2 [Yam3, Theorem 1.1]. *For a pair (m, n) satisfying the assumption, there exists a pair (p, q) satisfying the assumption such that $(A_{m,n}; mn, 0) \cong L(p^2, pq - 1)$.*

NOTATION 1.3. For integers x and N , we denote the multiple inverse of x modulo N by $\bar{x} \pmod N$, that is $x\bar{x} \equiv 1 \pmod N$. Note that, for a divisor $d (\geq 2)$ of N , both $x \pmod d$ and $\bar{x} \pmod d$ are uniquely determined by $x \pmod N$. We also use \bar{x} as a representing integer of $\bar{x} \pmod N$.

Our first main theorem is the answer to Problem 1.1 for the link $A_{m,n}$.

THEOREM 1.4. *We assume $r \in \mathbb{Q}$.*

- (1) *The result of $(r, 0)$ -surgery along $A_{m,n}$ is a lens space if and only if:*
 - (i) $r = mn$; or
 - (ii) $r = 7$ for $(m, n) = (2, 3)$.
- (2) *The resulting lens spaces are as follows:*
 - (i) $(A_{m,n}; mn, 0) \cong L((m + n)^2, m\bar{n})$, where $n\bar{n} \equiv 1 \pmod{(m + n)^2}$; and
 - (ii) $(A_{2,3}; 7, 0) \cong L(25, 7)$.

The ‘if part’ of Theorem 1.4(1)(i) follows from Theorem 1.2. Thus our aim is to show the ‘only if part’ of (1), and (2).

Our second main theorem is the answer to Problem 1.1 for the link $B_{p,q}$, which is in contrast to $A_{m,n}$.

THEOREM 1.5. *We set $\alpha/\beta \in \mathbb{Q}$, where α and β are coprime integers.*

- (1) *The result of $(\alpha/\beta, 0)$ -surgery along $B_{p,q}$ is a lens space if and only if $|\alpha - pq\beta| = 1$.*
- (2) *For α/β with $|\alpha - pq\beta| = 1$, the resulting lens space is $(B_{p,q}; \alpha/\beta, 0) \cong L(p^2\beta, \alpha)$.*

REMARK 1.6. We remark on surgeries along the mirror images of the links. Naturally, we have $B_{-p,-q} = B_{p,q}$, $B_{-p,q} = B_{p,-q}$ as unoriented links, and $B_{p,-q}$ is the mirror image of $B_{p,q}$. Theorem 1.5 can be extended to the cases $p < 0$ or/and $q < 0$. Similarly, Theorem 1.2 can be extended to the mirror image $A_{m,n}!$ of $A_{m,n}$. We note that $A_{m,n}!$ is not included in the family $\{A_{m,n}\}$.

We list some corollaries without the proofs.

COROLLARY 1.7. *The lens space $(A_{2,3}; 7, 0) \cong L(25, 7)$ cannot be obtained by any $(r, 0)$ -surgery along $B_{p,q}$.*

COROLLARY 1.8 (Integral lens space surgery along $B_{p,q}$). *Suppose $\beta = 1$ in Theorem 1.5. Then $(B_{p,q}; \alpha, 0)$ is a lens space if and only if $\alpha = pq - 1$ or $pq + 1$. The resulting lens space is $L(p^2, pq - 1)$ or $L(p^2, pq + 1)$, respectively.*

By Theorem 1.2 and Corollary 1.8, both $W^4(A_{m,n}; mn, \dot{0})$ and $W^4(B_{p,q}; pq - 1, \dot{0})$ represent $W_{p,q}$, under the correspondence between (m, n) and (p, q) in Theorem 1.2. Our second problem is the following question.

PROBLEM 1.9. Are $W^4(A_{m,n}; mn, \dot{0})$ and $W^4(B_{p,q}; pq - 1, \dot{0})$ diffeomorphic, homeomorphic or homotopic relative to the boundaries?

The lens space $(A_{2,3}; 7, 0)$ in Corollary 1.7 satisfies $L(25, 7) \cong -L(25, 7)$. On the other hand, on the integral lens space surgeries along $B_{p,q}$ in Corollary 1.8, it is easy to see that $L(p^2, pq - 1) \cong -L(p^2, pq + 1)$ and that $L(p^2, pq - 1) \not\cong L(p^2, pq + 1)$. Thus we have the following corollary.

COROLLARY 1.10. *Only in this corollary, we regard the link $A_{m,n}$ or $B_{p,q}$ as a knot K_1 in $S^1 \times S^2$, the result of 0-surgery along K_2 . We assume that $2 \leq m < n$, $p \geq 2$ and $q \geq 1$. Then, any knot $A_{m,n}$ is not isotopic to any knot $B_{p,q}$.*

Corollary 1.10 asserts that the attaching parts K_1 of the 2-handles of $W^4(A_{m,n}; mn, \dot{0})$ and $W^4(B_{p,q}; pq - 1, \dot{0})$ are not isotopic in $S^1 \times S^2$. Thus, the handle decompositions of the rational homology 4-balls do not move to each other by only handle slides of the 2-handles over the 1-handles. Problem 1.9 may be still open. During the final revision of the present paper, Baker informed the authors about recent results in [BBL] on (longitudinal) lens space surgeries along knots in $S^1 \times S^2$.

Our results can be regarded as lens space surgeries along 2-component links. For usual lens space surgery along knots, see [Ba, Ber, CGLS, Go]. We point out that Theorem 1.4 (on $A_{m,n}$) can also be obtained by the results ‘ $(A_{m,n}; \emptyset, 0)$ is a hyperbolic manifold’ in [DMM1, DMM2], and the cyclic surgery theorem in [CGLS]. If we use them, then the proof of the theorem can be shortened (see Section 8.1). The reason why we do not use them is to clarify effectivity of Alexander polynomials and Reidemeister–Turaev torsions from a technical point of view. As consequences, they preserve information of lens space surgeries completely in the present case, and our results are generalized to wider situations, see Theorems 8.1 and 8.2. The links $A_{m,n}$ are related to subfamilies of knots, called Type VII and Type VIII in Berge’s list [Ber],

see [Kad4, Yam1, Yam2, Yam4]. On the other hand, $(B_{p,q}; r, 0)$ with any r is a Seifert manifold (or a graph manifold), thus Theorem 1.5 looks like Moser's result [Mos] on lens space surgeries along torus knots. For an application to another 2-component link 'Berge's link', see [KY3] (an announcement of the present paper).

We also study a generalization of Theorem 1.4. To determine all Dehn surgeries along $A_{m,n}$ is a hard problem. To the best of the authors' knowledge, the complete answer is not given. From our present results and some known results, we would like to raise the following conjecture.

CONJECTURE 1.11. Let $M = (A_{m,n}; \alpha_1/\beta_1, \alpha_2/\beta_2)$ be the result of $(\alpha_1/\beta_1, \alpha_2/\beta_2)$ -surgery along $A_{m,n}$, where α_i and β_i ($i = 1, 2$) are coprime integers with $\beta_i \geq 1$. Then M is a lens space if and only if: (1) $\alpha_1/\beta_1 = mn$ and $\beta_2 = 1$; or (2) $\alpha_1/\beta_1 = 7$ for $(m, n) = (2, 3)$.

Ichihara [IS] informed the authors that, if we fix $\alpha_1/\beta_1 = mn$, then M is a lens space for any integer ($\beta_2 = 1$). He says that this can be shown by a method in [Yam3] (that is, a geometric method). In Section 7, we compute the Reidemeister torsions of M in the cases: (1) $\alpha_1/\beta_1 = mn$; and (2) $\alpha_1/\beta_1 = 7$ for $(m, n) = (2, 3)$, respectively. A partial affirmative answer for the 'only if part' of the case (1) is given.

In Section 2, we explain the Reidemeister torsion and its basic properties such as surgery formulae, d -norm and combinatorial Euler structure (that is, Reidemeister-Turaev torsion). In Section 3, we compute the Alexander polynomial of $A_{m,n}$. In Section 4, we compute the Reidemeister torsions of $(A_{m,n}; r, 0)$ by using the results in Sections 2 and 3. In Section 5, we prove the 'only if part' of Theorem 1.4(1) by using the Reidemeister-Turaev torsions, and Theorem 1.4(2) by using the values of the Reidemeister torsions, Theorem 1.2, and Kirby moves. In Section 6, we prove Theorem 1.5 by using Seifert structures of the link complement and the Reidemeister torsions. In Section 7, we study some lens space surgeries along $A_{m,n}$ other than $(r, 0)$ -surgery, related to Conjecture 1.11. In Section 8, we give an alternative proof of Theorem 1.4(1) under some assumption, and we generalize Theorems 1.4 and 1.5.

2. Reidemeister torsion

Our method to prove the main theorems is to deduce necessary conditions from the Reidemeister torsions of both the surgered manifolds and lens spaces. In this section, we state surgery formulae of the Reidemeister torsions (Section 2.1), and define derived invariants: one is the d -norm (Section 2.2), and the other is the *Reidemeister-Turaev torsion* which is a lift of the Reidemeister torsions by fixing a combinatorial Euler structure (Section 2.3).

2.1. Surgery formulae. For a precise definition of the Reidemeister torsion, the reader is referred to Turaev [Tur1, Tur2]. Throughout this paper, we use the following notations.

NOTATION 2.1 (For manifolds and homologies). Let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented μ -component link in a homology 3-sphere.

E_L	the complement of L ;
m_i, l_i	a meridian and a longitude of the i th component K_i ;
$[m_i], [l_i]$	their homology classes;
$\Delta_L(t_1, \dots, t_\mu)$	the Alexander polynomial of L , where t_i is represented by m_i ;
$(L; r_1, \dots, r_\mu)$	the result of (r_1, \dots, r_μ) -surgery along L ;
	where $r_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ is the surgery coefficient of K_i ;
V_i	the solid torus attached along K_i in the Dehn surgery;
$m'_i, [m'_i]$	a meridian of V_i , and its homology class;
$l'_i, [l'_i]$	an oriented core curve of V_i , and its homology class.

Let X be a finite CW complex and $\pi : \tilde{X} \rightarrow X$ its maximal abelian covering. Then \tilde{X} has a CW structure induced by that of X and π , and the cell chain complex \mathbf{C}_* of \tilde{X} has a $\mathbb{Z}[H]$ -module structure, where $H = H_1(X; \mathbb{Z})$ is the first homology of X . For an integral domain R and a ring homomorphism $\psi : \mathbb{Z}[H] \rightarrow R$, ‘the chain complex of \tilde{X} related with ψ ’, denoted by \mathbf{C}_*^ψ , is $\mathbf{C}_* \otimes_{\mathbb{Z}[H]} Q(R)$, where $Q(R)$ is the quotient field of R . The Reidemeister torsion of X related with ψ , denoted by $\tau^\psi(X)$, is calculated from \mathbf{C}_*^ψ , and is an element of $Q(R)$ determined up to multiplication of $\pm\psi(h)$ ($h \in H$). If $R = \mathbb{Z}[H]$ and ψ is the identity map, then we denote $\tau^\psi(X)$ by $\tau(X)$. We note that $\tau^\psi(X)$ is not zero if and only if \mathbf{C}_*^ψ is acyclic.

NOTATION 2.2 (For algebra). For a pair of elements A, B in $Q(R)$, if there exists an element $h \in H$ such that $A = \pm\psi(h)B$, then we denote the equality by $A \doteq B$. We will often take a field F and a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \rightarrow F$. We mainly use the d th cyclotomic fields $\mathbb{Q}(\zeta_d)$ as F , where ζ_d is a primitive d th root of unity.

For the first lemma, we need a little general setting: let E be a compact 3-manifold whose boundary ∂E consists of tori. We study the 3-manifold $M = E \cup V_1 \cup \dots \cup V_n$ obtained by attaching solid tori V_i to E by attaching maps $f_i : \partial V_i \rightarrow \partial E$ ($\text{Im}(f_i) \cap \text{Im}(f_j) = \emptyset$ for $i \neq j$). We let $\iota : E \hookrightarrow M$ denote the natural inclusion.

LEMMA 2.3 (Surgery formula I). *If $\psi([l'_i]) \neq 1$ for every $i = 1, \dots, n$, then we have*

$$\tau^\psi(M) \doteq \tau^{\psi'}(E) \prod_{i=1}^n (\psi([l'_i]) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_*$ (ι_* is a ring homomorphism induced by ι).

For the case of the link complement of a homology 3-sphere, the Reidemeister torsion is closely related with the Alexander polynomial.

LEMMA 2.4 (Milnor [Mil]). *For a μ -component link $L = K_1 \cup \dots \cup K_\mu$ in a homology 3-sphere,*

$$\tau(E_L) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & \text{if } \mu = 1, \\ \Delta_L(t_1, \dots, t_\mu) & \text{if } \mu \geq 2. \end{cases}$$

By Lemmas 2.3 and 2.4, we have the following result.

LEMMA 2.5 (Surgery formula II). (*Sakai [Sa] and Turaev [Tur1]*)

- (1) Let K be a knot in a homology 3-sphere. We set $M = (K; p/q)$ ($|p| \geq 2$), where p and q are coprime integers. Let T be a generator of $H_1(M)$ represented by a meridian of K , and $\psi_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ a ring homomorphism defined by $\psi_d(T) = \zeta_d$, where $d (\geq 2)$ is a divisor of p . Then we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod p$.

- (2) Let L be a μ -component link in a homology 3-sphere. We set $M = (L; p_1/q_1, \dots, p_\mu/q_\mu)$ ($\mu \geq 2$), where p_i and q_i are coprime integers for every $i = 1, \dots, \mu$. Let F be a field, $\psi : \mathbb{Z}[H_1(M)] \rightarrow F$ a ring homomorphism with $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ for every $i = 1, \dots, \mu$, where r_i and s_i are integers satisfying $p_i s_i - q_i r_i = -1$. Then we have

$$\tau^\psi(M) \doteq \Delta_L(\psi([m_1]), \dots, \psi([m_\mu])) \prod_{i=1}^\mu (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

EXAMPLE 2.6. The lens space $L(p, q)$ is obtained as $-p/q$ -surgery along the unknot. By Lemma 2.5(1), for a divisor $d \geq 2$ of p ,

$$\tau^{\psi_d}(L(p, q)) \doteq (\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod p$.

We recall the Torres formula for the Alexander polynomials.

LEMMA 2.7 (Torres formula). [*Tor*] Let $L = K_1 \cup \dots \cup K_\mu \cup K_{\mu+1}$ ($\mu \geq 1$) be an oriented $(\mu + 1)$ -component link, $L' = K_1 \cup \dots \cup K_\mu$ a μ -component sublink, and $\ell_i = \text{lk}(K_i, K_{\mu+1})$ ($i = 1, \dots, \mu$). Then we have

$$\Delta_L(t_1, \dots, t_\mu, 1) \doteq \begin{cases} \frac{t_1^{\ell_1} - 1}{t_1 - 1} \Delta_{L'}(t_1) & \text{if } \mu = 1, \\ (t_1^{\ell_1} \cdots t_\mu^{\ell_\mu} - 1) \Delta_{L'}(t_1, \dots, t_\mu) & \text{if } \mu \geq 2. \end{cases}$$

2.2. d -norm. Regarding algebraic fields, the reader is referred to Washington [*Was*], for example.

For an element x in the d th cyclotomic field $\mathbb{Q}(\zeta_d)$, the d -norm of x is defined as

$$N_d(x) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})} \sigma(x),$$

where $\text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ is the Galois group related with a Galois extension $\mathbb{Q}(\zeta_d)$ over \mathbb{Q} . The following proposition is well known.

PROPOSITION 2.8.

- (1) If $x \in \mathbb{Q}(\zeta_d)$, then $N_d(x) \in \mathbb{Q}$. The map $N_d : \mathbb{Q}(\zeta_d) \setminus \{0\} \rightarrow \mathbb{Q} \setminus \{0\}$ is a group homomorphism.
- (2) If $x \in \mathbb{Z}[\zeta_d]$, then $N_d(x) \in \mathbb{Z}$.

By easy calculations, we have the following lemma.

LEMMA 2.9.

- (1) $N_d(\pm\zeta_d) = \begin{cases} \pm 1 & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$
- (2) $N_d(1 - \zeta_d) = \begin{cases} \ell & \text{if } d \text{ is a power of a prime } \ell \geq 2, \\ 1 & \text{otherwise.} \end{cases}$

Regarding applications of d -norms, for example, see [Kad1, Kad2, Kad3, KMS, KY1, KY2].

Franz [Fz] showed the following lemma, and classified lens spaces by using it. We state a modified version (see [KY1]).

LEMMA 2.10 (Franz [Fz]). Let $p \geq 2$ be an integer, and $(\mathbb{Z}/p\mathbb{Z})^\times$ the multiplicative group of a ring $\mathbb{Z}/p\mathbb{Z}$. For $a_i, b_i \in (\mathbb{Z}/p\mathbb{Z})^\times$ ($i = 1, \dots, n$), suppose

$$\prod_{i=1}^n (\zeta_p^{a_i} - 1) \doteq \prod_{i=1}^n (\zeta_p^{b_i} - 1),$$

where ζ_p is a primitive p th root of unity. Then there exists a permutation σ of $\{1, \dots, n\}$ such that $a_i = \pm b_{\sigma(i)} \pmod p$ for all $i = 1, \dots, n$. In other words, $\{\pm a_i \pmod p\} = \{\pm b_i \pmod p\}$ as multiple sets.

We will use this lemma in Lemma 4.9 and in Section 8.

2.3. Combinatorial Euler structure (Reidemeister–Turaev torsion). Let M be a homology lens space with $H = H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then the Reidemeister torsion $\tau^{\psi_d}(M)$ of M related with ψ_d is determined up to multiplication by $\pm \zeta_d^m$ ($m \in \mathbb{Z}$), where $d \geq 2$ is a divisor of p and ψ_d is the same ring homomorphism as in Lemma 2.5(1). Once we fix a basis of a cell chain complex for the maximal abelian covering of M as a $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ -module, the value $\tau^{\psi_d}(M)$ is uniquely determined as an element of $\mathbb{Q}(\zeta_d)$ for every d . The choice of the basis up to ‘base change equivalence’ is called a *combinatorial Euler structure* of M (see Turaev [Tur2]). The Reidemeister torsion of a manifold with a fixed combinatorial Euler structure is called the *Reidemeister–Turaev torsion*.

We consider the sequence of the values $\tau^{\psi_d}(M)$ in $\mathbb{Q}(\zeta_d)$ of the Reidemeister–Turaev torsion for every divisor $d \geq 2$ of p , and regard them as a value sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ defined as below.

DEFINITION 2.11. A sequence of values $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ is a *value sequence (of degree p)* if $x_d \in \mathbb{Q}(\zeta_d)$ for every d . Two value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ are

equal $(\mathbf{x} = \mathbf{y})$ if $x_d = y_d$ for every d . We are mainly concerned with the value sequence of type $\mathbf{x} = \{F(\zeta_d)\}_{d|p, d \geq 2}$ for a rational function $F(t) \in \mathbb{Q}(t)$. In such a case, we say that \mathbf{x} is induced by $F(t)$ and that $F(t)$ is a lift of \mathbf{x} . A control of $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ by a trivial unit $u = \eta t^m \in \mathbb{Q}[t, t^{-1}]/(t^p - 1)$ is defined by

$$u\mathbf{x} = \{\eta \zeta_d^m x_d\}_{d|p, d \geq 2},$$

where $\eta = 1$ or -1 (constant) and $m \in \mathbb{Z}$. Two value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ are control equivalent if there is a trivial unit $u \in \mathbb{Q}[t, t^{-1}]/(t^p - 1)$ such that $\mathbf{y} = u\mathbf{x}$. A value sequence $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ is a real value sequence if x_d is a real number for every d .

Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then a sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ of the Reidemeister torsions of M with a combinatorial Euler structure is a value sequence of degree p . We call the value sequence a torsion sequence of M .

LEMMA 2.12.

- (1) Let M and M' be homeomorphic homology lens spaces with $H_1(M) \cong H_1(M') \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then torsion sequences $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ and $\{\tau^{\psi'_d}(M')\}_{d|p, d \geq 2}$ related with the corresponding representations ψ_d and ψ'_d (that is, $\psi_d = \psi'_d \circ h_*$, where h_* is the induced homomorphism of the homeomorphism) are control equivalent.
- (2) Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then we can control a torsion sequence of M into a real value sequence.

PROOF. (1) This is easy to see.

(2) Here we let ζ denote any d th primitive root (ζ_d) of unity. Since M is obtained by p/q -surgery along a knot K in a homology 3-sphere for some q (see [BL]), by Lemma 2.5(1),

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod{p}$. By the duality of the Alexander polynomial (see [Mil, Tur1, Tur2]), we may assume

$$\Delta_K(t) = \Delta_K(t^{-1}).$$

This is also a control of the combinatorial Euler structure of the exterior of K , which induces a control of a torsion sequence of M . We take an odd integer lift of \bar{q} . Then

$$\zeta^{(1+\bar{q})/2} \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

is a real number for every d . □

LEMMA 2.13. If two real value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ of degree p are control equivalent, satisfying $\mathbf{y} = u\mathbf{x}$ for a trivial unit $u = \eta t^m \in \mathbb{Q}[t, t^{-1}]/(t^p - 1)$, where $\eta = \pm 1$ and $m \in \mathbb{Z}$, then the possibilities for u are the following:

- (1) if p is odd, then $u = 1$ or -1 ;
- (2) if p is even, then $u = 1, -1, t^{p/2}$ or $-t^{p/2}$.

PROOF. Since the ratio $\zeta_p^m = \pm y_p/x_p$ is a real number, we have: (1) $m \equiv 0 \pmod p$ if p is odd; and (2) $m \equiv 0$ or $p/2 \pmod p$ if p is even. □

DEFINITION 2.14 (Symmetric Laurent polynomial). A Laurent polynomial $F(t) \in \mathbb{Q}[t, t^{-1}]$ is *symmetric* if it is of the form

$$F(t) = a_0 + \sum_{i=1}^{\infty} a_i(t^i + t^{-i}),$$

where a_i is a rational number for all $i = 1, 2, \dots$ and $a_i = 0$ for every sufficiently large i . Note that, if $F(t)$ is a symmetric Laurent polynomial, the induced value sequence $\{F(\zeta_d)\}_{d|p, d \geq 2}$ is a real value sequence. We are concerned with symmetric Laurent polynomials that are lifts (in $\mathbb{Q}[t, t^{-1}]$) of a polynomial in the quotient ring $\mathbb{Q}[t, t^{-1}]/(t^p - 1)$. We say that $F(t)$ (as above) is *reduced* if $a_i = 0$ for all $i > [p/2]$. We often *reduce* the symmetric polynomials by using $t^i + t^{-i} = t^{p+i} + t^{-(p+i)}$ modulo $(t^p - 1)$. We let $\text{red}(F(t))$ denote the reduction of $F(t)$ (that is, $\text{red}(F(t))$ is reduced and $\text{red}(F(t)) = F(t)$ in $\mathbb{Q}[t, t^{-1}]/(t^p - 1)$).

For a Laurent polynomial $F(t) \in \mathbb{Q}[t, t^{-1}]$, the *span* of $F(t)$ is the difference of the maximal degree and the minimal degree of $F(t)$, and we denote it by $\text{span}(F(t))$. Note that the span of a symmetric Laurent polynomial is always even, and that the span of a reduced symmetric Laurent polynomial is less than or equal to $2[p/2]$.

LEMMA 2.15. *Let $N \geq 2$ be an integer. Let $F(t)$ and $G(t)$ be symmetric Laurent polynomials and $\mathbf{x} = \{F(\zeta_d)\}_{d|N, d \geq 2}$, $\mathbf{y} = \{G(\zeta_d)\}_{d|N, d \geq 2}$ the induced real value sequences, respectively. If \mathbf{x} and \mathbf{y} are control equivalent, that is, $u\mathbf{x} = \mathbf{y}$ for a trivial unit u (here, $u = 1$ or -1 if N is odd, $u = 1, -1, t^{N/2}$ or $-t^{N/2}$ if N is even, by Lemma 2.13), then we have a congruence*

$$uF(t) \equiv G(t) \pmod{t^{N-1} + t^{N-2} + \dots + t + 1}.$$

Furthermore, assuming $\text{span}(G(t)) \leq 2([N/2] - 1)$:

- (1) In the case that $u = 1$ or -1 and $\text{span}(F(t)) \leq N - 2$, we have an identity $uF(t) = G(t)$ in $\mathbb{Q}[t, t^{-1}]$.
- (2) Otherwise (in the case that N is even and $u = \eta t^{N/2}$ with $\eta = 1$ or -1), if $\text{span}(\text{red}(t^{N/2}F(t))) \leq N - 2$, then we have $\text{red}(t^{N/2}F(t)) = \eta G(t)$ in $\mathbb{Q}[t, t^{-1}]$.

PROOF. By the Chinese remainder theorem, we have a ring isomorphism:

$$\mathbb{Q}[t, t^{-1}]/(t^{N-1} + t^{N-2} + \dots + t + 1) \cong \bigoplus_{d|N, d \geq 2} \mathbb{Q}(\zeta_d),$$

where $f(t)$ in the left-hand side maps to the value sequences $\{f(\zeta_d)\}_{d|N, d \geq 2}$ in the right-hand side. The isomorphism implies the required congruence.

Since $2([N/2] - 1) < N - 1 = \text{span}(t^{N-1} + t^{N-2} + \dots + t + 1)$, we have the identities. □

LEMMA 2.16. *Let $N \geq 2$ be an integer. Let $F(t), G(t)$ be symmetric Laurent polynomials and $\mathbf{x} = \{F(\zeta_d)\}_{d|N, d \geq 2}, \mathbf{y} = \{G(\zeta_d)\}_{d|N, d \geq 2}$ the induced real value sequences, respectively. If \mathbf{x} and \mathbf{y} are control equivalent, that is, $u\mathbf{x} = \mathbf{y}$ for a trivial unit u (here, $u = 1$ or -1 if N is odd, $u = 1, -1, t^{N/2}$ or $-t^{N/2}$ if N is even, by Lemma 2.13), and $F(1) = G(1) = 0$, then we have a congruence*

$$uF(t) \equiv G(t) \pmod{t^N - 1}.$$

Furthermore, assuming $\text{span}(G(t)) \leq 2\lfloor N/2 \rfloor$:

- (1) *In the case that $u = 1$ or -1 and $\text{span}(F(t)) \leq N - 1$, we have an identity $uF(t) = G(t)$ in $\mathbb{Q}[t, t^{-1}]$.*
- (2) *Otherwise (in the case that N is even and $u = \eta t^{N/2}$ with $\eta = 1$ or -1), we have $\text{red}(t^{N/2}F(t)) = \eta G(t)$ in $\mathbb{Q}[t, t^{-1}]$.*

PROOF. We use the same argument as in the proof of Lemma 2.15, but here we use the Chinese remainder theorem for the following ring isomorphism:

$$\mathbb{Q}[t, t^{-1}]/(t^N - 1) \cong \bigoplus_{d|N, d \geq 1} \mathbb{Q}(\zeta_d).$$

This concludes the proof. □

Note that $F(t)$ and $t^{N/2}F(t)$ induce control equivalent real value sequences by $u = t^{N/2}$, but $\text{red}(t^{N/2}F(t)) \neq F(t)$ in general. Thus we have to take into account the case (2) in Lemmas 2.15 and 2.16. Here, we study the relation between the coefficients of $F(t)$ and those of $\text{red}(t^{N/2}F(t))$.

LEMMA 2.17. *Let N be an even integer.*

$$\text{If } F(t) = a_0 + \sum_{i=1}^{N/2} a_i(t^i + t^{-i}) \text{ then } \text{red}(t^{N/2}F(t)) = b_0 + \sum_{i=1}^{N/2} b_i(t^i + t^{-i})$$

with

$$b_0 = 2a_{N/2}, \quad b_{N/2} = a_0/2 \quad \text{and} \quad b_j = a_{N/2-j} \quad (j = 1, 2, \dots, N/2 - 1).$$

PROOF. This is because

$$t^{N/2}(t^j + t^{-j}) = t^{N/2+j} + t^{N/2-j} \equiv t^{(N/2-j)} + t^{-(N/2-j)} \pmod{t^N - 1}.$$

This concludes the proof. □

This will be used in the proof of Lemma 4.9, see also Remark 4.7.

3. Alexander polynomial of $A_{m,n}$

We compute the Alexander polynomial of the link $A_{m,n}$.

DEFINITION 3.1. For a coprime positive pair (m, n) , we define a set $\mathfrak{S}(m, n)$ by

$$\begin{aligned} \mathfrak{S}(m, n) &:= (m\mathbb{Z} \cup n\mathbb{Z}) \cap \{k \in \mathbb{Z} \mid 0 \leq k \leq mn\} \\ &= \{0, m, 2m, \dots, nm\} \cup \{0, n, 2n, \dots, mn\}. \end{aligned}$$

Note that the cardinality of $\mathfrak{S}(m, n)$ is $m + n$. We sort the all elements in $\mathfrak{S}(m, n)$ as

$$0 = k_0 < k_1 < k_2 < \dots < k_{m+n-1} = mn \quad (k_i \in \mathfrak{S}(m, n)).$$

Here, k_1 is the smaller one in m and n .

The goal of this section is the following theorem.

THEOREM 3.2. *The Alexander polynomial of $A_{m,n}$ is*

$$\Delta_{A_{m,n}}(t, x) \doteq \sum_{i=0}^{m+n-1} t^{k_i} x^i,$$

where t (and x , respectively) is represented by a meridian of K_1 (that of K_2).

EXAMPLE 3.3. In the case $(m, n) = (3, 5)$, we have $\mathfrak{S}(3, 5) = \{0, 3, 5, 6, 9, 10, 12, 15\}$ and

$$\Delta_{A_{3,5}}(t, x) \doteq t^{15}x^7 + t^{12}x^6 + t^{10}x^5 + t^9x^4 + t^6x^3 + t^5x^2 + t^3x + 1.$$

3.1. Alexander matrix of $A_{m,n}$. We start the proof of Theorem 3.2 with the following lemma.

LEMMA 3.4. *The Alexander matrix of $A_{m,n}$ is*

$$I_{m+n-1} - xM(m, n)$$

and the Alexander polynomial of $A_{m,n}$ is obtained by

$$\Delta_{A_{m,n}}(t, x) \doteq \det(I_{m+n-1} - xM(m, n)),$$

where $M(m, n)$ is the $(m + n - 1) \times (m + n - 1)$ -matrix of the form

$$M(m, n) := \begin{bmatrix} O_{n-1,m-1} & \overrightarrow{T_{n-1}} & I_{n-1} \\ O_{m-1} & -t^n & O_{n-1} \\ t^n I_{m-1} & -t^n \overrightarrow{i_{m-1}} & O_{m-1,n-1} \end{bmatrix},$$

$\overrightarrow{T_{n-1}}$, $\overrightarrow{i_{m-1}}$ are the following column vectors of size $(n - 1) \times 1$ and $(m - 1) \times 1$ respectively

$$\overrightarrow{T_{n-1}} := \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^{n-1} \end{bmatrix}, \quad \overrightarrow{i_{m-1}} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$O_{s,s'}$ (and o_s , respectively) is the zero matrix of size $s \times s'$ (and of size $1 \times s$) and I_s is the identity matrix of size $s \times s$.

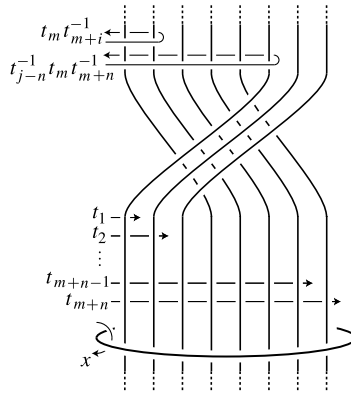


FIGURE 3. Generators t_i, t_j (ex. $(m, n) = (3, 5)$).

EXAMPLE 3.5.

$$M(3, 5) = \begin{bmatrix} 0 & 0 & -t & 1 & 0 & 0 & 0 \\ 0 & 0 & -t^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t^3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -t^4 & 0 & 0 & 0 & 1 \\ 0 & 0 & -t^5 & 0 & 0 & 0 & 0 \\ t^5 & 0 & -t^5 & 0 & 0 & 0 & 0 \\ 0 & t^5 & -t^5 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$I - xM(3, 5) = \begin{bmatrix} 1 & 0 & tx & -x & 0 & 0 & 0 \\ 0 & 1 & t^2x & 0 & -x & 0 & 0 \\ 0 & 0 & 1 + t^3x & 0 & 0 & -x & 0 \\ 0 & 0 & t^4x & 1 & 0 & 0 & -x \\ 0 & 0 & t^5x & 0 & 1 & 0 & 0 \\ -t^5x & 0 & t^5x & 0 & 0 & 1 & 0 \\ 0 & -t^5x & t^5x & 0 & 0 & 0 & 1 \end{bmatrix}.$$

PROOF OF LEMMA 3.4. We assumed that $2 \leq m < n$ (Section 1). We take the generators of the fundamental group $\pi_1(S^3 \setminus A_{m,n})$ as in Figure 3. Then we have a presentation of the group:

$$\left\langle \begin{matrix} t_1, t_2, \dots, t_{m+n} \\ x \end{matrix} \left| \begin{matrix} L_i := t_i x t_m t_{m+i}^{-1} x^{-1} & (i = 1, 2, \dots, n) \\ R_j := t_j x t_{j-n}^{-1} t_m t_{m+n}^{-1} x^{-1} & (j = n + 1, n + 2, \dots, m + n) \end{matrix} \right. \right\rangle$$

$$= \left\langle \begin{matrix} t_1, t_2, \dots, t_{m+n-1} \\ x \end{matrix} \left| \begin{matrix} L_i := t_i x t_m t_{m+i}^{-1} x^{-1} & (i = 1, 2, \dots, n - 1) \\ C := t_n x t_m x t_m^{-1} x^{-1} t_n^{-1} x^{-1} \\ R_j := t_j x t_{j-n}^{-1} x^{-1} t_n^{-1} & (j = n + 1, \dots, m + n - 1) \end{matrix} \right. \right\rangle,$$

where we canceled $t_{m+n} = x^{-1}t_nxt_m$ by the relation L_n . Then R_{m+n} is changed to the relation C , and the other R_j are also changed. The nonzero free differentials of the relations L_i and R_j by t_* are:

$$\begin{aligned} \frac{\partial L_i}{\partial t_i} &= 1, & \frac{\partial L_i}{\partial t_m} &= \delta_{im} + t_ix, & \frac{\partial L_i}{\partial t_{m+i}} &= -t_ixt_mt_{m+i}^{-1}, \\ \frac{\partial R_j}{\partial t_j} &= 1, & \frac{\partial R_j}{\partial t_{j-n}} &= -t_jxt_{j-n}^{-1}, & \frac{\partial R_j}{\partial t_n} &= -t_jxt_{j-n}^{-1}x^{-1}t_n^{-1}, \end{aligned} \tag{3.1}$$

where δ_{im} is Kronecker’s delta. The nonzero free differentials of C by t_* are

$$\frac{\partial C}{\partial t_m} = t_nx - t_nxt_mt_m^{-1}, \quad \frac{\partial C}{\partial t_n} = 1 - t_nxt_mt_m^{-1}x^{-1}t_n^{-1}. \tag{3.2}$$

We let γ denote the Hurewicz epimorphism $\gamma : \pi_1(S^3 \setminus A_{m,n}) \rightarrow H_1(S^3 \setminus A_{m,n}; \mathbb{Z}) \cong \langle t, x \mid - \rangle$, defined as $\gamma(t_i) = t^i, \gamma(x) = x$. We redefine $L_n := C$ and $L_i := R_i$ for $n + 1 \leq i \leq m + n - 1$. Then the well-known formula ([Kaw, Lemma 7.3.2 (page 93)]) on the Alexander polynomial of links says

$$\det \left[\gamma \left(\frac{\partial L_i}{\partial t_j} \right) \right] \doteq (1 - x) \Delta_{A_{m,n}}(t, x),$$

where j (columns) runs in $1 \leq j \leq m + n - 1$.

A submatrix from the first row to the $(n - 1)$ th row of $[\gamma(\partial L_i/\partial t_j)]$ coincides with $I - xM(m, n)$ by the first half of (3.1). In the n th row of $[\gamma(\partial L_i/\partial t_j)]$, nonzero entries are only

$$\gamma \left(\frac{\partial C}{\partial t_m} \right) = (1 - x)t^n x \quad \text{at } (n, m) \quad \text{and} \quad \gamma \left(\frac{\partial C}{\partial t_n} \right) = (1 - x) \quad \text{at } (n, n),$$

by (3.2). Thus the n th row of $[\gamma(\partial L_i/\partial t_j)]$ coincides with $(1 - x)$ times the n th row of $I - xM(m, n)$. We add $1/(1 - x)$ times the n th row of $[\gamma(\partial L_i/\partial t_j)]$ to each j th row of $[\gamma(\partial L_i/\partial t_j)]$ with $j \geq n + 1$. The resulting j th row coincides with the j th row of $I - xM(m, n)$ by the second half of (3.1). Therefore, we have

$$\det \left[\gamma \left(\frac{\partial L_i}{\partial t_j} \right) \right] \doteq (1 - x) \cdot \det(I - xM(m, n)).$$

This concludes the proof. □

REMARK 3.6. The matrix $M(m, n)$ can also be obtained by the Burau representation (see Birman [Bir, page 121]) of the braid group.

3.2. Properties of $\mathfrak{S}(m, n)$. We need some properties on elements k_i of $\mathfrak{S}(m, n)$, see Definition 3.1. They will be also used in Sections 4.2, 5.4.

DEFINITION 3.7. We define u_j ($j = 0, 1, 2, \dots, n$) and w_j ($j = 0, 1, 2, \dots, m$) by

$$k_{u_j} = jm \quad \text{and} \quad k_{w_j} = jn.$$

It is easy to see the following proposition.

PROPOSITION 3.8.

- (1) $k_i + k_{m+n-1-i} = mn$, ($i = 0, 1, \dots, m + n - 1$).
- (2) $u_0 = w_0 = 0$, $u_n = w_m = m + n - 1$, and

$$u_j = \left\lfloor \frac{jm}{n} \right\rfloor + j \quad (j = 1, 2, \dots, n - 1),$$

$$w_j = \left\lfloor \frac{jn}{m} \right\rfloor + j \quad (j = 1, 2, \dots, m - 1),$$

where $\lfloor \cdot \rfloor$ is the Gaussian symbol.

- (3) Both u_j, w_j are increasing sequences.
- (4) $u_j + u_{n-j} = w_j + w_{m-j} = m + n - 1$.
- (5)

$$\sum_{i=0}^{m+n-1} t^{k_i} x^i = 1 + \sum_{i=1}^{n-1} t^{im} x^{u_i} + \sum_{j=1}^{m-1} t^{jn} x^{w_j} + t^{mn} x^{m+n-1}.$$

NOTATION 3.9. For an integer N , we denote by $[N]_n$ the unique integer satisfying

$$[N]_n \equiv N \pmod{n} \quad \text{and} \quad 0 \leq [N]_n \leq n - 1.$$

We let σ (and its inverse σ' , respectively) denote the bijection

$$\sigma, \sigma' : \{0, 1, 2, \dots, n - 1\} \rightarrow \{0, 1, 2, \dots, n - 1\}$$

defined by $\sigma(i) := [i \bar{m}]_n$ (and $\sigma'(i) := [im]_n$), where \bar{m} is regarded as an integral lift of $\bar{m} \pmod{n}$. We also define

$$\rho(i) := \frac{\sigma(i)m - i}{n} \quad (i = 1, 2, \dots, n - 1).$$

By the definition of $\sigma(i)$, $\rho(i)$ is an integer.

LEMMA 3.10.

- (1) $\sigma(0) = 0$, $\sigma(m) = 1$ and $\sigma(n - m) = n - 1$.
- (2) For $i = 1, 2, \dots, n - 1$ and $i \neq n - m$, we have $u_{\sigma(i)+1} - u_{\sigma(i)} = 1$ or 2 . Furthermore, $u_{\sigma(i)+1} - u_{\sigma(i)} = 2$ if and only if $n - m + 1 \leq i \leq n - 1$.
- (3) $\sigma(m + i) = \sigma(i) + 1$, $u_{\sigma(m+i)} = u_{\sigma(i)} + 1$, ($i = 1, 2, \dots, n - m - 1$).
- (4) $\sigma(n - m + i) = \sigma(i) - 1$, $u_{\sigma(n-m+i)} = u_{\sigma(i)} - 2$, ($i = 1, 2, \dots, m - 1$).
- (5) $\rho : \{1, 2, \dots, m - 1\} \rightarrow \{1, 2, \dots, m - 1\}$ is a bijection.
- (6) $w_{\rho(i)} = u_{\sigma(i)} - 1$, ($i = 1, 2, \dots, m - 1$).

PROOF. (1) This is easy to see.

(2) Suppose that $i = 1, 2, \dots, n - 1$ and $i \neq n - m$. Since σ is a bijection, by (1) and Proposition 3.8(3), we have $1 \leq \sigma(i) \leq n - 2$ and $u_{\sigma(i)+1} - u_{\sigma(i)} = 1$ or 2 . We

set $j = \sigma(i)$. It holds that $u_{j+1} - u_j = 2$ if and only if there exists an integer j' such that $jm < j'n < (j + 1)m$, which implies $j'n - m < jm < j'n$ and

$$n - m < [jm]_n < n.$$

Since $[jm]_n = \sigma'(j) = \sigma' \circ \sigma(i) = i$, we have $n - m + 1 \leq i \leq n - 1$.

(3) Suppose that $i = 1, 2, \dots, n - m - 1$. We have $1 \leq \sigma(i) \leq n - 2$ again. Since $\sigma(m + i) \equiv (m + i)\bar{m} \equiv \sigma(i) + 1 \pmod n$, we have $\sigma(m + i) = \sigma(i) + 1$. By (2), we have $u_{\sigma(m+i)} = u_{\sigma(i)+1} = u_{\sigma(i)} + 1$.

(4) Suppose that $i = 1, 2, \dots, m - 1$. We have $2 \leq \sigma(i) \leq n - 1$. Since $\sigma(n - m + i) \equiv (n - m + i)\bar{m} \equiv \sigma(i) - 1 \pmod n$, we have $\sigma(n - m + i) = \sigma(i) - 1$. By (2) and $n - m + 1 \leq n - m + i \leq n - 1$, we have $u_{\sigma(n-m+i)} = u_{\sigma(i)} - 2$.

(5) Suppose that $i = 1, 2, \dots, m - 1$. Since $\sigma(i)m = [i\bar{m}]_n m \equiv i \pmod n$, and $0 \leq \sigma(i) \leq n - 1$,

$$0 \leq \rho(i) = \frac{\sigma(i)m - i}{n} \leq \frac{(n - 1)m}{n} < m,$$

and hence the image of the map ρ is included in $\{0, 1, 2, \dots, m - 1\}$. By the definition, we have $\rho(i)n \equiv -i \pmod m$, thus ρ is bijective. In fact, $\rho(i) = [(m - i)\bar{n}]_m$.

(6) Suppose that $i = 1, 2, \dots, m - 1$. By (4), we have $u_{\sigma(n-m+i)} = u_{\sigma(i)-1} = u_{\sigma(i)} - 2$. Then, as in the proof of (2), there exists an integer j such that $w_j = u_{\sigma(i)} - 1$ such that

$$(\sigma(i) - 1)m = k_{u_{\sigma(i)-1}} < k_{w_j} = jn < k_{u_{\sigma(i)}} = \sigma(i)m.$$

Since $\sigma(i)m - i \equiv 0 \pmod n$, we have $jn = \sigma(i)m - i$ and $j = \rho(i)$. □

DEFINITION 3.11. We define the monomial e_i of t, x by

$$e_i = t^{\sigma(i)m-i} x^{u_{\sigma(i)}}, \quad (i = 1, 2, \dots, n - 1).$$

LEMMA 3.12.

- (1) $e_m = x$ and $e_{n-m} = t^{mn-n} x^{m+n-2}$.
- (2) $e_{m+i} = xe_i$ ($i = 1, 2, \dots, n - m - 1$).
- (3) $e_{n-m+i} = t^{-n} x^{-2} e_i$ ($i = 1, 2, \dots, m - 1$).
- (4) $t^{\sigma(i)m-i} x^{u_{\sigma(i)}-1} = t^{\rho(i)n} x^{w_{\rho(i)}} (i = 1, 2, \dots, m - 1)$.

PROOF. They are proved by Lemma 3.10(1), (3), (4) and (6) respectively. □

3.3. Proof of Theorem 3.2 (Alexander polynomial of $A_{m,n}$). We regard

$$\sum_{i=0}^{m+n-1} t^{k_i} x^i = 0 \tag{3.3}$$

as an algebraic equation over $\mathbb{Z}[t, t^{-1}]$, and x as a root of the equation.

LEMMA 3.13. *The algebraic Equation (3.3) over $\mathbb{Z}[t, t^{-1}]$ has no multiple root.*

PROOF. If we substitute $t = 1$ into the Equation (3.3), then we have

$$\sum_{i=0}^{m+n-1} x^i = 0.$$

Since this equation has no multiple root, and its degree is equal to that of (3.3) ($=m + n - 1$), the Equation (3.3) has no multiple root. \square

Let \mathbf{v} be a row vector of size $1 \times (m + n - 1)$, decomposed as

$$\mathbf{v} = [\mathbf{e} \quad f \quad \mathbf{g}],$$

where f is a scalar and

$$\mathbf{e} = [e_1 \quad e_2 \quad \cdots \quad e_{n-1}], \quad \mathbf{g} = [g_1 \quad g_2 \quad \cdots \quad g_{m-1}]$$

are row vectors of size $1 \times (n - 1)$ and $1 \times (m - 1)$, respectively. Then the matrix $M(m, n)$ in Lemma 3.4 satisfies

$$\mathbf{v} \cdot M(m, n) = [t^n \mathbf{g} \quad (-\mathbf{e} \cdot \overrightarrow{T_{n-1}} - t^n f - t^n \mathbf{g} \cdot \overrightarrow{i_{m-1}}) \quad \mathbf{e}]. \tag{3.4}$$

LEMMA 3.14. *If we take $e_i = t^{\sigma(i)m-i} x^{u_{\sigma(i)}}$ ($i = 1, 2, \dots, n - 1$) as in Definition 3.11, $f = x e_{n-m}$ and $g_i = t^{-n} x^{-1} e_i$ ($i = 1, 2, \dots, m - 1$), then we have*

$$\mathbf{v} \cdot M(m, n) = x^{-1} \mathbf{v}.$$

PROOF. By Lemma 3.12 and the definition of e_i and g_i , we can see immediately that the i th entry of $\mathbf{v} \cdot M(m, n)$ is equal to that of $M(m, n)$ multiplied by x^{-1} except the case $i = m$. The m th entry of (3.4) is computed as

$$\begin{aligned} -\mathbf{e} \cdot \overrightarrow{T_{n-1}} - t^n f - t^n \mathbf{g} \cdot \overrightarrow{i_{m-1}} &= -\sum_{i=1}^{n-1} t^i e_i - t^n f - t^n \sum_{j=1}^{m-1} g_j \\ &= -\sum_{i=1}^{n-1} t^{\sigma(i)m} x^{u_{\sigma(i)}} - t^{mn} x^{m+n-1} \\ &\quad - \sum_{j=1}^{m-1} t^{\rho(j)n} x^{w_{\rho(j)}} \\ &= 1 \end{aligned}$$

by Definition 3.11, Lemma 3.12, bijectivity of σ and ρ , Equation (3.3) and Proposition 3.8(5). We have the lemma. \square

PROOF OF THEOREM 3.2. By Lemma 3.14, x^{-1} is an eigenvalue of the matrix $M(m, n)$, where x is a root of (3.3). Since the degree of Equation (3.3) and the size of $M(m, n)$ are identical ($=m + n - 1$), and Equation (3.3) has no multiple root by Lemma 3.13,

$$\sum_{i=0}^{m+n-1} t^{ki} x^i = x^{m+n-1} \cdot \det(x^{-1} I_{m+n-1} - M(m, n)).$$

By Lemma 3.4, we have the result. □

4. Reidemeister torsions of $(A_{m,n}; r, 0)$ and key lemmas

We compute the Reidemeister torsions of $(A_{m,n}; r, 0)$. The goal is Lemma 4.3.

4.1. The first homology of $(A_{m,n}; r, 0)$. We calculate the first homology of $M = (A_{m,n}; \alpha/\beta, 0)$. Let E denote the complement of $A_{m,n}$. We consider $M = E \cup V_1 \cup V_2$, see notation in Section 2.1. We set $M_1 := E \cup V_1 \subset M$.

From now on, we always assume that $\gcd(\alpha, \beta) = 1, \beta > 0$, and

$$\gcd(m + n, \alpha) = 1,$$

which is equivalent to the condition for the first homology $H_1(M; \mathbb{Z})$ to be finite cyclic by the elementary divisor theory. Then the order is $(m + n)^2\beta$: $H_1(M; \mathbb{Z}) \cong \mathbb{Z}/(m + n)^2\beta\mathbb{Z}$.

We determine the first homologies of E, M_1 and M , define generators and study relations. First, $H_1(E)$ is a free abelian group of rank two generated by $[m_1]$ and $[m_2]$:

$$H_1(E) \cong \langle [m_1], [m_2] \mid - \rangle \cong \mathbb{Z}^2.$$

We have

$$[l_1] = [m_2]^{m+n} \quad \text{and} \quad [l_2] = [m_1]^{m+n}. \tag{4.1}$$

Next, we attach V_1 to E to make M_1 . We take integers γ, δ such that $\alpha\delta - \beta\gamma = -1$, and fix the meridian-longitude system m'_1, l'_1 of the solid torus V_1 . In $H_1(M_1)$, we have the relations (4.1) and

$$[m'_1] = [m_1]^\alpha [l_1]^\beta = 1 \quad \text{and} \quad [l'_1] = [m_1]^\gamma [l_1]^\delta.$$

Thus,

$$\begin{aligned} H_1(M_1) &\cong \langle [m_1], [m_2] \mid [m_1]^\alpha [m_2]^{(m+n)\beta} = 1 \rangle \\ &\cong \langle T \mid - \rangle \cong \mathbb{Z}, \end{aligned}$$

where $T = [m_1]^{\gamma'} [m_2]^{\delta'}$ by taking integers γ', δ' satisfying $\alpha\delta' - \beta\beta\gamma' = -1$. By the relations above,

$$\begin{aligned} [m_1] &= [m_1]^{-\alpha\delta' + (m+n)\beta\gamma'} \\ &= ([m_1]^\alpha [m_2]^{(m+n)\beta})^{-\delta'} ([m_1]^{\gamma'} [m_2]^{\delta'})^{(m+n)\beta} = T^{(m+n)\beta}, \\ [m_2] &= [m_2]^{-\alpha\delta' + (m+n)\beta\gamma'} \\ &= ([m_1]^\alpha [m_2]^{(m+n)\beta})^{\gamma'} ([m_1]^{\gamma'} [m_2]^{\delta'})^{-\alpha} = T^{-\alpha}, \\ [l'_1] &= [m_1]^\gamma [l_1]^\delta \\ &= [m_1]^\gamma [m_2]^{(m+n)\delta} = T^{(m+n)\beta\gamma - (m+n)\alpha\delta} = T^{m+n}. \end{aligned} \tag{4.2}$$

Finally, we attach V_2 to M_1 to make M . By (4.1) and (4.2) in $H_1(M)$,

$$[m'_2] = [l_2] = [m_1]^{m+n} = T^{(m+n)^2\beta} = 1, \quad [l'_2] = [m_2] = T^{-\alpha},$$

and

$$H_1(M) \cong \langle T \mid T^{(m+n)^2\beta} = 1 \rangle \cong \mathbb{Z}/(m+n)^2\beta\mathbb{Z}.$$

4.2. Reidemeister torsion of $(A_{m,n}; r, \mathbf{0})$. In this subsection, we compute the Reidemeister torsion of $(A_{m,n}; r, \mathbf{0})$. The goal is Lemma 4.3.

First, by the surgery formula II (Lemma 2.5) and the results on the first homology in the last subsection,

$$\tau(M_1) \doteq \Delta_{A_{m,n}}(T^{(m+n)\beta}, T^{-\alpha})(T^{m+n} - 1)^{-1}. \tag{4.3}$$

By the Alexander polynomial in Theorem 3.2 and (4.3),

$$\begin{aligned} \tau(M_1) &\doteq \left(\sum_{i=0}^{m+n-1} T^{k_i(m+n)\beta - i\alpha} \right) (T^{m+n} - 1)^{-1} \\ &= \frac{\sum_{i=0}^{m+n-1} (T^{k_i(m+n)\beta - i\alpha} - T^{-i\alpha})}{T^{m+n} - 1} + \frac{\sum_{i=0}^{m+n-1} T^{-i\alpha}}{T^{m+n} - 1} \\ &= \sum_{i=0}^{m+n-1} \left(T^{-i\alpha} \cdot \frac{T^{k_i(m+n)\beta} - 1}{T^{m+n} - 1} \right) \\ &\quad + \frac{T^{-(m+n)\alpha} - 1}{T^{m+n} - 1} \cdot (T^{-\alpha} - 1)^{-1}. \end{aligned} \tag{4.4}$$

Note that $(T^{ab} - 1)/(T^b - 1)$ is the polynomial $1 + T^b + T^{2b} + \dots + T^{(a-1)b}$.

Next, let $d \geq 2$ be a divisor of $(m+n)^2\beta$. It holds that $\gcd(d, \alpha) = 1$. By the surgery formula I (Lemma 2.3) and the results on the first homology, the Reidemeister-Turaev torsion of M is

$$\tau^{\psi_d}(M) \doteq \tau^{\rho_d}(M_1)(\zeta_d^\alpha - 1)^{-1}, \tag{4.5}$$

where $\rho_d := \psi_d \circ \iota$ is the composition of a ring homomorphism $\iota : \mathbb{Z}[H_1(M_1)] \rightarrow \mathbb{Z}[H_1(M)]$ induced from the natural inclusion, and a ring homomorphism $\psi_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ such that $\psi_d(T) = \zeta_d$.

We take d as a divisor of $(m+n)$ and a ring homomorphism $\psi'_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ such that $\psi'_d(T) = \zeta_d^{-\bar{\alpha}}$, where $\alpha\bar{\alpha} \equiv 1 \pmod{m+n}$. Then d is still a divisor of $(m+n)^2\beta$, and $\zeta_d^{-\bar{\alpha}}$ is still a primitive d th root of unity, since $\gcd(d, \alpha) = 1$. By (4.4) and (4.5),

$$\tau^{\psi'_d}(M) \doteq \left\{ \beta(\zeta_d - 1) \sum_{i=0}^{m+n-1} k_i \zeta_d^i - \alpha \right\} (\zeta_d - 1)^{-2}. \tag{4.6}$$

DEFINITION 4.1. For a divisor $d \geq 2$ of $(m+n)$, and the primitive d th root $(\zeta = \zeta_d)$ of unity, we define

$$R(m, n) := (\zeta - 1) \sum_{i=0}^{m+n-1} k_i \zeta^i.$$

By (4.6), the Reidemeister-Turaev torsion of M is expressed as

$$\tau^{\psi_d}(M) \doteq \{\beta R(m, n) - \alpha\}(\zeta - 1)^{-2}.$$

LEMMA 4.2.

- (1) $R(m, n)$ is a real number.
- (2) $R(m, n) = mn + \frac{1}{2} \sum_{i=1}^{m+n-1} (k_{i-1} - k_i)(\zeta^i + \zeta^{-i})$.
- (3) $R(m, n) = m(n + 1) + \sum_{j=1}^{m-1} (m - s_j)(\zeta^{w_j} + \zeta^{-w_j})$,
where s_j is defined by $s_j := [jn]_m$ for an integer j , see notation in Section 3.2.
- (4) $R(m, n) = m(n + 1) + \sum_{j=1}^{m-1} (m - j)(\xi^j + \xi^{-j})$.
- (5) $R(m, n) = \xi^{-(m-1)} \cdot ((\xi^m - 1)/(\xi - 1))^2 + mn = |(\xi^m - 1)/(\xi - 1)|^2 + mn$,
where $\xi = \zeta^{\bar{m}}$ with $m\bar{m} \equiv 1 \pmod{m+n}$.

PROOF. (1) The complex conjugate $\overline{R(m, n)}$ of $R(m, n)$ is

$$\begin{aligned} \overline{R(m, n)} &= (\zeta^{-1} - 1) \sum_{i=0}^{m+n-1} k_i \zeta^{-i} = (\zeta - 1) \sum_{i=0}^{m+n-1} -k_i \zeta^{-i-1} \\ &= (\zeta - 1) \sum_{i=0}^{m+n-1} (mn - k_i) \zeta^{m+n-i-1} \\ &= (\zeta - 1) \sum_{i=0}^{m+n-1} k_{m+n-i-1} \zeta^{m+n-i-1} \\ &= (\zeta - 1) \sum_{i=0}^{m+n-1} k_i \zeta^i = R(m, n) \end{aligned}$$

by Proposition 3.8(1) and the equality $1 + \zeta + \dots + \zeta^{m+n-1} = 0$.

(2) First,

$$\begin{aligned} R(m, n) &= \sum_{i=0}^{m+n-1} k_i (\zeta^{i+1} - \zeta^i) = \sum_{i=1}^{m+n} k_{i-1} \zeta^i - \sum_{i=0}^{m+n-1} k_i \zeta^i \\ &= mn + \sum_{i=1}^{m+n-1} (k_{i-1} - k_i) \zeta^i. \end{aligned}$$

By the proof of (1),

$$\begin{aligned} R(m, n) &= \frac{1}{2} \{R(m, n) + \overline{R(m, n)}\} \\ &= mn + \frac{1}{2} \sum_{i=1}^{m+n-1} (k_{i-1} - k_i)(\zeta^i + \zeta^{-i}). \end{aligned}$$

Before the proof of (3)–(5), we prove the following claim.

CLAIM (Properties of s_j).

- (1) The map from j to $s_j (= [jn]_m)$ is a bijection on $\{1, 2, \dots, m - 1\}$ to itself.
- (2) It holds that $s_j + s_{m-j} = m$.

- (3) There exists a unique element h in $\{1, 2, \dots, m - 1\}$ such that $\gcd(h, m) = 1$ and $s_h = 1$. It holds that $w_h \equiv -\bar{m} \pmod{m + n}$.
- (4) For the same h in (3) and each element a in $\{1, 2, \dots, m - 1\}$,

$$s_{ah} = a \quad \text{and} \quad w_{ah} \equiv aw_h \pmod{m + n},$$

where we regard w_{ah} as w_j with $j = [ah]_m$, precisely.

PROOF OF CLAIM. (1) The map is induced by the multiplication of n (that is, $j \mapsto jn$) over $(\mathbb{Z}/m\mathbb{Z}) \setminus \{0\}$. It is a bijection, since $\gcd(m, n) = 1$.

(2) This is easy to see.

(3) By (1), there exists a unique element h in $\{1, 2, \dots, m - 1\}$ such that $s_h = 1$. In fact, it holds that $h \equiv \bar{n} \pmod{m}$. We have $\gcd(h, m) = 1$. The second half is shown by

$$-mw_h = -m\left(h + \left[\frac{hn}{m}\right]\right) \equiv hn - m\left[\frac{hn}{m}\right] = [hn]_m = s_h = 1 \pmod{m + n},$$

see Proposition 3.8(2).

(4) Since $s_{ah} \equiv ahn \equiv a \pmod{m}$, we have $s_{ah} = a$, for a in $\{1, 2, \dots, m - 1\}$. Since

$$mw_{ah} = mah + m\left[\frac{ahn}{m}\right] \equiv -\left(ahn - m\left[\frac{ahn}{m}\right]\right) = -[ahn]_m = -s_{ah} = -a \pmod{m + n}.$$

This concludes the proof. □

(3) We go back to the expression (2). We divide the set $\{1, 2, \dots, m + n - 1\}$ of indices of k_i , into M, R and L according to whether k_{i-1} and/or k_i belongs to $m\mathbb{Z} \cap \mathfrak{S}(m, n)$.

$$\{1, 2, \dots, m + n - 1\} = M \cup R \cup L \quad (\text{a disjoint union})$$

Definition of the subset	Parameter j	$k_{i-1} - k_i$
$M := \{i \mid k_{i-1} \in m\mathbb{Z} \text{ and } k_i \in m\mathbb{Z}\}$	—	$-m$
$R := \{i \mid k_{i-1} \in m\mathbb{Z} \text{ and } k_i \notin m\mathbb{Z}\}$	$k_i = jn$	$m\left[\frac{jn}{m}\right] - jn$
$L := \{i \mid k_{i-1} \notin m\mathbb{Z} \text{ and } k_i \in m\mathbb{Z}\}$	$k_{i-1} = jn$	$jn - m\left(\left[\frac{jn}{m}\right] + 1\right)$

Note that the case both $k_{i-1} \notin m\mathbb{Z}$ and $k_i \notin m\mathbb{Z}$ (in other words, the case that both k_{i-1} and k_i belong to $n\mathbb{Z}$) never occurs, since $m < n$ (Section 1).

For each $i \in R$, there exists j with $1 \leq j < m$ such that $k_i = jn$, equivalently $i = w_j$. Then, by Proposition 3.8(4), $i' := w_{m-j} + 1$ belongs to L and it holds that $i' = (m + n - 1 - w_j) + 1 = m + n - i$. The correspondence between $i \in R$ and $i' \in L$ above is one-to-one. It also holds that $\zeta^{i'} = \zeta^{w_{m-j}+1} = \zeta^{m+n-i} = \zeta^{-i}$ and

$$\begin{aligned} k_{i-1} - k_i &= m\left[\frac{jn}{m}\right] - jn = -[jn]_m = -s_j, \\ k_{i'-1} - k_{i'} &= (m - j)n - m\left(\left[\frac{(m - j)n}{m}\right] + 1\right) \\ &= [(m - j)n]_m - m = s_{m-j} - m = -s_j, \end{aligned}$$

by Claim (2). Thus

$$\begin{aligned} & (k_{i-1} - k_i)(\zeta^i + \zeta^{-i}) + (k_{i'-1} - k_{i'}) (\zeta^{i'} + \zeta^{-i'}) \\ &= -2s_j(\zeta^i + \zeta^{-i}) \\ &= -2s_j(\zeta^{w_j} + \zeta^{-w_j}), \end{aligned}$$

and

$$\begin{aligned} & \sum_{i \in R} (k_{i-1} - k_i + m)(\zeta^i + \zeta^{-i}) + \sum_{i \in L} (k_{i-1} - k_i + m)(\zeta^i + \zeta^{-i}) \\ &= \sum_{i \in R} \{ (k_{i-1} - k_i + m)(\zeta^i + \zeta^{-i}) + (k_{i'-1} - k_{i'} + m)(\zeta^{i'} + \zeta^{-i'}) \} \\ &= 2 \sum_{j=1}^{m-1} (m - s_j)(\zeta^{w_j} + \zeta^{-w_j}). \end{aligned}$$

Thus, using $1 + \zeta + \zeta^2 + \dots + \zeta^{m+n-1} = 0$,

$$\begin{aligned} R(m, n) &= mn + \frac{1}{2} \sum_{i=1}^{m+n-1} (k_{i-1} - k_i)(\zeta^i + \zeta^{-i}) \\ &= mn + \frac{1}{2} \sum_{i=1}^{m+n-1} (k_{i-1} - k_i)(\zeta^i + \zeta^{-i}) + \frac{1}{2} \sum_{i=0}^{m+n-1} m(\zeta^i + \zeta^{-i}) \\ &= mn + m + \frac{1}{2} \sum_{i=1}^{m+n-1} (k_{i-1} - k_i + m)(\zeta^i + \zeta^{-i}) \\ &= m(n + 1) + \frac{1}{2} \sum_{i \in M \cup R \cup L} (k_{i-1} - k_i + m)(\zeta^i + \zeta^{-i}) \\ &= m(n + 1) + \sum_{j=1}^{m-1} (m - s_j)(\zeta^{w_j} + \zeta^{-w_j}). \end{aligned}$$

(4) We take $h(=\bar{n} \pmod m)$ in Claim (3). By Claim (4),

$$\begin{aligned} R(m, n) &= m(n + 1) + \sum_{j=1}^{m-1} (m - s_j)(\zeta^{w_j} + \zeta^{-w_j}) \\ &= m(n + 1) + \sum_{a=1}^{m-1} (m - s_{ah})(\zeta^{w_{ah}} + \zeta^{-w_{ah}}) \\ &= m(n + 1) + \sum_{a=1}^{m-1} (m - a)(\zeta^{-a\bar{m}} + \zeta^{a\bar{m}}) \\ &= m(n + 1) + \sum_{a=1}^{m-1} (m - a)(\xi^a + \xi^{-a}), \end{aligned}$$

where we set $\xi = \zeta^{\bar{m}}$, which is also a d th primitive root of unity.

(5) Using elementary calculus

$$\sum_{a=1}^{m-1} (m-a)x^a = \frac{x^m - mx - 1}{x - 1} + \frac{x^m - 1}{(x - 1)^2} = x \frac{x^m - 1}{(x - 1)^2} - m \frac{x}{x - 1},$$

we have

$$R(m, n) = \xi^{-(m-1)} \cdot \left(\frac{\xi^m - 1}{\xi - 1} \right)^2 + mn.$$

The proof of Lemma 4.2 is completed. □

The result of this subsection is summarized as the following result.

LEMMA 4.3. *Let $M = (A_{m,n}; \alpha/\beta, 0)$ and $d \geq 2$ a divisor of $m + n$. Then the Reidemeister–Turaev torsion of M related with $\psi'_d(T) = \zeta_d^{-\bar{\alpha}}$ is*

$$\begin{aligned} \tau^{\psi'_d}(M) &\doteq \xi^{-m} \{ \beta R(m, n) - \alpha \} (\xi^m - 1)^{-2} \\ &= \xi^{-m} \left\{ \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - (\alpha - mn\beta) \right\} (\xi^m - 1)^{-2}, \end{aligned}$$

where $\xi = \zeta^{\bar{m}}$ (thus $\xi^m = \zeta$) is a primitive d th root of unity.

4.3. Necessary conditions. We study some necessary conditions for $(A_{m,n}; \alpha/\beta, 0)$ to be a lens space by the Reidemeister–Turaev torsions.

LEMMA 4.4. *Suppose that $(A_{m,n}; \alpha/\beta, 0)$ is a lens space. Then there exist integers i and j such that $\gcd(i, m + n) = \gcd(j, m + n) = 1$ and*

$$\{ \beta R(m, n) - \alpha \} (\xi^m - 1)^{-2} \doteq (\xi^i - 1)^{-1} (\xi^j - 1)^{-1}, \tag{4.7}$$

equivalently,

$$\left\{ \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - \alpha' \right\} (\xi^m - 1)^{-2} \doteq \frac{1}{(\xi^i - 1)(\xi^j - 1)}, \tag{4.8}$$

where $\alpha' = \alpha - mn\beta$, and ξ is a primitive d th root of unity.

The equalities (4.7), (4.8) correspond to two expressions of $\tau^{\psi'_d}(M)$ in Lemma 4.3.

LEMMA 4.5. *Let $d \geq 2$ be a divisor of $m + n$. Suppose that $(A_{m,n}; \alpha/\beta, 0)$ is a lens space. Then we have:*

- (1) $|N_d(\beta R(m, n) - \alpha)| = 1$, where N_d is the d -norm, see Section 2.2;
- (2) $\alpha' = \alpha - mn\beta \geq 0$.

PROOF. (1) We take the d -norm of the equality (4.7). Since $N_d(\xi^m - 1) = N_d(\xi^i - 1) = N_d(\xi^j - 1) \neq 0$ by Proposition 2.8(1) and Lemma 2.9, we have the result.

(2) Suppose that the integer $\alpha' = \alpha - mn\beta < 0$. Then we have

$$\beta R(m, n) - \alpha = \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - \alpha' > 1,$$

hence $|N_d(\beta R(m, n) - \alpha)| > 1$. By (1), we have the result. □

Fixing the combinatorial Euler structure, we will regard (4.8) as a control equivalence of the value sequences of degree $m + n$, in the sense of Lemma 2.12(1). Note that the first factor in the left-hand side is a real value. On the right-hand side, we have to control (i, j) to use Lemmas 2.15 or 2.16.

Conditions on (i, j) and (e, f) . We can take i and j satisfying $1 \leq i \leq j \leq (m + n - 1)/2$. If $i + j$ is odd (then $m + n$ is odd), then we replace j with $m + n - j$ and denote it by j again. Then, as a condition on (i, j) , we may assume

$$1 \leq i \leq j \leq m + n - 1, \quad 2 \leq i + j \leq m + n - 1 \quad \text{and } i + j \text{ is even.} \tag{4.9}$$

From now on, we regard equality (4.8) as a control equivalence between the real value sequences

$$u\xi^{-(i+j)/2} \left\{ \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - \alpha' \right\} (\xi^i - 1)(\xi^j - 1) = \xi^{-m}(\xi^m - 1)^2, \tag{4.10}$$

where $u = \pm 1$, or $\pm \xi^{(m+n)/2}$ only if $m + n$ is even, by Lemma 2.13. We define the integers

$$e := \frac{j - i}{2} \quad \text{and} \quad f := \frac{i + j}{2}. \quad \text{They satisfy } 0 \leq e < f \leq (m + n - 1)/2. \tag{4.11}$$

Using (e, f) , we can rewrite (4.10) as

$$\begin{aligned} & u\{\beta(\xi^m + \xi^{-m}) - \alpha'(\xi + \xi^{-1}) + 2(\alpha' - \beta)\} \{(\xi^f + \xi^{-f}) - (\xi^e + \xi^{-e})\} \\ & = (\xi^{m+1} + \xi^{-(m+1)}) - 2(\xi^m + \xi^{-m})(\xi^{m-1} + \xi^{-(m-1)}) - 2(\xi + \xi^{-1}) + 4. \end{aligned} \tag{4.12}$$

We define two symmetric Laurent polynomials

$$\begin{aligned} F(t) &= \{\beta(t^m + t^{-m}) - \alpha'(t + t^{-1}) + 2(\alpha' - \beta)\} \{(t^f + t^{-f}) - (t^e + t^{-e})\}, \\ G(t) &= (t^{m+1} + t^{-(m+1)}) - 2(t^m + t^{-m}) + (t^{m-1} + t^{-(m-1)}) - 2(t + t^{-1}) + 4, \end{aligned} \tag{4.13}$$

then (4.10) means that the two real value sequences (of degree $m + n$) induced by $F(t)$ and $G(t)$ are control equivalent, see Section 2.3. Note that $F(1) = G(1) = 0$. By Lemma 2.16, (4.12) lifts to a congruence of the symmetric Laurent polynomials.

LEMMA 4.6 (Necessary condition). *Suppose that $(A_{m,n}; \alpha/\beta, 0)$ is a lens space. We set $\alpha' = \alpha - mn\beta$. Then there exist integers e and f such that $0 \leq e < f \leq (m + n - 1)/2$, and the following congruence holds.*

- (1) *If $m + n$ is odd, we have $F(t) \equiv \pm G(t) \pmod{t^{m+n} - 1}$.*
- (2) *If $m + n$ is even, we have $F(t) \equiv \pm G(t)$ or $F(t) \equiv \pm t^{(m+n)/2} G(t) \pmod{t^{m+n} - 1}$.*

If $m + n$ is even, then $\text{span}(G(t)) = 2(m + 1) \leq m + n$, since the pair m, n is coprime, thus both are odd and $m + 2 \leq n$. Furthermore, the congruence also induces an identity

$$(i) \text{red}(F(t)) = \pm G(t) \quad \text{or} \quad (ii) \text{red}(t^{(m+n)/2} F(t)) = \pm G(t), \tag{4.14}$$

as in the second half of Lemma 2.16. We will regard it as an equation of (e, f) on the surgery coefficient α/β for $M = (A_{m,n}; \alpha/\beta, 0)$ to be a lens space. Suppose that M is a lens space, then there exists a solution (e, f) of the equation. We mainly use its contraposition. If the equation has no solution (e, f) , then M is not a lens space. The case (2) looks troublesome. To prove that M is not a lens space, we have to show that neither (i) nor (ii) has a solution. Fortunately, we only have to show one of them.

REMARK 4.7. In either case $m + n$ being odd or even, to prove that $(A_{m,n}; \alpha/\beta, 0)$ is not a lens space, it is sufficient to show that $\text{red}(F(t)) = \pm G(t)$ has no solution (e, f) , because we can prove the following.

LEMMA 4.8. *In the case (2) $m + n$ is even in Lemma 4.6, if the equation (i) $\text{red}(F(t)) = \pm G(t)$ in (4.14) has a solution, the other equation (ii) $\text{red}(t^{(m+n)/2}F(t)) = \mp G(t)$ has a solution, and vice versa.*

PROOF. We concentrate on the factor $\{(t^f + t^{-f}) - (t^e + t^{-e})\}$ of $F(t)$. We transform (e, f) to (e', f') by

$$e' = \frac{m+n}{2} - f \quad \text{and} \quad f' = \frac{m+n}{2} - e,$$

which satisfies the same condition $0 \leq e' < f' \leq (m+n-1)/2$ as (4.11). For a solution (e, f) of the equation $\text{red}(F(t)) = \pm G(t)$, its transformation (e', f') is a solution of $\text{red}(t^{(m+n)/2}F(t)) = \mp G(t)$, and vice versa. □

In Section 5.3, we will prove that $(e, f) = (0, 1)$ with $\alpha' = 0$ is the only solution for the equation $\text{red}(F(t)) = \pm G(t)$ in general cases (see Lemma 4.9(1) below). Note that $\alpha' = 0$ implies $\alpha/\beta = mn$, which is related to the lens space surgery in Theorem 1.2.

Using the expression of $R(m, n)$ in Lemma 4.2(4), we can prove the following lemma.

LEMMA 4.9.

- (1) *The condition $\alpha/\beta = mn$ (that is, $\alpha' = 0$) is equivalent to $(e, f) = (0, 1)$ (that is, $i = j = 1$).*
- (2) *In (4.8), if $(e, f) = (0, m)$ (that is, $i = j = m$), then there is no root for α/β .*

PROOF. (1) Suppose that $\alpha/\beta = mn$. Then we have $\tau^{\psi'_d}(M) \doteq (\xi - 1)^{-2}$ by Lemma 4.3, and equality (4.8) becomes

$$\frac{1}{(\xi - 1)^2} \doteq \frac{1}{(\xi^i - 1)(\xi^j - 1)}.$$

We have $i = j = 1$ by the Franz lemma (Lemma 2.10).

Conversely, suppose that $i = j = 1$. Then equality (4.7) can be written as

$$\beta R(m, n) - \alpha = u\xi^{-(m-1)} \left(\frac{\xi^m - 1}{\xi - 1} \right)^2,$$

where $u = \pm 1$ or $\pm \xi^{(m+n)/2}$. Using the expression of $R(m, n)$ in Lemma 4.2(4) and the calculus in the proof of Lemma 4.2(5),

$$\beta m(n + 1) - \alpha + \sum_{j=1}^{m-1} \beta(m - j)(\xi^j + \xi^{-j}) = u \left\{ m + \sum_{j=1}^{m-1} (m - j)(\xi^j + \xi^{-j}) \right\}.$$

By taking the symmetric polynomial lift and Lemma 2.15 (note that the span is $2(m - 1) \leq 2([(m + n)/2] - 1)$), the case $u = \pm \xi^{(m+n)/2}$ does not occur, by Lemma 2.17. We have $\beta = 1$ and $\alpha = mn$.

(2) Suppose that $i = j = m$. Then equality (4.7) becomes

$$\beta R(m, n) - \alpha = u,$$

where $u = \pm 1$ or $\pm \xi^{(m+n)/2}$. By the same method as above,

$$\beta m(n + 1) - \alpha + \sum_{j=1}^{m-1} \beta(m - j)(\xi^j + \xi^{-j}) = u.$$

By Lemmas 2.15 and 2.17, we have $\beta = 0$. Hence there is no root for α/β . □

LEMMA 4.10. *Suppose that $\alpha' > 0$.*

- (1) $\gcd(\alpha', \beta) = 1$.
- (2) *If $\alpha' = \beta$, then the congruence $F(t) \equiv \pm G(t) \pmod{t^{m+n} - 1}$ has a unique solution $(m, n) = (2, 3)$.*

PROOF. (1) By the Euclidean algorithm,

$$\gcd(\alpha', \beta) = \gcd(\alpha - mn\beta, \beta) = \gcd(\alpha, \beta) = 1.$$

(2) Suppose $\alpha' = \beta$. Then, by (1), we have $\alpha' = \beta = 1$ and the polynomials (4.13) are

$$F(t) = t^{-m-f}(t^{m+1} - 1)(t^{m-1} - 1)(t^i - 1)(t^j - 1),$$

$$G(t) = t^{-m-1}(t^m - 1)^2(t - 1)^2.$$

The congruence $F(t) \equiv \pm G(t) \pmod{t^{m+n} - 1}$ implies

$$F(\zeta) \doteq G(\zeta),$$

where ζ is a primitive $(m + n)$ th root of unity. Suppose $\gcd(m - 1, m + n) \geq 2$ or $\gcd(m + 1, m + n) \geq 2$. Then the left-hand side of the equation above is 0 for some d . Hence we have $\gcd(m - 1, m + n) = 1$ and $\gcd(m + 1, m + n) = 1$. By the Franz lemma [Fz] (Lemma 2.10),

$$\{\pm(m - 1), \pm(m + 1), \pm i, \pm j \pmod{m + n}\} = \{\pm 1, \pm 1, \pm m, \pm m \pmod{m + n}\}$$

as multiple sets. It has a unique solution $(m, n) = (2, 3)$ with $(i, j) = (1, 3)$. □

5. Proof of Theorem 1.4 (Lens space surgeries along $A_{m,n}$)

The ‘if part’ of Theorem 1.4(1) follows from Theorem 1.2, thus our main purpose is to prove the ‘only if part’. We study the condition on α/β for the Equations (4.10) or (4.14)(i) to have a solution (i, j) or (e, f) , respectively. Our proof is divided into three cases: $m = 2$ (Section 5.1), $n = m + 1$ (Section 5.2), and the general case where $m \geq 3$ and $n \geq m + 2$ (Section 5.3). Note that the first two cases contain the exceptional case $(A_{2,3}; 7, 0)$. In Section 5.4 and 5.5, we prove Theorem 1.4(2) by using the values of the Reidemeister torsions. We also use Kirby moves.

To make expressions of symmetric Laurent polynomials short, we use the notation $\langle t^x \rangle = t^x + t^{-x}$ for any integer x . We regard $\langle t^x \rangle$ as $\langle t^{-x} \rangle$ if $x < 0$, and $\langle t^0 \rangle = 2$. For the terminologies ‘reduce, reduction (denoted by $\text{red}(P(t))$)’ of symmetric Laurent polynomials, see Definition 2.14.

5.1. The case $m = 2$. In this case, n and $m + n = n + 2$ are odd. Let ξ denote any d th root of unity, where d is a divisor of $n + 2$ with $d \geq 2$. We have $R(2, n) = |\xi + 1|^2 + mn = \xi + \xi^{-1} + mn + 2$ by Lemma 4.2(5), thus the Equation (4.10), divided by $\xi^{-1}(\xi - 1)^2$ as a value sequence, becomes

$$\xi^{-(i+j-2)/2} \cdot \{\beta(\xi + \xi^{-1}) - \alpha''\} \frac{(\xi^i - 1)(\xi^j - 1)}{(\xi - 1)^2} = \eta \xi^{-1} \cdot \frac{(\xi^2 - 1)^2}{(\xi - 1)^2}, \tag{5.1}$$

where $\alpha'' = \alpha' - 2\beta = \alpha - 2(n + 1)\beta$, $\eta = \pm 1$ and (i, j) satisfies the condition (4.9) in the last section. We regard (5.1) as an equality between real value sequences, defined in Section 2.3.

(1) The case $2 \leq i + j < n + 1$. Note that $i + j$ is even, see (4.9).

By Lemma 2.15, the equalities (5.1) lift to a congruence and

$$t^{-(i+j-2)/2} \cdot \{\beta(t + t^{-1}) - \alpha''\} \frac{(t^i - 1)(t^j - 1)}{(t - 1)^2} = \eta t^{-1} \cdot \frac{(t^2 - 1)^2}{(t - 1)^2}.$$

Note that $(t^x - 1)/(t - 1)$ is a polynomial for an integer x and that both hand sides are symmetric Laurent polynomials. The span of the left-hand side is $i + j \leq 2((m + n)/2) - 1 = 2n$. From $\beta > 0$, we have $i = j = 1$, $\beta(t + t^{-1}) - \alpha'' = t^{-1}(t + 1)^2$, $\beta = 1$, $\alpha'' = -2$, and $\alpha = 2n$.

(2) The case $i + j = n + 1$ ($=2f$, see (4.11) in Section 4.3).

Then $e = (j - i)/2$ is an integer satisfying $1 \leq e < (n + 1)/2$. The Equation (4.10) is

$$\xi^{-(n+1)/2} \cdot \{\beta(\xi + \xi^{-1}) - \alpha''\} (\xi^i - 1)(\xi^j - 1) = \eta \xi^{-2} \cdot (\xi^2 - 1)^2.$$

By Lemma 2.16, it lifts to

$$\begin{aligned} &(\beta - \alpha'')(t^{(n+1)/2} + t^{-(n+1)/2}) + \beta(t^{(n-1)/2} + t^{-(n-1)/2}) \\ &\quad - \beta(t^{e+1} + t^{-(e+1)}) + \alpha''(t^e + t^{-e}) - \beta(t^{e-1} + t^{-(e-1)}) \\ &= \eta\{(t^2 + t^{-2}) - 2\}, \end{aligned} \tag{5.2}$$

which is, using the notation $\langle t^x \rangle = t^x + t^{-x}$,

$$(\beta - \alpha'')\langle t^{(n+1)/2} \rangle + \beta\langle t^{(n-1)/2} \rangle - \beta\langle t^{e+1} \rangle + \alpha''\langle t^e \rangle - \beta\langle t^{e-1} \rangle = \eta(\langle t^2 \rangle - 2).$$

Here we used $\langle t^{(n+3)/2} \rangle \equiv \langle t^{(n+1)/2} \rangle \pmod{(t^{n+2} - 1)}$. Note that the span of the left-hand side is at most $n + 1 = 2[(m + n)/2]$. We have $e = 1, \eta = \beta = 1$ and $n = 3$. Then (5.2) becomes

$$-\alpha''(t^2 + t^{-2}) + (1 + \alpha'')(t + t^{-1}) - 2 = (t^2 + t^{-2}) - 2.$$

Hence we have $\alpha'' = -1$ and $\alpha = 7$. In this case, $(i, j) = (1, 3)$. This corresponds to the case $(A_{2,3}; 7, 0)$, and it is a lens space, see Section 5.5.

5.2. The case $n = m + 1$. In this case, $m + n = 2m + 1$ is odd. Let ξ denote any d th root of unity, where d is a divisor of $2m + 1$ with $d \geq 2$. We use $\zeta = \xi^m$ as in Lemma 4.3, then $\xi = \zeta^{-2}$. The Reidemeister torsion in Lemma 4.3 becomes

$$\begin{aligned} \tau^{\psi_d}(M) &= \xi^{-m} \left\{ \beta \left| \frac{\xi^m - 1}{\xi - 1} \right|^2 - \alpha' \right\} (\xi^m - 1)^2 \\ &= \zeta \left\{ \beta \zeta \left(\frac{\zeta - 1}{\zeta^2 - 1} \right)^2 - \alpha' \right\} (\zeta - 1)^{-2} \\ &= -\zeta^2 \left\{ \alpha' \zeta^{-1} \left(\frac{\zeta^2 - 1}{\zeta - 1} \right)^2 - \beta \right\} (\zeta^2 - 1)^{-2} \\ &= -\zeta^2 \{ \alpha'(\zeta + \zeta^{-1}) - \beta \} (\zeta^2 - 1)^{-2}. \end{aligned}$$

We apply the same argument to this equality as in Section 4.3 and take (i, j) satisfying the condition (4.9). The equality of the real value sequence (4.10) is

$$\zeta^{-(i+j)/2} \left\{ \alpha' \zeta^{-1} \left(\frac{\zeta^2 - 1}{\zeta - 1} \right)^2 - \beta \right\} (\zeta^i - 1)(\zeta^j - 1) = \eta \zeta^{-2} (\zeta^2 - 1)^2.$$

Divided by $\zeta^{-1}(\zeta - 1)^2$, it induces a similar equation to (5.1). We can apply the same argument as in Section 5.1. Instead of (5.1), we study

$$\zeta^{-(i+j-2)/2} \cdot \left\{ \alpha'(\zeta + \zeta^{-1}) - \beta' \right\} \frac{(\zeta^i - 1)(\zeta^j - 1)}{(\zeta - 1)^2} = \eta \zeta^{-1} \cdot \frac{(\zeta^2 - 1)^2}{(\zeta - 1)^2}, \tag{5.3}$$

where $\beta' = \beta - 2\alpha' (= \beta - 2(\alpha - mn\beta))$. Equation (5.3) is obtained from (5.1) by changing ξ to ζ, β to α' and α'' to β , respectively. Thus, using the correspondence, we can study their roots by the same argument. In the last subsection, $\beta = 0$ was not allowed, but here $\alpha' = 0$ is allowed (Lemma 4.5).

- (1) The case $\alpha' = 0$ ($\alpha/\beta = m(m + 1) = mn$).
In this case, $\alpha/\beta = m(m + 1)$ is a root by Lemma 4.9(1).
- (2) The case $\alpha' \geq 1$. The argument is similar to the case $m = 2$ in Section 5.1.
 - (i) The case $2 \leq i + j \leq 2m - 2$.

There is no root for α', β , by the argument with (1) in Section 5.1, since the corresponding root $(\alpha', \beta') = (1, -2)$ implies $\beta = 0$, which is not allowed.

(ii) The case $i + j = 2m$.

Then $e = (j - i)/2$ is an integer satisfying $1 \leq e < m$. The argument is similar to that of (2) in Section 5.1. We have

$$(\alpha' - \beta')\langle t^m \rangle + \alpha'\langle t^{m-1} \rangle - \alpha'\langle t^{e+1} \rangle + \beta'\langle t^e \rangle - \alpha'\langle t^{e-1} \rangle = \eta(\langle t^2 \rangle - 2).$$

Here we used $\langle t^{m+1} \rangle \equiv \langle t^m \rangle \pmod{t^{2m+1} - 1}$. The span of the left-hand side is at most $2m = 2[(m + n)/2]$. Corresponding to that the Equation (5.1) has a root $(\beta, \alpha) = (-1, 1)$, this equation has a root $(\alpha', \beta') = (1, -1)$ only if $m = 2$, which implies $(\alpha, \beta) = (7, 1)$. This corresponds to the case $(A_{2,3}; 7, 0)$, and it is a lens space, see Section 5.5.

5.3. The case $m \geq 3$ and $n \geq m + 2$. Let $P(t)$ be a symmetric Laurent polynomial in Definition 2.14 of the form:

$$P(t) = a_0 + \sum_{i=1}^{\infty} a_i(t^i + t^{-i}) = a_0 + \sum_{i=1}^{\infty} a_i\langle t^i \rangle.$$

We call a_0 the *constant term*, $a_i\langle t^i \rangle$ the *ith term*, and a_i the *ith coefficient*. When $P(t)$ is considered in $\mathbb{Q}[t, t^{-1}]/(t^N - 1)$, $P(t)$ (as above) is *reduced* if $a_i = 0$ for all $i > [N/2]$. We denote the reduction of $P(t)$ by $\text{red}(P(t))$.

ASSUMPTION (of this subsection). Let (m, n) be a fixed pair of positive coprime integers with $m \geq 3$ and $n \geq m + 2$. We assume that $\beta \geq 1$, $\alpha' = \alpha - mn\beta \geq 0$, see Lemma 4.5. We set $F(t)$ and $G(t)$ as:

$$\begin{aligned} F(t) &= \{\beta\langle t^m \rangle - \alpha'\langle t^1 \rangle + 2(\alpha' - \beta)\}\{\langle t^f \rangle - \langle t^e \rangle\}, \\ G(t) &= \langle t^{m+1} \rangle - 2\langle t^m \rangle + \langle t^{m-1} \rangle - 2\langle t^1 \rangle + 4 \end{aligned}$$

in (4.13) in the last section. From now on, we fix $N = m + n$. Note that $G(t)$ is already reduced. Let

$$\text{red}(F(t)) = \pm G(t) \tag{5.4}$$

be an equation for (e, f) . Existence of an integral solution is a necessary condition for $(A_{m,n}; \alpha/\beta, 0)$ to be a lens space. We show that there exist no integral solution of (5.4) with $0 \leq e < f \leq (m + n - 1)/2$ for $\alpha' > 0$, see Lemma 4.9(1) for the case $\alpha' = 0$. By Lemma 4.10, we assume that $\alpha' \neq \beta$.

Using $\langle t^a \rangle \cdot \langle t^b \rangle = \langle t^{a+b} \rangle + \langle t^{a-b} \rangle$,

$$\begin{aligned} F(t) &= \beta\langle t^{m+f} \rangle + \beta\langle t^{m-f} \rangle - \alpha'\langle t^{f+1} \rangle - \alpha'\langle t^{f-1} \rangle + 2(\alpha' - \beta)\langle t^f \rangle \\ &\quad - \beta\langle t^{m+e} \rangle - \beta\langle t^{m-e} \rangle + \alpha'\langle t^{e+1} \rangle + \alpha'\langle t^{e-1} \rangle - 2(\alpha' - \beta)\langle t^e \rangle. \end{aligned}$$

This consists of ten nonzero terms, but may not be reduced. On the other hand, $G(t)$ has five terms and is already reduced. Thus our problem is ‘Which term in $F(t)$ is

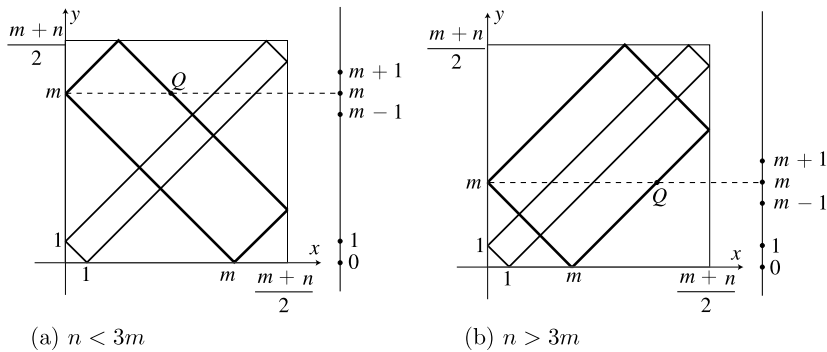


FIGURE 4. Graph of the degrees I.

reduced to a term in $G(t)$?' and 'Which terms in $F(t)$ are canceled with other terms?' On the terms $\langle t^{m+f} \rangle, \langle t^{m-f} \rangle, \langle t^{m+e} \rangle$ and $\langle t^{m-e} \rangle,$

$$\text{red}(\langle t^{m+x} \rangle) = \begin{cases} \langle t^{m+x} \rangle & \text{if } x \leq (n-m)/2, \\ \langle t^{n-x} \rangle & \text{if } x > (n-m)/2, \end{cases} \quad \text{red}(\langle t^{m-x} \rangle) = \langle t^{|m-x|} \rangle,$$

for $x = e$ or f (thus $0 \leq x \leq (m+n-1)/2$). The term $\langle t^{f+1} \rangle$ is already reduced except only one case.

Case F. If $(m+n)$ is odd and $f = (m+n-1)/2$, it holds that $\text{red}(\langle t^{f+1} \rangle) = \langle t^f \rangle.$

In Case F, it holds that $f - m = (n - m - 1)/2 \geq 1/2$, thus we have the following lemma.

LEMMA 5.1. *In Case F, it holds that $f \geq m + 1$. Furthermore, for an integer $a \geq 1$, $f = m + a$ is equivalent to $n = m + 2a + 1$.*

We will often take care of this exceptional case. The other five terms are already reduced. It is easy to see the following result.

LEMMA 5.2.

- (1) Neither $\langle t^{m+x} \rangle$ nor $\langle t^{x+1} \rangle$ can be reduced to the constant term.
- (2) The term $\langle t^{2m} \rangle$ ($\langle t^{m+x} \rangle$ with $x = m$) can be reduced to neither the constant term nor the first term.

The graph of the degrees of $\text{red}(\langle t^{m+x} \rangle), \text{red}(\langle t^{m-x} \rangle), \text{red}(\langle t^{x+1} \rangle)$ and $\text{red}(\langle t^{x-1} \rangle)$ are useful, see Figure 4. We are interested in only the points whose coordinates are integers. We set the rectangle formed by $y = \text{deg}(\text{red}(\langle t^{m+x} \rangle))$ and $y = \text{deg}(\text{red}(\langle t^{m-x} \rangle))$ as R . The intersection points of R and the line $y = k$ correspond to solving the equation $\text{red}(\langle t^{m \pm x} \rangle) = \langle t^k \rangle$. The point Q is one of the intersections for $k = m$ other than $(0, m)$. The (x, y) -coordinate of Q is as follows.

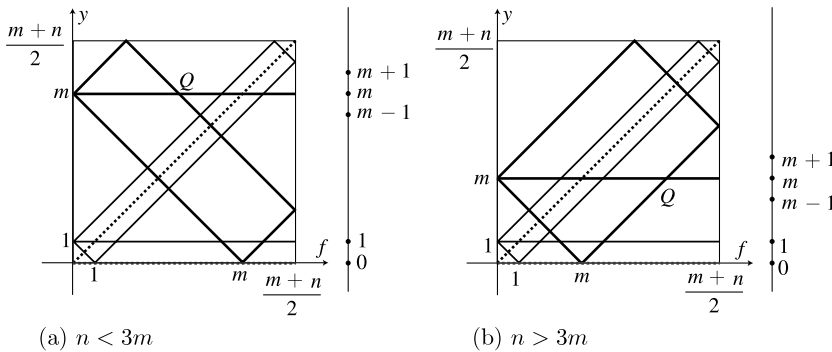


FIGURE 5. Graph of the degrees II.

If $n < 3m$, then $Q(n - m, m)$, which corresponds to $\text{red}(\langle t^{m+x} \rangle) = \langle t^m \rangle$.

If $n > 3m$, then $Q(2m, m)$, which corresponds to $\text{red}(\langle t^{m-x} \rangle) = \langle t^m \rangle$.

Here, note that $n = 3m$ contradicts coprimeness of m and n .

LEMMA 5.3 (The case $\text{red}(\langle t^{m \pm x} \rangle) = \langle t^m \rangle$). By x_Q we denote the x -coordinate ($= n - m$ or $2m$) of Q . Then we have:

- (1) $x_Q \neq m$;
- (2) $x_Q = m + a$ only if $n = 2m + a$, for $a = -2, -1, 1$ or 2 ;
- (3) $x_Q = a$ only if $n = m + a$, for $a = 2$ or 3 .

Since $G(t)$ has the nonzero constant term, by Lemma 5.2(1),

$$\{e, f\} \cap \{0, 1, m\} \neq \emptyset.$$

The proof is divided into the five cases: (1) $e = 0$; (2) $e = 1$ with $m \geq 4$; (3) $e = m$; (4) ' $e \neq 0, 1$ and $f = m$ '; and (5) $e = 1$ with $m = 3$. Note that $f = 0$ is impossible and that the case $f = 1$ is included by the case $e = 0$, because of the assumption $e < f$.

Case 1: $e = 0$.

$$F(t) = \beta \langle t^{m+f} \rangle + \beta \langle t^{m-f} \rangle - \alpha' \langle t^{f+1} \rangle - \alpha' \langle t^{f-1} \rangle + 2(\alpha' - \beta) \langle t^f \rangle - 2\beta \langle t^m \rangle + 2\alpha' \langle t^1 \rangle - 4(\alpha' - \beta).$$

Since $G(t)$ has a nontrivial $(m + 1)$ - and $(m - 1)$ -term, by Lemma 5.3, we have (taking into account Case F)

$$f \in \{1, x_Q + \epsilon, m, m + 1, m + 2\} \cap \{1, x_Q - \epsilon, m - 2, m - 1, m\},$$

where $\epsilon = -1$ if $n < 3m$ (and $\epsilon = +1$ if $n > 3m$, respectively). Figure 5 helps to understand this. Thus we have four cases: (i) $f = 1$; (ii) $f = m$; (iii) $f = m + 2$ (if $n = 2m + 1$ ($x_Q = m + 1$)); or (iv) $f = m - 2$ (if $n = 2m - 1$ ($x_Q = m - 1$)) by Lemma 5.3(2).

In each case, Case F (that is, $\text{red}(\langle t^{f+1} \rangle) = \langle t^f \rangle$) does not occur because $f \leq m$ in (i), (ii) and (iv), and because $f \neq (m+n-1)/2$ in (iii), by Lemma 5.1.

Subcase(1-i): $(e, f) = (0, 1)$.

$$F(t) = \beta \langle t^{m+1} \rangle - 2\beta \langle t^m \rangle + \beta \langle t^{m-1} \rangle - \alpha' \langle t^2 \rangle + 2(2\alpha' - \beta) \langle t^1 \rangle - 2(3\alpha' - 2\beta).$$

Since $\alpha' \neq 0$, to cancel $-\alpha' \langle t^2 \rangle$, we need $m = 3$ and $\alpha' = \beta$, which contradicts the assumption. (If we admit $\alpha' = 0$, then $(e, f) = (0, 1)$ is a solution, see Lemma 4.9(1).)

Subcase(1-ii): $(e, f) = (0, m)$.

$$F(t) = \beta \langle t^{2m} \rangle - \alpha' \langle t^{m+1} \rangle + 2(\alpha' - 2\beta) \langle t^m \rangle - \alpha' \langle t^{m-1} \rangle + 2\alpha' \langle t^1 \rangle - 2(2\alpha' - 3\beta).$$

Since $2m > m + 1$, the $2m$ th term is not reduced. By Lemma 5.2(2) and the ratio of the $(m+1)$ th and $(m-1)$ th coefficients ($=1$), we have $\text{red}(\langle t^{2m} \rangle) = \langle t^m \rangle$ and $n - m = m$. This contradicts coprimeness of m and n .

Subcase(1-iii): $(e, f) = (0, m+2)$ with $n = 2m + 1$. It holds that $\text{red}(\langle t^{2m+2} \rangle) = \langle t^{m-1} \rangle$.

$$F(t) = -\alpha' \langle t^{m+3} \rangle + 2(\alpha' - \beta) \langle t^{m+2} \rangle - \alpha' \langle t^{m+1} \rangle - 2\beta \langle t^m \rangle + \beta \langle t^{m-1} \rangle \\ + \beta \langle t^2 \rangle + 2\alpha' \langle t^1 \rangle - 4(\alpha' - \beta).$$

Since $m \geq 3$ (thus $m+2 \leq (m+n)/2$) and $\alpha' \neq \beta$, the $(m+2)$ th term cannot be canceled. We have a contradiction.

Subcase(1-iv): $(e, f) = (0, m-2)$ with $n = 2m - 1$. If $m = 3$, we go back to (1-i), thus we assume $m \geq 4$. It holds that $\text{red}(\langle t^{2m-2} \rangle) = \langle t^{m+1} \rangle$.

$$F(t) = \beta \langle t^{m+1} \rangle - 2\beta \langle t^m \rangle - \alpha' \langle t^{m-1} \rangle + 2(\alpha' - \beta) \langle t^{m-2} \rangle - \alpha' \langle t^{m-3} \rangle \\ + \beta \langle t^2 \rangle + 2\alpha' \langle t^1 \rangle - 4(\alpha' - \beta).$$

All terms are already reduced. By the signs of the $(m+1)$ th and $(m-1)$ th coefficients, we have a contradiction.

Case 2: $e = 1$ with $m \geq 4$.

$$F(t) = \beta \langle t^{m+f} \rangle + \beta \langle t^{m-f} \rangle - \alpha' \langle t^{f+1} \rangle - \alpha' \langle t^{f-1} \rangle + 2(\alpha' - \beta) \langle t^f \rangle \\ - \beta \langle t^{m+1} \rangle - \beta \langle t^{m-1} \rangle + \alpha' \langle t^2 \rangle - 2(\alpha' - \beta) \langle t^1 \rangle + 2\alpha'.$$

Recall the assumption $f > e = 1$. Comparing $F(t)$ and $G(t)$, we have that at least one term in $F(t)$ is equal to or reduced to the m th term, and that the term $\alpha' \langle t^2 \rangle$ is canceled with another term whose coefficient is negative. Here we used $m \geq 4$ (that is, $\langle t^{m-1} \rangle \neq \langle t^2 \rangle$). We have (taking into account Case F)

$$f \in \{m-1, m, m+1, x_Q\} \cap \{2, 3\},$$

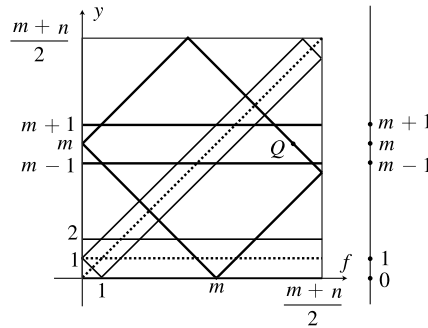


FIGURE 6. Graph of the degrees III.

where x_Q is the value defined in Lemma 5.3. Figure 6 helps to understand this. Since $f \leq 3$, Case F does not occur by Lemma 5.1. Thus we have three cases: (i) $m = 4$ and $f = 3$, (ii) $f = 2$ (if $n = m + 2$ ($x_Q = 2$)), (iii) $f = 3$ (if $n = m + 3$ ($x_Q = 3$)), see Lemma 5.3(3).

Subcase(2-i): $m = 4$ and $(e, f) = (1, 3)$.

$$F(t) = \beta\langle t^7 \rangle - \beta\langle t^5 \rangle - \alpha'\langle t^4 \rangle + (2\alpha' - 3\beta)\langle t^3 \rangle - (2\alpha' - 3\beta)\langle t^1 \rangle + 2\alpha'.$$

The term $\beta\langle t^7 \rangle$ has to be reduced: $\text{red}(\langle t^7 \rangle) = \langle t^{n-3} \rangle$ and $3 \leq n - 3 \leq 5$. By coprimeness of m and n , we have $n = 7$. By the fifth and fourth coefficients, we have a contradiction.

Subcase(2-ii): $m \geq 4, f = 2$ (if $n = m + 2$ ($x_Q = 2$)). It holds that $\text{red}(\langle t^{m+2} \rangle) = \langle t^m \rangle$.

$$F(t) = -\beta\langle t^{m+1} \rangle + \beta\langle t^m \rangle - \beta\langle t^{m-1} \rangle + \beta\langle t^{m-2} \rangle - \alpha'\langle t^3 \rangle + (3\alpha' - 2\beta)\langle t^2 \rangle - (3\alpha' - 2\beta)\langle t^1 \rangle + 2\alpha'.$$

All terms are already reduced. Considering the ratio of the $(m + 1)$ -st and m th coefficients, we have a contradiction.

Subcase(2-iii): $m \geq 4, f = 3$ (if $n = m + 3$ ($x_Q = 3$)). It holds that $\text{red}(\langle t^{m+3} \rangle) = \langle t^m \rangle$.

$$F(t) = -\beta\langle t^{m+1} \rangle + \beta\langle t^m \rangle - \beta\langle t^{m-1} \rangle + \beta\langle t^{m-3} \rangle - \alpha'\langle t^4 \rangle + 2(\alpha' - \beta)\langle t^3 \rangle - 2(\alpha' - \beta)\langle t^1 \rangle + 2\alpha'.$$

All terms are already reduced. If $m \geq 5$, then we have a contradiction by the same method as in the last case. If $m = 4$,

$$\text{red}(F(t)) = -\beta\langle t^5 \rangle - (\alpha' - \beta)\langle t^4 \rangle + (2\alpha' - 3\beta)\langle t^3 \rangle - (2\alpha' - 3\beta)\langle t^1 \rangle + 2\alpha'.$$

Considering the ratio of the first two coefficients, $-\beta : -(\alpha' - \beta) = 1 : -2$, that is, $\beta = -\alpha' < 0$. We have a contradiction.

In the rest of the proof, we will often use the following lemma.

LEMMA 5.4. *Even if $\text{red}(\langle t^a \rangle) = \langle t^b \rangle$, the sum $\pm\beta\langle t^a \rangle \pm \alpha'\langle t^b \rangle$ of any signs never cancels by the reduction.*

This lemma is easily shown by the assumption $\alpha' > 0, \beta > 0$ and $\alpha' \neq \beta$.

Case 3: $e = m$.

$$F(t) = \beta\langle t^{m+f} \rangle + \beta\langle t^{m-f} \rangle - \alpha'\langle t^{f+1} \rangle - \alpha'\langle t^{f-1} \rangle + 2(\alpha' - \beta)\langle t^f \rangle - \beta\langle t^{2m} \rangle - 2\beta + \alpha'\langle t^{m+1} \rangle + \alpha'\langle t^{m-1} \rangle - 2(\alpha' - \beta)\langle t^m \rangle.$$

Recall the assumption $f > e = m$. Comparing $F(t)$ and $G(t)$, at least one term in $F(t)$ is equal to or reduced to the first term. By Lemma 5.2(2), we have two cases: (i) $f = m + 1$ (it can be in Case F) and (ii) $\text{red}(\langle t^{m+f} \rangle) = \langle t^1 \rangle$. But in the latter case, $f = n - 1 \leq (m + n)/2$ (see the graph in Figure 4), thus $n = m + 2$ and $f = m + 1$. We only have to study the case (i).

Subcase(3-i): $(e, f) = (m, m + 1)$.

$$F(t) = \beta\langle t^{2m+1} \rangle - \beta\langle t^{2m} \rangle - \alpha'\langle t^{m+2} \rangle + (3\alpha' - 2\beta)\langle t^{m+1} \rangle - (3\alpha' - 2\beta)\langle t^m \rangle + \alpha'\langle t^{m-1} \rangle + \beta\langle t^1 \rangle - 2\beta.$$

We focus on $-\alpha'\langle t^{m+2} \rangle$. Neither $\beta\langle t^{2m+1} \rangle - \alpha'\langle t^{m+2} \rangle$ nor $-\beta\langle t^{2m} \rangle - \alpha'\langle t^{m+2} \rangle$ cancel by the reduction by Lemma 5.4. Thus, if $m + 2 \leq (m + n)/2$ (that is, $n \geq m + 4$), the term $-\alpha'\langle t^{m+2} \rangle$ is left after the reduction, which is a contradiction. We have $n = m + 2$ or $n = m + 3$.

First we assume $n = m + 2$, then $\text{red}(\beta\langle t^{2m+1} \rangle - \beta\langle t^{2m} \rangle - \alpha'\langle t^{m+2} \rangle) = \beta\langle t^1 \rangle - \beta\langle t^2 \rangle - \alpha'\langle t^m \rangle$. Considering the ratio of the $(m + 1)$ th and $(m - 1)$ th coefficient, we need $\alpha' = \beta$. We have a contradiction. Next we assume $n = m + 3$, which implies $m \geq 4$. Then $\text{red}(\beta\langle t^{2m+1} \rangle - \beta\langle t^{2m} \rangle - \alpha'\langle t^{m+2} \rangle) = \beta\langle t^2 \rangle - \beta\langle t^3 \rangle - \alpha'\langle t^{m+1} \rangle$. (It is in Case F.) The second coefficient is $\beta \neq 0$, so we have a contradiction.

Case 4: $f = m$. It is not in Case F by Lemma 5.1.

$$F(t) = -\beta\langle t^{m+e} \rangle - \beta\langle t^{m-e} \rangle + \alpha'\langle t^{e+1} \rangle + \alpha'\langle t^{e-1} \rangle - 2(\alpha' - \beta)\langle t^e \rangle + \beta\langle t^{2m} \rangle + 2\beta - \alpha'\langle t^{m+1} \rangle - \alpha'\langle t^{m-1} \rangle + 2(\alpha' - \beta)\langle t^m \rangle.$$

Recall the assumption $e < f = m$. Comparing $F(t)$ and $G(t)$, we have that at least one term in $F(t)$ is equal to or reduced to the first term. By Lemma 5.2(2), $\langle t^{2m} \rangle$ is not reduced to the first term. It does not hold that $\text{red}(\langle t^{m+e} \rangle) = \langle t^1 \rangle$, because it implies $e = n - 1$, which contradicts to $e < f = m$. Thus we need $\text{red}(\langle t^{m-e} \rangle) = \langle t^1 \rangle$, that is, $e = m - 1$.

$$F(t) = \beta\langle t^{2m} \rangle - \beta\langle t^{2m-1} \rangle - \alpha'\langle t^{m+1} \rangle + (3\alpha' - 2\beta)\langle t^m \rangle - (3\alpha' - 2\beta)\langle t^{m-1} \rangle + \alpha'\langle t^{m-2} \rangle - \beta\langle t^1 \rangle + 2\beta.$$

One of the first two terms has to cancel $\alpha' \langle t^{m-2} \rangle$, but this is impossible, because neither $\beta \langle t^{2m} \rangle + \alpha' \langle t^{m-2} \rangle$ nor $-\beta \langle t^{2m-1} \rangle + \alpha' \langle t^{m-2} \rangle$ cancel by Lemma 5.4.

Case 5: $e = 1$ with $m = 3$. Note that $G(t) = \langle t^4 \rangle - 2 \langle t^3 \rangle + \langle t^2 \rangle - 2 \langle t^1 \rangle + 4$.

$$F(t) = \beta \langle t^{3+f} \rangle + \beta \langle t^{3-f} \rangle - \alpha' \langle t^{f+1} \rangle - \alpha' \langle t^{f-1} \rangle + 2(\alpha' - \beta) \langle t^f \rangle - \beta \langle t^4 \rangle + (\alpha' - \beta) \langle t^2 \rangle - 2(\alpha' - \beta) \langle t^1 \rangle + 2\alpha'.$$

The proof is divided into four cases: (i) $f = 2$; (ii) $f = 3$; (iii) $f = 4$ or (iv) $f \geq 5$.

Subcase(5-i): $(e, f) = (1, 2)$.

$$F(t) = \beta \langle t^5 \rangle - \beta \langle t^4 \rangle - \alpha' \langle t^3 \rangle + 3(\alpha' - \beta) \langle t^2 \rangle - 3(\alpha' - \beta) \langle t^1 \rangle + 2\alpha'.$$

The term $\beta \langle t^5 \rangle$ has to be reduced: $\text{red}(\langle t^5 \rangle) = \langle t^{n-2} \rangle$ and $n - 2 \geq m \geq 3$. Considering the ratio of the first coefficient and the constant term, we have $-3(\alpha' - \beta) : 2\alpha' = -2 : 4$, that is, $2\alpha' = 3\beta$. Considering the ratio of the second and the first coefficients, we have a contradiction.

Subcase(5-ii): $(e, f) = (1, 3)$.

$$F(t) = \beta \langle t^6 \rangle - (\alpha' + \beta) \langle t^4 \rangle + 2(\alpha' - \beta) \langle t^3 \rangle - \beta \langle t^2 \rangle - 2(\alpha' - \beta) \langle t^1 \rangle + 2(\alpha' + \beta).$$

The term $\beta \langle t^6 \rangle$ has to be reduced: $\text{red}(\langle t^6 \rangle) = \langle t^{n-3} \rangle$ and $n - 3 \geq 2$. Considering the ratio of the first coefficient and the constant term, we have $-2(\alpha' - \beta) : 2(\alpha' + \beta) = -2 : 4$, that is, $\alpha' = 3\beta$.

$$F(t) = \beta \cdot (\langle t^6 \rangle - 4 \langle t^4 \rangle + 4 \langle t^3 \rangle - \langle t^2 \rangle - 4 \langle t^1 \rangle + 8).$$

In any of the cases $\text{red}(\langle t^6 \rangle) = \langle t^4 \rangle, \langle t^3 \rangle$ or $\langle t^2 \rangle$, it does not hold that $\text{red}(F(t)) = \pm G(t)$. We have a contradiction.

Subcase(5-iii): $(e, f) = (1, 4)$.

$$F(t) = \beta \langle t^7 \rangle - \alpha' \langle t^5 \rangle + (2\alpha' - 3\beta) \langle t^4 \rangle - \alpha' \langle t^3 \rangle + (\alpha' - \beta) \langle t^2 \rangle - (2\alpha' - 3\beta) \langle t^1 \rangle + 2\alpha'.$$

By Lemma 5.4, both terms $\beta \langle t^7 \rangle$ and $-\alpha' \langle t^5 \rangle$ have to be reduced. This is possible only if $(m + n)/2 < 5$, that is, $n < 7$. Since $n \geq m + 2 = 5$ and n is coprime to $m = 3$, we have $n = 5$. Then $\text{red}(\langle t^7 \rangle) = \langle t^1 \rangle$ and $\text{red}(\langle t^5 \rangle) = \langle t^3 \rangle$. Considering the ratio of the fourth and the second coefficients, we have $(2\alpha' - 3\beta) = (\alpha' - \beta)$, that is, $\alpha' = 2\beta$. In this case, we lose the first term: $\text{red}(\beta \langle t^7 \rangle - (2\alpha' - 3\beta) \langle t^1 \rangle) = 0$. We have a contradiction.

Subcase(5-iv): $e = 1$ and $f \geq 5$.

$$F(t) = \beta \langle t^{f+3} \rangle - \alpha' \langle t^{f+1} \rangle + 2(\alpha' - \beta) \langle t^f \rangle - \alpha' \langle t^{f-1} \rangle + \beta \langle t^{f-3} \rangle - \beta \langle t^4 \rangle + (\alpha' - \beta) \langle t^2 \rangle - 2(\alpha' - \beta) \langle t^1 \rangle + 2\alpha'.$$

The f th term has to be canceled. If it is canceled by the $(f + 3)$ th term, then the $(f + 1)$ th term is left after the reduction. If it is canceled by the $(f + 1)$ th term, which is in Case F, we have $\alpha' = 2\beta$, $n = 2f - 2$, and $\text{red}(\langle t^{f+3} \rangle) = \langle t^{f-2} \rangle$ with $f - 2 \geq 3$, thus

$$\text{red}(F(t)) = \beta \cdot (-2\langle t^{f-1} \rangle + \langle t^{f-2} \rangle + \langle t^{f-3} \rangle - \langle t^4 \rangle + \langle t^2 \rangle - 2\langle t^1 \rangle + 4).$$

If $f > 5$, then the top degree is $f - 1 > 4$, so we have a contradiction. Otherwise $f = 5$,

$$\text{red}(F(t)) = \beta \cdot (-3\langle t^4 \rangle + \langle t^3 \rangle + 2\langle t^2 \rangle - 2\langle t^1 \rangle + 4).$$

The ratio of the coefficients is different from that of $G(t)$. We have a contradiction.

In any of the cases, we have a contradiction. The proof of Theorem 1.4(1) is completed.

5.4. Type of the lens space $(A_{m,n}; mn, 0)$. We verify the second term $(=m\bar{m})$ of the lens space $(A_{m,n}; mn, 0) \cong L((m + n)^2, m\bar{m})$ by the Reidemeister torsion. Here we are also interested in the possibility of the transformation between the parameters (m, n) and (p, q) .

We use the notations $\mathfrak{S}(m, n) = \{k_i \mid i = 0, 1, \dots, m + n - 1\}$, u_j ($j = 0, 1, \dots, n$) and w_j ($j = 0, 1, \dots, m$) defined in Section 3.

LEMMA 5.5.

- (1) We set $k'_i = k_i(m + n) - imn$ ($i = 0, 1, \dots, m + n - 1$). Then we have $k'_i + k'_{m+n-1-i} = mn$ and $k'_0 = 0 \leq k'_i \leq k'_{m+n-1} = mn$.
- (2) $\mathfrak{S}(m, n) = \{k'_i \mid i = 0, 1, \dots, m + n - 1\}$.

PROOF. (1) By the definition of k'_i and Proposition 3.8(1),

$$\begin{aligned} k'_i + k'_{m+n-1-i} &= k_i(m + n) - imn + k_{m+n-1-i}(m + n) - (m + n - 1 - i)mn \\ &= (k_i + k_{m+n-1-i})(m + n) - (m + n - 1)mn \\ &= mn(m + n) - (m + n - 1)mn = mn. \end{aligned}$$

By Proposition 3.8(2), every element $k_i \in \mathfrak{S}(m, n)$ is uniquely expressed in the form k_{u_j} or k_{w_j} . It is easy to see that $k'_0 = 0$ and $k'_{m+n-1} = mn$.

(i) The case $i = u_j$ ($j = 1, 2, \dots, n - 1$). Since $k_i = k_{u_j} = jm$ by Proposition 3.8(2) and $\text{gcd}(m, n) = 1$,

$$k'_i = jm(m + n) - \left(\left[\frac{jm}{n} \right] + j \right) mn = mn \left(\frac{jm}{n} - \left[\frac{jm}{n} \right] \right) > 0.$$

Since $k'_i + k'_{m+n-1-i} = mn$, we have $0 < k'_i < mn$.

(ii) The case $i = w_j$ ($j = 1, 2, \dots, m - 1$). Since $k_i = k_{w_j} = jn$ by Proposition 3.8(2) and $\text{gcd}(m, n) = 1$,

$$k'_i = jn(m + n) - \left(\left[\frac{jn}{m} \right] + j \right) mn = mn \left(\frac{jn}{m} - \left[\frac{jn}{m} \right] \right) > 0.$$

Since $k'_i + k'_{m+n-1-i} = mn$, we have $0 < k'_i < mn$. Therefore, we have the result.

(2)(i) The case $i = u_j$ ($j = 1, 2, \dots, n - 1$). By the proof of (1)(i), k'_i is of the form $k'_i = j'm$ for some $j' \in \mathbb{Z}$ and $j' \equiv jm \pmod n$ is uniquely determined.

(ii) The case $i = w_j$ ($j = 1, 2, \dots, m - 1$). By the proof of (1)(ii), k'_i is of the form $k'_i = j'n$ for some $j' \in \mathbb{Z}$ and $j' \equiv jn \pmod m$ is uniquely determined.

From (i), (ii) and (1), the set $\{k'_i \mid i = 0, 1, \dots, m + n - 1\}$ consists of $m + n$ distinct elements, and we have the result. \square

LEMMA 5.6. *For a coprime positive pair (m, n) , the Alexander polynomial $\Delta_{A_{m,n}}(t, x)$ of $A_{m,n}$ satisfies that*

$$\Delta_{A_{m,n}}(T^{m+n}, T^{-mn}) \doteq \frac{(T^{mn} - 1)(T^{m+n} - 1)}{(T^m - 1)(T^n - 1)}.$$

PROOF. As we remarked in Section 1, the first component K_1 of $A_{m,n}$ is the (m, n) -torus knot $T_{m,n}$. By Theorem 3.2 and the Torres formula (Lemma 2.7),

$$\Delta_{A_{m,n}}(t, 1) \doteq \frac{t^{m+n} - 1}{t - 1} \cdot \Delta_{T_{m,n}}(t) \doteq \frac{(t^{mn} - 1)(t^{m+n} - 1)}{(t^m - 1)(t^n - 1)} = \sum_{i=0}^{m+n-1} t^{k_i}.$$

By Theorem 3.2, it holds that

$$\Delta_{A_{m,n}}(T^{m+n}, T^{-mn}) \doteq \sum_{i=0}^{m+n-1} T^{k_i(m+n)-imn} = \sum_{i=0}^{m+n-1} T^{k'_i},$$

and hence we have $\Delta_{A_{m,n}}(T^{m+n}, T^{-mn}) \doteq \Delta_{A_{m,n}}(T, 1)$ by Lemma 5.5(2). \square

PROOF OF $(A_{m,n}; mn, 0) \cong L((m + n)^2, m\bar{n})$. Assuming that $M = (A_{m,n}; mn, 0)$ is a lens space $L(P, Q)$, it is clear that $P = (m + n)^2$ by the first homology. By Lemma 5.6 and the formulae (4.3) and (4.4),

$$\begin{aligned} \tau(M_1) &\doteq \Delta_{A_{m,n}}(T^{m+n}, T^{-mn})(T^{m+n} - 1)^{-1} = \frac{T^{mn} - 1}{(T^m - 1)(T^n - 1)}, \\ \tau^{\psi_{(m+n)^2}}(M) &\doteq (\zeta_{(m+n)^2}^m - 1)^{-1}(\zeta_{(m+n)^2}^n - 1)^{-1}. \end{aligned}$$

Because of the ambiguity of the Reidemeister torsion, the parameters of the lens space are almost decided but the orientation (the choice \pm) is left undecided:

$$M \cong L((m + n)^2, \pm m\bar{n}),$$

where \bar{n} is taken modulo $(m + n)^2$.

Using the knowledge of existence of a lens space surgery $(A_{m,n}; mn, 0) \cong L(p^2, pq - 1)$ in Theorem 1.2, we can determine the sign of $\pm m\bar{n}$ as follows. Suppose that it is $-m\bar{n}$. Then we have $pq - 1 \equiv -m\bar{n} \pmod{p^2}$, and

$$pq \equiv 1 - m\bar{n} \equiv \bar{n}(n - m) \pmod{p^2}.$$

Since $\gcd(pq, p^2) = p = m + n$ is changed to $\gcd(\bar{n}(n - m), (m + n)^2) = m + n$ (we regard \bar{n} as an integral lift), this contradicts that $\gcd(\bar{n}, m + n) = 1$ and $\gcd(n - m, m + n) = \gcd(n - m, 2m) \leq 2m < m + n$. \square

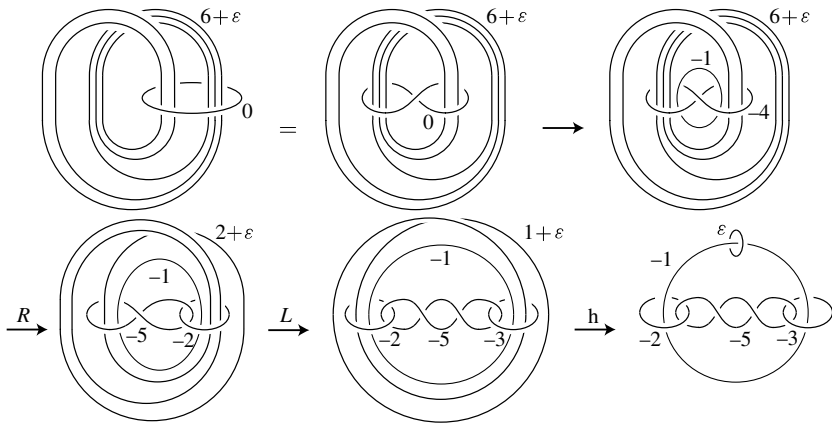


FIGURE 7. $(A_{2,3}; 7, 0)$ is $L(25, 7) I$.

5.5. Type of the lens space $(A_{2,3}; 7, 0)$. Suppose that $(m, n) = (2, 3)$ and $\alpha/\beta = 7$. Then we have

$$\Delta_{A_{2,3}}(t, x) = 1 + t^2x + t^3x^2 + t^4x^3 + t^6x^4$$

by Theorem 3.2, and (4.3) is computed as

$$\tau(M_1) \doteq \Delta_{A_{2,3}}(T^5, T^{-7})(T^5 - 1)^{-1} = \frac{1 + T^3 + T + T^{-1} + T^2}{T^5 - 1} \doteq \frac{1}{T - 1}.$$

Hence we have

$$\tau^{M_{25}}(M) \doteq (\zeta_{25} - 1)^{-1}(\zeta_{25}^7 - 1)^{-1}$$

by (4.4). Assuming that $(A_{2,3}; 7, 0)$ is a lens space, only by our method of Reidemeister-Turaev torsion, we can say that the lens space is homeomorphic to $L(25, 7) \cong L(25, -7)$.

By Kirby calculus in Figures 7 and 8, we have $(A_{2,3}; 7, 0) \cong L(25, 7)$. For the moves in Figure 7, see [Yam3]. By setting $\varepsilon = 0$, we have $(A_{2,3}; 6, 0) \cong L(25, 9)$. In the moves ‘f’ in Figure 8, we used the formula in Figure 9 on a -2 -framed unknot. In the second f move, the union of two (3- and -4 -framed) components are regarded as x , and get unlinked by the positive full twist.

6. Proof of Theorem 1.5 (Lens space surgeries along $B_{p,q}$)

In this section, we prove Theorem 1.5 on lens space surgeries along $B_{p,q}$.

6.1. Alexander polynomial of $B_{p,q}$. The goal of this subsection is the following lemma.

LEMMA 6.1. *The Alexander polynomial of the link $B_{p,q}$ is*

$$\Delta_{B_{p,q}}(t, x) \doteq \frac{t^{pq}x^p - 1}{t^q x - 1},$$

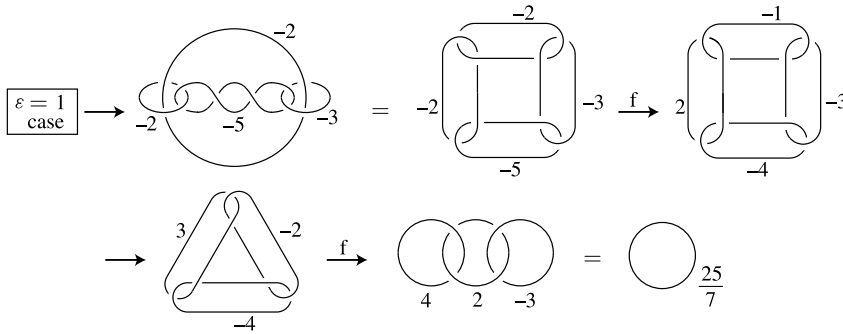


FIGURE 8. $(A_{2,3}; 7, 0)$ is $L(25, 7)$ II.

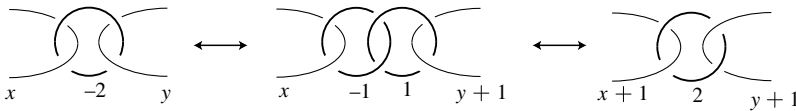


FIGURE 9. Kirby move f.

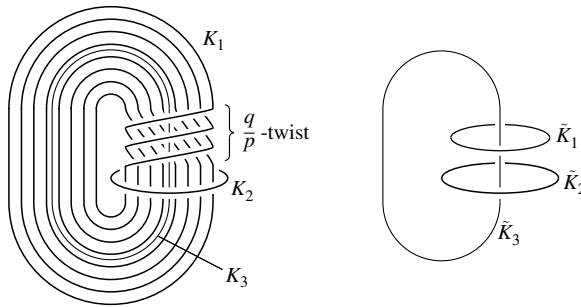


FIGURE 10. Link L (ex. $(p, q) = (8, 3)$) and \tilde{L} .

where t (and x , respectively) is represented by the meridian of the torus knot component K_1 (that of the unknotted component K_2).

PROOF. We add the third component K_3 to $B_{p,q} = K_1 \cup K_2$ such that $H := K_2 \cup K_3$ is a Hopf link and K_1 is isotopic to a simple closed loop in the level torus under the identification of the complement of H and $T^2 \times (-1, 1)$. We set $L := B_{p,q} \cup K_3$ and use the notations defined in Section 2.1. The complement E_L of L is homeomorphic to that $E_{\tilde{L}}$ of the connected sum $\tilde{L} = \tilde{K}_1 \cup \tilde{K}_2 \cup \tilde{K}_3$ of two Hopf links (\tilde{L} is ‘ $B_{0,1} \cup K_3$ ’ in a sense), see Figure 10.

Our strategy of the proof is as follows: first we prove the Alexander polynomial of L by studying the transformation of the homologies from the easier \tilde{L} to L , and use the Torres formula (Lemma 2.7) on the sublink $B_{p,q}$ of L .

We take a homeomorphism $f : E_{\tilde{L}} \rightarrow E_L$ that carries each regular neighborhood of \tilde{K}_i to that of K_i ($i = 1, 2, 3$). We will use a tilde for the notation of each component of \tilde{L} . We denote the meridian-longitude system of \tilde{K}_i by \tilde{m}_i, \tilde{l}_i , and its Alexander polynomial of \tilde{L} by $\Delta_{\tilde{L}}(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)$. In $H_1(E_{\tilde{L}})$ and $H_1(E_L)$, the longitudes are presented by meridians:

$$\begin{aligned}
 [l_1] &= [m_2]^p [m_3]^q, & [l_2] &= [m_1]^p [m_3], & [l_3] &= [m_1]^q [m_2], \\
 [\tilde{l}_1] &= [\tilde{m}_3], & [\tilde{l}_2] &= [\tilde{m}_3], & [\tilde{l}_3] &= [\tilde{m}_1][\tilde{m}_2].
 \end{aligned}
 \tag{6.1}$$

We note that

$$[m_2]^p [l_2]^q = [m_2]^p ([m_1]^p [m_3])^q = [m_1]^{pq} [l_1].
 \tag{6.2}$$

We take integers a, b satisfying $aq - bp = 1$. The isomorphism $f_* : H_1(E_{\tilde{L}}) \rightarrow H_1(E_L)$ induced by f satisfies:

$$\begin{aligned}
 f_*([\tilde{m}_1]) &= [m_1], & f_*([\tilde{l}_1]) &= [m_2]^p [l_2]^q = [m_1]^{pq} [l_1], \\
 f_*([\tilde{m}_2]) &= [m_2]^a [l_2]^b, & f_*([\tilde{l}_2]) &= [m_2]^p [l_2]^q, \\
 f_*([\tilde{m}_3]) &= [m_3]^q [l_3]^p, & f_*([\tilde{l}_3]) &= [m_3]^b [l_3]^a.
 \end{aligned}
 \tag{6.3}$$

Thus, setting $t_i := [m_i]$ and $\tilde{t}_i := [\tilde{m}_i]$ ($i = 1, 2, 3$), by (6.1)–(6.3),

$$f_*(\tilde{t}_1) = t_1, \quad f_*(\tilde{t}_2) = t_1^{pb} t_2^a t_3^b, \quad f_*(\tilde{t}_3) = t_1^{pq} t_2^p t_3^q.
 \tag{6.4}$$

The Alexander polynomial of \tilde{L} is known:

$$\Delta_{\tilde{L}}(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \doteq \tilde{t}_3 - 1.$$

Substituting (6.4) into this, we have the Alexander polynomial of L :

$$\Delta_L(t_1, t_2, t_3) \doteq t_1^{pq} t_2^p t_3^q - 1.$$

Hence, by the Torres formula (Lemma 2.7), we have the Alexander polynomial of $B_{p,q}$:

$$\Delta_{B_{p,q}}(t, x) \doteq \frac{\Delta_L(t, x, 1)}{t^q x - 1} \doteq \frac{t^{pq} x^p - 1}{t^q x - 1}.$$

This concludes the proof. □

6.2. Conditions from the first homology of $(B_{p,q}; r, \mathbf{0})$. We set $M = (B_{p,q}; \alpha/\beta, 0)$. The arguments are parallel to Section 4.1. We let E denote the complement of $B_{p,q}$, and regard $M = E \cup V_1 \cup V_2$, see the notations in Section 2.1. We set $M_1 = E \cup V_1$.

We assume that $\gcd(p, \alpha) = 1$, which is equivalent to the condition for $H_1(M; \mathbb{Z})$ to be finite cyclic. Then we have $H_1(M; \mathbb{Z}) \cong \mathbb{Z}/p^2\beta\mathbb{Z}$.

We determine the first homologies of E, M_1 and M , and represent some elements by the generators. First, $H_1(E)$ is a free abelian group of rank two generated by $[m_1]$ and $[m_2]$:

$$H_1(E) \cong \langle [m_1], [m_2] \mid - \rangle \cong \mathbb{Z}^2.$$

We have

$$[l_1] = [m_2]^p \quad \text{and} \quad [l_2] = [m_1]^p. \tag{6.5}$$

We attach V_1 to E to make M_1 . We take integers γ, δ such that $\alpha\delta - \beta\gamma = -1$, and fix the meridian-longitude system m'_1, l'_1 of the solid torus V_1 . In $H_1(M_1)$,

$$[m'_1] = [m_1]^\alpha [l_1]^\beta = 1, \quad [l'_1] = [m_1]^\gamma [l_1]^\delta$$

and (6.5) also hold. Thus,

$$\begin{aligned} H_1(M_1) &\cong \langle [m_1], [m_2] \mid [m_1]^\alpha [m_2]^{p\beta} = 1 \rangle \\ &\cong \langle T \mid - \rangle \cong \mathbb{Z}, \end{aligned}$$

where $T = [m_1]^\gamma [m_2]^\delta$ by taking integers γ', δ' satisfying $\alpha\delta' - p\beta\gamma' = -1$. By the relations above,

$$\begin{aligned} [m_1] &= [m_1]^{-\alpha\delta' + p\beta\gamma'} \\ &= ([m_1]^\alpha [m_2]^{p\beta})^{-\delta'} ([m_1]^\gamma [m_2]^\delta)^{p\beta} = T^{p\beta}, \\ [m_2] &= [m_2]^{-\alpha\delta' + p\beta\gamma'} \\ &= ([m_1]^\alpha [m_2]^{p\beta})^{\gamma'} ([m_1]^\gamma [m_2]^\delta)^{-\alpha} = T^{-\alpha}, \\ [l'_1] &= [m_1]^\gamma [l_1]^\delta \\ &= [m_1]^\gamma [m_2]^{p\delta} = T^{p\beta\gamma - p\alpha\delta} = T^p. \end{aligned} \tag{6.6}$$

Next, we attach V_2 to M_1 to make M . By (6.5) and (6.6) in $H_1(M)$,

$$[m'_2] = [l_2] = [m_1]^p = T^{p^2\beta} = 1, \quad [l'_2] = [m_2] = T^{-\alpha},$$

and

$$H_1(M) \cong \langle T \mid T^{p^2\beta} = 1 \rangle \cong \mathbb{Z}/p^2\beta\mathbb{Z}.$$

6.3. Proof of Theorem 1.5. By the surgery formula II (Lemma 2.5)(2), Lemma 6.1 and (6.6),

$$\tau(M_1) \doteq \Delta_{B_{p,q}}(T^{p\beta}, T^{-\alpha})(T^p - 1)^{-1} \doteq \frac{T^{p(-\alpha + pq\beta)} - 1}{(T^{-\alpha + pq\beta} - 1)(T^p - 1)}. \tag{6.7}$$

Let $\iota : \mathbb{Z}[H_1(M_1)] \rightarrow \mathbb{Z}[H_1(M)]$ be a ring homomorphism induced from the natural inclusion, $\psi_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ a ring homomorphism such that $\psi_d(T) = \zeta_d$, and set $\rho_d := \psi_d \circ \iota$, where d is a divisor of p . Note that $\gcd(d, \alpha) = 1$. Then by the surgery formula I (Lemma 2.3), (6.6) and (6.7),

$$\tau^{\psi_d}(M) \doteq \tau^{\rho_d}(M_1)(\zeta_d^\alpha - 1)^{-1} \doteq (\alpha - pq\beta)(\zeta_d^\alpha - 1)^{-2}.$$

Suppose that M is a lens space. Then its Reidemeister torsion is equal to that of a lens space (Example 2.6), that is, there exist integers i and j such that $\gcd(p, i) = \gcd(p, j) = 1$ and

$$(\alpha - pq\beta)(\zeta_d^\alpha - 1)^{-2} \doteq (\zeta_d^i - 1)^{-1}(\zeta_d^j - 1)^{-1}.$$

By taking the d -norm (see Section 2.2) of both sides, we have a necessary condition

$$|\alpha - pq\beta| = 1$$

by Lemma 2.9.

Conversely, suppose $|\alpha - pq\beta| = 1$ and set $\varepsilon = \alpha - pq\beta (= \pm 1)$. We can prove that $(B_{p,q}; \alpha/\beta, 0)$ is homeomorphic to the lens space by Kirby–Rolfsen moves [Ro] as follows (see [Mos], see also [IS, Proposition 4.1]): M is the result of $(\alpha/\beta, 0, \infty)$ -surgery along $L = B_{p,q} \cup K_3$. It is homeomorphic to that of $(\varepsilon/\beta, -p/a, -a/p)$ -surgery ($aq - bp = 1$) along \tilde{L} because of the identification between E_L and $E_{\tilde{L}}$ in (6.3), and that of $(-p/a, -(a + \varepsilon p\beta)/p)$ -surgery along the Hopf link by $(-\varepsilon\beta)$ -twist on \hat{K}_1 . Therefore, we have $M \cong L(p^2\beta, \alpha)$. \square

6.4. Other surgeries along $B_{p,q}$. The link $B_{p,q}$ is included in a family of Burde–Murasugi’s links [BM], whose complements admit structures of Seifert fiber spaces. The complement of $B_{p,q} \cup K_3$ in Section 6.1 is homeomorphic to that of the $(3, 3)$ -torus link (and to that of the link in Figure 10 also). Thus we can determine all Dehn surgeries along $B_{p,q}$ by a result of the first author and Shimozawa [KS] on Dehn surgeries along torus links.

Let $(B_{p,q}; \alpha_1/\beta_1, \alpha_2/\beta_2)$ be the result of $(\alpha_1/\beta_1, \alpha_2/\beta_2)$ -surgery along $B_{p,q}$, where α_i and β_i ($i = 1, 2$) are coprime integers with $\beta_i \geq 1$.

THEOREM 6.2. *We set $\varepsilon_1 = \alpha_1 - \beta_1 pq$, $\varepsilon_2 = \alpha_2 q - \beta_2 p$, $P = \alpha_1 \alpha_2 - \beta_1 \beta_2 p^2$, and a and b as coprime integers such that $aq - bp = 1$. Then the result of surgery $M = (B_{p,q}; \alpha_1/\beta_1, \alpha_2/\beta_2)$ with $p \geq 2$ is as follows.*

- (1) *If $\varepsilon_1 \varepsilon_2 \neq 0$, then M is a Seifert fiber space over S^2 with at most three singular fibers whose multiplicities are $|\varepsilon_1|$, $|\varepsilon_2|$ and p .*
- (2) *Suppose $\varepsilon_1 \varepsilon_2 \neq 0$. Then M is a lens space if and only if $|\varepsilon_1| = 1$ or $|\varepsilon_2| = 1$. The resulting lens spaces are*

$$L(P, -\beta_2 \varepsilon_1 - \beta_1 \varepsilon_2 q) \cong L(P, -\alpha_1 \beta_2 - \alpha_2 \beta_1 q^2 + 2\beta_1 \beta_2 pq)$$

in both cases.

- (3) (i) *If $\varepsilon_1 = 0$ (that is, $\alpha_1/\beta_1 = pq$), then M is a connected sum of lens spaces*

$$L(\varepsilon_2, \alpha_2 b - \beta_2 a) \# L(p, -q).$$

- (ii) *If $\varepsilon_2 = 0$ (that is, $\alpha_2/\beta_2 = p/q$), then M is a connected sum of lens spaces*

$$L(\varepsilon_1, -\beta_1) \# L(p, -q).$$

Here we regard $L(\pm 1, Q) (\cong S^3)$ and $L(0, Q) (\cong S^1 \times S^2)$ for any Q as lens spaces.

PROOF. Let $L_{p,q}$ be the link L in the proof of Lemma 6.1. Then $L_{1,1}$ is the $(3, 3)$ -torus link. We set $L_{1,1} = \hat{L} = \hat{K}_1 \cup \hat{K}_2 \cup \hat{K}_3$. Let \hat{m}_i and \hat{l}_i be a meridian and a longitude of \hat{K}_i ($i = 1, 2, 3$), respectively.

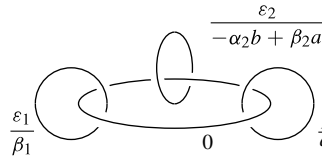


FIGURE 11. Seifert fiber space $(B_{p,q}; \alpha_1/\beta_1, \alpha_2/\beta_2)$.

Let $h : E_L \rightarrow E_{\hat{L}}$ be a homeomorphism inducing an isomorphism $h_* : H_1(E_L) \rightarrow H_1(E_{\hat{L}})$ such that

$$\begin{aligned} h_*([m_1]) &= [\hat{m}_1], & h_*([l_1]) &= [\hat{m}_1]^{-pq+1}[\hat{l}_1], \\ h_*([m_2]) &= [\hat{m}_2]^{q-b}[\hat{l}_2]^{-b}, & h_*([l_2]) &= [\hat{m}_2]^{-p+a}[\hat{l}_2]^a, \\ h_*([m_3]) &= [\hat{m}_3]^a[\hat{l}_3]^{-p+a}, & h_*([l_3]) &= [\hat{m}_3]^{-b}[\hat{l}_3]^{q-b}. \end{aligned}$$

By the relations,

$$\begin{aligned} h_*([m_1]^{\alpha_1} [l_1]^{\beta_1}) &= [\hat{m}_1]^{\alpha_1 - \beta_1(pq-1)}[\hat{l}_1]^{\beta_1}, \\ h_*([m_2]^{\alpha_2} [l_2]^{\beta_2}) &= [\hat{m}_2]^{\alpha_2(q-b) - \beta_2(p-a)}[\hat{l}_2]^{-\alpha_2 b + \beta_2 a}, \\ h_*([m_3]) &= [\hat{m}_3]^a[\hat{l}_3]^{-p+a}, \end{aligned}$$

and hence M is the result of $(\{\alpha_1 - \beta_1(pq - 1)\}/\beta_1, \{\alpha_2(q - b) - \beta_2(p - a)\}/(-\alpha_2 b + \beta_2 a), a/(-p + a))$ -surgery along the $(3, 3)$ -torus link. By [KS, Theorem 3.3(3)], if $\epsilon_1 \epsilon_2 \neq 0$, then M is a Seifert fiber space over S^2 with at most three singular fibers whose indices are

$$\left(-1; \frac{\beta_1}{\epsilon_1}, \frac{-\alpha_2 b + \beta_2 a}{\epsilon_2}, \frac{-p + a}{p}\right) = \left(0; \frac{\beta_1}{\epsilon_1}, \frac{-\alpha_2 b + \beta_2 a}{\epsilon_2}, \frac{a}{p}\right),$$

see Figure 11. Other cases are obtained from it. □

REMARK 6.3. The convention on lens spaces in [KS] is different from that in the present paper. Theorem 1.5 corresponds to the case $\alpha_2 = 0$ and $\beta_2 = 1$ in Theorem 6.2(2), and we can deduce Theorem 1.5(2) from Theorem 6.2(2) after some deformations.

7. Lens space surgeries along $A_{m,n}$ other than $(r, 0)$ -surgery

In contrast to the case of $B_{p,q}$ in Section 6.4, to determine all Dehn surgeries along $A_{m,n}$ is a hard problem. Related to Conjecture 1.11, in Sections 7.1 and 7.2, we compute the Reidemeister torsions of M in the cases: (1) $\alpha_1/\beta_1 = mn$; and (2) $\alpha_1/\beta_1 = 7$ for $(m, n) = (2, 3)$, respectively.

7.1. The case $(A_{m,n}; mn, r)$. For $A_{m,n} = K_1 \cup K_2$, we compute the Reidemeister torsions of $M = (A_{m,n}; mn, r)$. We set $r = \alpha/\beta$, where α and β are coprime integers with $\beta \geq 1$, and $P = (m + n)^2 \beta - mn\alpha$. Let $d(\geq 2)$ be a divisor of P .

We use the notations E , M_1 and M defined in Section 2.1. By the similar way to Section 4.1, we have two lemmas.

LEMMA 7.1. *We take integers u and v as $(m + n)u - mnv = 1$, and set $T = [m_1]^u [m_2]^v$ in $H_1(M_1)$. Then:*

- (1) T is a generator of $H_1(M_1)$;
- (2) $[m_1] = T^{m+n}$ and $[m_2] = T^{-mn}$.

LEMMA 7.2. *Let T be the generator of $H_1(M)$ which is induced by that of $H_1(M_1)$ in Lemma 7.1. Let $\psi_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ be a ring homomorphism such that $\psi_d(T) = \zeta_d$. Then the core of V_2 (that is, $[l_2]$) is mapped to $\zeta_d^{(m+n)^2\delta - mn\gamma}$ by ψ_d , where γ and δ are integers such that $\alpha\delta - \beta\gamma = -1$.*

LEMMA 7.3. *The integer $(m + n)^2\delta - mn\gamma$ is coprime to P .*

PROOF. We have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -mn \\ (m + n)^2 \end{pmatrix} = \begin{pmatrix} (m + n)^2\beta - mn\alpha \\ (m + n)^2\delta - mn\gamma \end{pmatrix}. \tag{7.1}$$

Since $\alpha\delta - \beta\gamma = -1$, the matrix is invertible over \mathbb{Z} , thus we have

$$\gcd((m + n)^2\beta - mn\alpha, (m + n)^2\delta - mn\gamma) = \gcd((m + n)^2, mn) = 1.$$

This concludes the proof. □

THEOREM 7.4. *The Reidemeister torsion of M related with ζ_d is as follows:*

$$\tau^{\psi_d}(M) \doteq \frac{\zeta_d^{mn} - 1}{(\zeta_d^m - 1)(\zeta_d^n - 1)(\zeta_d^{(m+n)^2\delta - mn\gamma} - 1)}.$$

PROOF. We use the surgery formula II (Lemma 2.5). By Lemmas 7.2 and 7.3, $\zeta_d^{(m+n)^2\delta - mn\gamma}$ is a primitive d th root of unity. By the Alexander polynomial of $A_{m,n}$ in Theorem 3.2, Lemmas 7.1, 7.2 and 5.6, we have the result. □

By the Franz lemma (Lemma 2.10), $\tau^{\psi_d}(M)$ is equal to the Reidemeister torsion of a lens space if and only if

$$m \equiv \pm 1, \quad n \equiv \pm 1 \quad \text{or} \quad (m + n)^2\delta - mn\gamma \equiv \pm mn \pmod{P}.$$

LEMMA 7.5.

- (1) *If $\beta = 1$, then M has the same Reidemeister torsions as a lens space $L(P, \pm m\bar{n})$, where $n\bar{n} \equiv 1 \pmod{P}$.*
- (2) *$(m + n)^2\delta - mn\gamma \equiv \pm mn \pmod{P}$ is equivalent to $\beta \equiv \pm 1 \pmod{P}$.*
- (3) *Suppose that $\alpha < 0$, and M has the same Reidemeister torsions as a lens space. Then we have $\beta = 1$.*

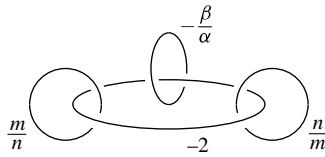


FIGURE 12. Seifert manifold $(A_{m,n}; mn, \alpha/\beta)$.

PROOF. (1) If $\beta = 1$, then we can take $\gamma = 1, \delta = 0$, thus $(m + n)^2\delta - mn\gamma \equiv \pm mn \pmod P$.

(2) Suppose that $(m + n)^2\delta - mn\gamma \equiv \pm mn \pmod P$. Then by (7.1), it holds that

$$\begin{pmatrix} -mn \\ (m + n)^2 \end{pmatrix} \equiv \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \begin{pmatrix} 0 \\ \pm mn \end{pmatrix} = \begin{pmatrix} \pm\beta mn \\ \mp\alpha mn \end{pmatrix} \pmod P,$$

hence $\beta \equiv \pm 1 \pmod P$. Conversely, suppose that $\beta \equiv \pm 1 \pmod P$. Then we can take $\gamma \equiv \pm 1, \delta \equiv 0 \pmod P$, thus $(m + n)^2\delta - mn\gamma \equiv \pm mn \pmod P$.

(3) $P = (m + n)^2\beta - mn\alpha = (m + n)^2\beta + mn|\alpha|$. By the assumption $2 \leq m < n$, it holds that $m \not\equiv \pm 1$ and $n \not\equiv \pm 1 \pmod P$. Since $0 < \beta < P - mn|\alpha|$, we have $\beta = 1$. \square

Extending the Kirby calculus in [Yam3], one can prove the following lemma.

LEMMA 7.6 (An extension of [Yam3]). *The surgered manifold $(A_{m,n}; mn, \alpha/\beta)$ is a Seifert fiber space over S^2 with at most three singular fibers whose indices are*

$$\left(-2; \frac{n}{m}, \frac{m}{n}, -\frac{\alpha}{\beta}\right),$$

see Figure 12.

As a corollary, we can prove some lens space surgeries.

COROLLARY 7.7. *The surgered manifold $(A_{m,n}; mn, \alpha/\beta)$ is a lens space if and only if $\beta = 1$ (thus $P = (m + n)^2 - \alpha mn$) and the lens space is $L(P, m\bar{n})$, where $n\bar{n} \equiv 1 \pmod P$.*

PROOF (Using Lemma 7.6). We apply [KS, Lemma 2.2] to the description in Figure 12 in the case $\beta = 1$. Then the manifold is $L(P, Q)$ with

$$P = (m + n)^2 - \alpha mn, \quad Q = mx + \{m(\alpha - 2) - n\}y,$$

where x, y are coprime integers satisfying $nx + my = 1$, and it holds that

$$\begin{aligned} nQ &= mnx + \{mn(\alpha - 2) - n^2\}y \\ &\equiv mnx + \{(m + n)^2 - 2mn - n^2\}y \pmod P \\ &= mnx + m^2y \\ &= m(nx + my) \\ &= m. \end{aligned}$$

\square

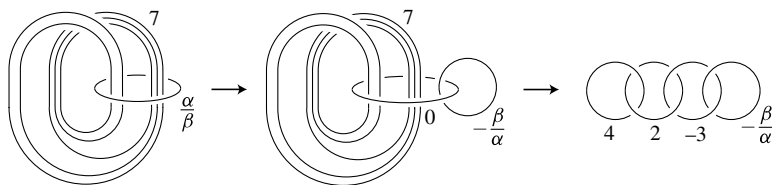


FIGURE 13. Lens space $(A_{2,3}; 7, \alpha/\beta)$.

7.2. The case $(A_{2,3}; 7, r)$. We set $(m, n) = (2, 3)$, $r = \alpha/\beta$, where α and β are coprime integers with $\beta \geq 1$ and $P = 25\beta - 7\alpha$. Let $d(\geq 2)$ be a divisor of P . We compute the Reidemeister torsions of $M = (A_{2,3}; 7, r)$.

THEOREM 7.8. *Let $\psi_d : \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$ be a ring homomorphism which maps a generator of $H_1(M)$ to ζ_d . Then we have*

$$\tau^{\psi_d}(M) \doteq (\zeta_d - 1)^{-1} (\zeta_d^{7\gamma - 25\delta} - 1)^{-1},$$

where integers γ and δ satisfy $\alpha\delta - \beta\gamma = -1$.

In particular, M has the same Reidemeister torsions as a lens space $L(P, \pm(7\gamma - 25\delta))$.

We can verify the lens space surgeries.

LEMMA 7.9. *The surgered manifold $(A_{2,3}; 7, \alpha/\beta)$ is a lens space $L(25\beta - 7\alpha, 2\alpha - 7\beta)$.*

PROOF. We modify the Kirby calculus in Figures 7 and 8 as in Figure 13. □

8. Final remarks

In this section, we give some remarks concerning our results.

8.1. Alternative proof for Theorem 1.4(1). At the beginning of our study, the authors did not know whether $(A_{m,n}; \emptyset, 0)$ is proved to be non-Seifert, and our method by Reidemeister-Turaev torsion works without the knowledge. In fact, we are interested in the condition on the Alexander polynomial for a link to admit a lens space surgery.

In the preparation of this paper, K. Motegi informed us that $(A_{m,n}; \emptyset, 0)$ is proved to be non-Seifert. We give an alternative rough proof of Theorem 1.4(1) by assuming $(A_{m,n}; \emptyset, 0)$ is non-Seifert and by using the cyclic surgery theorem [CGLS].

PROOF OF THEOREM 1.4(1) (Using the results in [DMM1, DMM2, CGLS]). We set $M = (A_{m,n}; r, 0)$ ($r \in \mathbb{Q}$). By the cyclic surgery theorem [CGLS], only $r = mn - 1$ or $mn + 1$ can be a solution of Problem 1.1 other than $r = mn$.

Case 1: $r = mn - 1$.

This case does not occur by Lemma 4.5(2).

Case 2: $r = mn + 1$.

By Lemma 4.3,

$$\tau_d^{\psi'}(M) \doteq (\xi^{m-1} - 1)(\xi^{m+1} - 1)(\xi - 1)^{-2}(\xi^m - 1)^{-2},$$

where ξ is a primitive d th root of unity ($d \mid m + n$ and $d \geq 2$). If $M = (A_{m,n}; mn + 1, 0)$ is a lens space, there exist integers i and j such that $\gcd(i, m + n) = \gcd(j, m + n) = 1$, as in (4.7):

$$(\xi^{m-1} - 1)(\xi^{m+1} - 1)(\xi^i - 1)(\xi^j - 1) \doteq (\xi - 1)^2(\xi^m - 1)^2.$$

Similarly to the proof of Lemma 4.10, using the Franz lemma [Fz] (Lemma 2.10),

$$\{\pm(m - 1), \pm(m + 1), \pm i, \pm j \bmod m + n\} = \{\pm 1, \pm 1, \pm m, \pm m \bmod m + n\}$$

as multiple sets. It has a unique solution $(m, n) = (2, 3)$ with $(i, j) = (1, 3)$. Thus we have $r = 7$. □

8.2. Algebraic generalization. Our main theorems can be extended to the cases of 2-component links in homology 3-spheres with the same Alexander polynomials as $A_{m,n}$ and $B_{p,q}$.

THEOREM 8.1. *Let $L_{m,n}$ be a 2-component link in a homology 3-sphere with the same Alexander polynomial as $A_{m,n}$ (Theorem 3.2). If $(L_{m,n}; r, 0)$ is a lens space, then we have $r = mn$, or $r = 7$ for $(m, n) = (2, 3)$. Moreover, we have $(L_{m,n}; mn, 0) \cong L((m + n)^2, \pm m\bar{n})$ or $(L_{2,3}; 7, 0) \cong L(25, 7)$ respectively, where $n\bar{n} \equiv 1 \pmod{m + n}^2$.*

THEOREM 8.2. *Let $L'_{p,q}$ be a 2-component link in a homology 3-sphere with the same Alexander polynomial as $B_{p,q}$ (Lemma 6.1). If $(L'_{p,q}; \alpha/\beta, 0)$ is a lens space, then we have $|\alpha - pq\beta| = 1$. Furthermore, in this case, we have $(L'_{p,q}; \alpha/\beta, 0) \cong L(p^2\beta, \pm\alpha)$.*

We remark that the converses of the theorems above do not always hold in general.

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