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Nagoya Math. J.
Vol. 99 (1985), 159–172

IDEALS WITH SLIDING DEPTH

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Introduction

We study here a class of ideals of a Cohen-Macaulay ring $\{R, \mathfrak{m}\}$ somewhat intermediate between complete intersections and general Cohen-Macaulay ideals. Its definition, while a bit technical, rapidly leads to the development of its elementary properties. Let $I = (x_1, \cdots, x_n) = (\boldsymbol{x})$ be an ideal of R and denote by $H_*(\boldsymbol{x})$ the homology of the ordinary Koszul complex $K_*(\boldsymbol{x})$ built on the sequence $\boldsymbol{x}$. It often occurs that the depth of the module H_i, $i > 0$, increases with i (as usual, we set depth $(0) = \infty$). We shall say that I satisfies *sliding depth* if

$$\text{(SD)} \qquad \operatorname{depth} H_i(x) \geq \dim(R) - n + i, \qquad i \geq 0.$$

This definition depends solely on the number of elements in the sequence $\boldsymbol{x}$. This property localizes (cf. [9]) and is an invariant of even linkage (cf. [10]).

An extreme case of this property is given by a complete intersection. A more general instance of it is that where all the modules H_i are Cohen-Macaulay, a situation that was dubbed *strongly* Cohen-Macaulay ideals (cf. [11]).

These ideals have appeared earlier in two settings:

(i) The investigation of arithmetical properties of the Rees algebra of I

$$S = \mathscr{R}(I) = \oplus I^s,$$

and of the associated graded ring

$$G = \operatorname{gr}_I(R) = \oplus I^s/I^{s+1}.$$

It was shown in [7], [8] and [16] that for ideals satisfying (SD) and such that for each prime P containing I, height $(P) = \operatorname{ht}(I) \geq v(I_p) =$ minimum number of generators of the localization I_p, both S and G are Cohen-Macaulay. In addition, if R is a Gorenstein ring, G will be Goren-

Received June 27, 1984.
(*) Partially supported by NSF grant DMS-8301870.

stein precisely when I is strongly Cohen-Macaulay ([9, (6.5)]).

(ii) The other context is that of a generalization and corrections by Huneke ([11]) of a result of Artin-Nagata on residual Cohen-Macaulayness ([1]), i.e. conditions under which for a subideal $J \subset I$, $J : I$ is Cohen-Macaulay, $(J : I) \cap I = J$ and $\operatorname{ht}((J : I) + I) > \operatorname{ht}(I)$. It connects with the notion of linkage—when J is a complete intersection—by requiring that I be a strongly Cohen-Macaulay ideal. In turn our extension shows that the assertions of the theorem are intertwined with the sliding depth condition.

Our goals here are the following:

(i) In Section 1 we demark more precisely the distinction between strongly Cohen-Macaulay ideals and ideals with (SD). This is more conveniently done if I is generated by a d-sequence—for ideals with (SD) this is essentially equivalent to requiring that $v(I_p) \leq \operatorname{ht}(P)$, for prime ideals $P \supset I$. If one further assumes that R is Gorenstein, and $v(I_p) < \operatorname{ht}(P) - 1$ for primes with $\operatorname{ht}(P) > \operatorname{ht}(I) + 2$, then I is strongly Cohen-Macaulay. This was proved by Huneke ([11]) using the duality of [6]. We reinforce this result by replacing the last inequality by $v(I_p) < \operatorname{ht}(P)$. It still follows from [6] but depends on some quirks of the Koszul complex. The next case—i.e. $v(I_p) \leq \operatorname{ht}(P)$—is however critical. What precisely overcomes it is not well-known. Some conditions we impose involve the conormal module I/I^2.

(ii) In Section 2 we discuss examples of Cohen-Macaulay prime ideals of codimension three in a regular local ring R, that have (SD), but are not strongly Cohen-Macaulay. It will rely on properties of the divisor class group of R/I. In particular we shall see that if I is the ideal generated by the $n - 1$ sized minors of a generic, symmetric, $n \times n$ matrix then I is syzygetic (cf. [7]). For $n = 3$ we have the desired example. Its Rees algebra $\mathscr{R}(I)$ is even integrally closed.

We also record an extension of a result of Serre asserting that Gorenstein ideals of codimension two are complete intersections. More generally, one can show that if I is a Cohen-Macaulay of codimension two, then the canonical module of R/I cannot have 2-torsion.

(iii) In Section 3 the generalization of Huneke's theorem to ideals with sliding depth is given. Some of its elements may be used to construct ideals with sliding depth of a fixed height and various projective dimensions.

We thank Craig Huneke and Aron Simis for several conversations, and also Giuseppe Valla for raising one of our motivating questions.

§ 1. Strongly Cohen-Macaulay ideals

The rings considered throughout will be Noetherian, commutative with an identity. For notation, terminology and basic results—especially those dealing with Koszul complexes and Cohen-Macaulay rings—we shall use [13].

It is convenient to rephrase the condition (SD) for an ideal I in terms of the depths of the cycles and boundaries of the associated Koszul complex. Assume that R is a Cohen-Macaulay local ring of dimension d and that I is generated by the sequence $x = \{x_1, \cdots, x_n\}$; put $g = \mathrm{ht}(I)$. Denote by Z_i and B_i the modules of cycles and boundaries of the associated Koszul complex K_*. If one uses the defining exact sequences

$$0 \longrightarrow Z_{i+1} \longrightarrow K_{i+1} \longrightarrow B_i \longrightarrow 0$$
$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0$$

the depth conditions (SD) and (SCM = strongly Cohen-Macaulay) translate as follows:

$$\mathrm{depth}\,(Z_i) \geq \begin{cases} \min\{d, d-n+i+1\}, & \text{for (SD)} \\ \min\{d, d-g+2\}, & \text{for (SCM)}. \end{cases}$$

We look at the case $i = n - g$ to examine the role of duality. From now on we assume that R is a Gorenstein ring.

Proposition 1.1. *Let R be a Gorenstein local ring of dimension d and I be a Cohen-Macaulay ideal of height g generated by n elements. Then* $\mathrm{depth}\,(Z_{n-g}) \geq \min\{d, d-g+2\}$.

Proof. If $g = 0$, $Z_n = 0 : I = \mathrm{Hom}_R(R/I, R)$ is Cohen-Macaulay since R/I is a Cohen-Macaulay module and R is Gorenstein.

If $g = 1$, the exact sequence

$$0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

yields ($^*E$ denotes the R-dual $\mathrm{Hom}\,(E, R)$):

$$0 \longrightarrow Z^*_{n-1} \longrightarrow B^*_{n-1} \longrightarrow \mathrm{Ext}^1(H_{n-1}, R) \longrightarrow \mathrm{Ext}^1(Z_{n-1}, R) \longrightarrow 0$$

Since $B^*_{n-1} = R$ and $\mathrm{Ext}^1(H_{n-1}, R) = R/I$ by duality, we get an exact sequence

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$$0 \longrightarrow R/Z_{n-1}^* \xrightarrow{\phi} R/I \longrightarrow \operatorname{Ext}^1(Z_{n-1}, R) \longrightarrow 0\,.$$

Since Z_{n-1} is a second syzygy module, the last module has support at primes of height greater than two. In the identification $B_{n-1}^* = R$, ϕ maps Z_{n-1}^* maps exactly onto I: To see this it suffices to localize at any prime P (necessarily of height 1) associated to either Z_{n-1}^* or I. Thus ϕ is essentially the multiplication of R/I into itself via a regular element of the Cohen-Macaulay ring R/I. By the remark above on the support of $\operatorname{Ext}^1(Z_{n-1}, R)$, ϕ is an isomorphism.

If $g > 1$, consider the sequence

$$0 \longrightarrow B_{n-g} \longrightarrow Z_{n-g} \longrightarrow H_{n-g} \longrightarrow 0\,.$$

Here B_{n-g} has depth $d - g + 1$ while H_{n-g} has depth $d - g$ being the canonical module of R/I. The exact sequence says that depth $(Z_{n-g}) \geq d - g$. We now test the vanishing of the modules $\operatorname{Ext}^i(Z_{n-g}, R)$ for $i = g,\ g - 1$. From above we obtain the homology sequence

$$\begin{aligned}&\operatorname{Ext}^{g-1}(H_{n-g}, R) \longrightarrow \operatorname{Ext}^{g-1}(Z_{n-g}, R) \longrightarrow \operatorname{Ext}^{g-1}(B_{n-g}, R) \longrightarrow \\ &\operatorname{Ext}^{g}(H_{n-g}, R) \longrightarrow \operatorname{Ext}^{g}(Z_{n-g}, R) \longrightarrow \operatorname{Ext}^{g}(B_{n-g}, R)\,.\end{aligned}$$

Here $\operatorname{Ext}^{g-1}(B_{n-g}, R) = R/I$ from the exactness of the tail of the Koszul complex. On the other hand $\operatorname{Ext}^g(B_{n-g}, R) = \operatorname{Ext}^{g-1}(H_{n-g}, R) = 0$, while $\operatorname{Ext}^g(H_{n-g}, R) = R/I$ since R is a Gorenstein ring. Thus we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^{g-1}(Z_{n-g}, R) \longrightarrow R/I \xrightarrow{\phi} R/I \longrightarrow \operatorname{Ext}^{g}(Z_{n-g}, R) \longrightarrow 0\,.$$

Localizing at primes of height g and $g + 1$, we get that ϕ is an isomorphism since Z_{n-g} is a second syzygy module and the desired assertion follows. □

COROLLARY 1.2 (see [2]). *Let I be a Cohen-Macaulay ideal of height g that can be generated by $n = g + 2$ elements. Then I is strongly Cohen-Macaulay.*

Remark. If $n = g + 3$ even the condition (SD) may fail to hold; see Section 2.

COROLLARY 1.3. *Let I be an ideal satisfying* (SD). *If R/I satisfies Serre's condition S_2, then I is Cohen-Macaulay.*

Proof. (SD) implies that the canonical module of R/I, H_{n-g}, is Cohen-Macaulay. But the argument above shows that $R/I = \operatorname{Ext}^g(H_{n-g}, R)$ given the condition S_2. □

The main result of this section is the following criterion for (SCM).

THEOREM 1.4. *Let R be a Gorenstein local ring and let I be a Cohen-Macaulay ideal. If I satisfies* (SD) *and* $\nu(I_p) \leq \max\{\mathrm{ht}(I), \mathrm{ht}(P) - 1\}$ *for each prime ideal $P \supset I$, then I is strongly Cohen-Macaulay.*

Proof. Since (SD) and the other conditions localize (cf. [9]), we may assume that I is (SCM) on the punctured spectrum of R. By adding a set of indeterminates to R and to I, we may assume the height g of I is larger than $n - g + 1$, $n =$ minimum number of generators of the new ideal. This clearly leaves the Koszul homology and (SD) unchanged. The net effect however is that we have a Koszul complex K_* whose acyclic tail is longer than the remainder of the complex.

(i) In the conditions above, H_{n-g-i} is the H_{n-g}-dual of H_i [11]; to use the theorem of duality of [6]—see also [11]—one has to verify that the left hand side of the inequality

$$\begin{aligned}\mathrm{depth}\,(H_i) + \mathrm{depth}\,(H_{n-g-i}) &\geq (d - n + i) + (d - n + n - g - i)\\ &= (d - g) + (d - n)\end{aligned}$$

exceeds $(d - g) + 1$. If, therefore, $n < d - 1$, it will follow that each H_i is Cohen-Macaulay.

(ii) To set the tone of the argument in case $n = d - 1$, we examine H_1. Here $\mathrm{depth}\,(H_{n-g-1}) \geq d - g - 1$ and $\mathrm{depth}\,(H_1) \geq 2$; we will strengthen the first inequality. Suppose it cannot be done and consider the exact sequence

$$0 \longrightarrow B_{n-g-1} \longrightarrow Z_{n-g-1} \longrightarrow H_{n-g-1} \longrightarrow 0\,.$$

By (1.1) $\mathrm{depth}\,(B_{n-g-1}) \geq d - g + 1$ so that if $\mathrm{depth}\,(H_{n-g-1}) = d - g - 1$ then $\mathrm{depth}\,(Z_{n-g-1}) = d - g - 1$ as well. It will follow that $\mathrm{depth}\,(B_{n-g-2}) = d - g - 2$. A similar sequence for $i = n - g - 2$, again by duality, says that $\mathrm{depth}\,(H_{n-g-2}) = d - g$ or $d - g - 2$. In either case we get that $\mathrm{depth}\,(Z_{n-g-2}) = d - g - 2$. We repeat this argument until we get

$$\mathrm{depth}\,(B_1) = \mathrm{depth}\,(B_{n-g-(n-g-1)}) = d - g - (n - g - 1) = d - n + 1 = 2\,.$$

Since $\mathrm{depth}\,(Z_1) = d - g + 2 > 2$, we get a contradiction.

(iii) To set up the induction routine, suppose we have shown that H_k and H_{n-g-k} are Cohen-Macaulay; we show that $\mathrm{depth}\,(Z_{n-g-k}) \geq d - g + 2$.

The argument is similar to (1.1). We have the exact homology sequence

$$0 \longrightarrow \mathrm{Ext}^{g-1}(Z_{n-g-k}, R) \longrightarrow \mathrm{Ext}^{g-1}(B_{n-g-k}, R) \longrightarrow \mathrm{Ext}^{g}(H_{n-g-k}, R) \longrightarrow \mathrm{Ext}^{g}(Z_{n-g-k}, R) \longrightarrow 0\,,$$

since $\mathrm{depth}(B_{n-g-k}) \geq d - g + 1$, by induction. But we also have the isomorphisms $\mathrm{Ext}^{g-1}(B_{n-g-k}, R) = \mathrm{Ext}^{g-2}(Z_{n-g-k+1}, R) = \mathrm{Ext}^{g-2}(B_{n-g-k+1}, R) = \cdots = \mathrm{Ext}^{g-k-1}(B_{n-g}, R)$. (This is possible by our 'increase' in g.) This last module however, from the self-duality in the Koszul complex, is nothing but H_k. Since $\mathrm{Ext}^{g}(H_{n-g-k}, R)$ is also a Cohen-Macaulay module, as in (1.1) we conclude that $\mathrm{depth}(Z_{n-g-k}) \geq d - g + 2$. □

It is clear that one only needs this strengthened (SD) to hold in the lower half range of i. In this regard we have

COROLLARY 1.5. *Let I be a Cohen-Macaulay ideal with* (SD). *If I is a syzygetic ideal and I/I^2 is a torsion-free R/I-module then H_1 is a Cohen-Macaulay module.*

Proof. The syzygetic condition on I (cf. [15]) simply means that the natural sequence

$$H_1 \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

is exact on the left. In such case H_1 satisfies S_2, and the argument above goes through. □

Remark. If R is not a Gorenstein ring (1.5) does not always hold.

§ 2. Codimension three

We exhibit examples of Cohen-Macaulay ideals of height 3 in regular local rings, generated by d-sequences, satisfying (SD) but not (SCM). Since it is known that ideals in the linkage class of a complete intersection are (SCM) [10], we look at non-Gorenstein ideals. For an ideal I with a presentation

$$0 \longrightarrow Z \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

one has the following exact sequences

$$0 \longrightarrow \mathrm{Tor}_1(I, R/I) \longrightarrow Z/IZ \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

and

$$\Lambda^2 I \longrightarrow \mathrm{Tor}_1(I, R/I) \longrightarrow \delta(I) \longrightarrow 0$$

https://doi.org/10.1017/S0027763000021553 Published online by Cambridge University Press

where $\delta(I)$ is defined by the associated exact sequence

$$0 \longrightarrow \delta(I) \longrightarrow H_1 \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0\,,$$

cf. [15]. As remarked, I is called syzygetic if $\delta(I) = 0$. If 2 is invertible in R, we can further add that $\mathrm{Tor}_1(I, R/I) = \Lambda^2 I \oplus \delta(I)$.

THEOREM 2.1. *Let R be a regular local ring of dimension at least 6 with $2R = R$ and let I be a Cohen-Macaulay ideal of height 3. Denote by W the canonical module of R/I and let $W^* = \mathrm{Hom}_{R/I}(W, R/I)$. Assume that I is syzygetic on the punctured spectrum of R. If W^* has depth at least 3, then I is syzygetic.*

Proof. Let

$$0 \longrightarrow R^p \xrightarrow{\ \psi\ } R^m \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

be a minimal resolution of I. By assumption $\delta(I)$ is a module of finite length so that we only have to show that $\mathrm{Tor}_1(I, R/I)$ has depth at least 1. Denote by Z the first-order syzygies of I. We have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_2(I, R/I) \longrightarrow (R/I)^p \xrightarrow{\ \psi\otimes R/I\ } (R/I)^m \longrightarrow Z/IZ \longrightarrow 0\,.$$

On the other hand, $W = \mathrm{coker}\,(\psi^*) = \mathrm{coker}\,(\psi^*\otimes(R/I))$, so that $\mathrm{Tor}_2(I, R/I)$ is identified to W^* (see [4, supplement] for general comparisons between these two modules). It follows that Z/IZ—and $\mathrm{Tor}_1(I, R/I)$ along with it—has the required depth. □

For the next two corollaries the hypothesis $2R = R$ is in force.

COROLLARY 2.2. *Let I be the ideal generated by the $(n-1)$-sized $(n>1)$ minors of a generic, symmetric $n \times n$ matrix. Then I is syzygetic.*

Proof. The assumption is that $R = k[[x_{ij}]]$, where k = field and x_{ij}, $1 \leq i, j \leq n$, are indeterminates and the entries of a symmetric matrix $= \phi$. The hypothesis on the punctured spectrum follows by induction and the discussion in [12] of such ideals. On the other hand, Goto [3] proved that R/I is integrally closed with divisor class group $Z/(2)$, generated by the class of W. □

Remark. Let I be the ideal generated by the 2×2 minors of a generic 2×4 matrix. In view of the Plücker relations, I is not syzygetic. Since I is a complete intersection on the punctured spectrum of the corresponding ring, W^* must have depth 2.

COROLLARY 2.3. *Let I be the ideal generated by the* 2×2 *minors of a generic, symmetric* 3×3 *matrix* ϕ. *Then*:

(a) *I is generated by a d-sequence, satisfies* (SD) *but not* (SCM).

(b) *The Rees algebra of I, $\mathscr{R}(I)$, is an integrally closed, Cohen-Macaulay domain.*

(c) *The associated graded ring of I, $\mathrm{gr}_I(R)$, is a non-reduced, non-Gorenstein, Cohen-Macaulay ring.*

Proof. Let d be the determinant of the matrix ϕ. It is easily verified that $dx_{ij} \in I^2$ for each entry of ϕ; since $d \notin I^2$, the class of d in I/I^2 is annihilated by the maximal ideal of R. Since I is syzygetic by (2.2), $\operatorname{depth}(H_1) = 1$. Furthermore, as $d^2 \in I^3$, $\mathrm{gr}_I(R)$ is non-reduced.

(a) We compute the depths of the modules Z_i, $i = 1, 2$ and 3, of the Koszul complex on the canonical 6 generators of I. Since $\operatorname{depth}(H_1) = 1$, $\operatorname{depth}(Z_2) = 1 + \operatorname{depth}(B_1) = 3$. On the other hand, $\operatorname{depth}(Z_3) = 5$ by (1.1), so that I satisfies (SD) but not (SCM). Moreover, since I is also a complete intersection on the punctured spectrum of R, the approximation complex of I is acyclic and thus I is generated by a d-sequence (cf. [8]).

(b) and (c) follow now from [9, (6.5)], for the Cohen-Macaulay assertions. That $\mathscr{R}(I)$ is integrally closed can be verified either by a direct application of the Jacobian criterion—$\mathscr{R}(I)$ can be presented as a quotient $R[T_{ij}]/J$, with J derived from the explicit resolution of I—or more rapidly in the following manner. Since $\mathscr{R}(I)$ is Cohen-Macaulay, by Serre's normality criterion it suffices to check the localizations at its height 1 primes. Let P be such a prime and $\mathfrak{p} = P \cap R$. If $\mathfrak{p} \neq \mathfrak{m} =$ maximal ideal of R there is no difficulty since $I_{\mathfrak{p}}$ is a complete intersection. If $\mathfrak{p} = \mathfrak{m}$, $P = \mathfrak{m}\mathscr{R}(I)$. Let Q be the corresponding prime of $R[T_{ij}]$ – i.e. $Q = \mathfrak{m}R[T_{ij}]$. Looking at the image of J in the vector space $(Q/Q^2)_Q$ one easily gets that it has the desired rank 5. □

The crucial hypothesis of (2.2) never occurs in codimension two.

THEOREM 2.4. *Let R be a regular local ring and let I be a Cohen-Macaulay ideal of height* 2 *which is generically a complete intersection. If the class of W in the divisor class monoid of R/I is* 2*-torsion, then I is a complete intersection.*

Proof. Let

$$0 \longrightarrow R^{n-1} \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

be a resolution of I. Tensoring over with R/I we obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(I, R/I) \longrightarrow (R/I)^{n-1} \longrightarrow H_1 \longrightarrow 0\,,$$

since I is syzygetic (cf. [15]). As in the proof of (2.1), $\mathrm{Tor}_1(I, R/I) = W^*$; if the class of W is 2-torsion, we have the exact sequence

$$0 \longrightarrow W \longrightarrow (R/I)^{n-1} \longrightarrow H_1 \longrightarrow 0\,.$$

Since H_1 is Cohen-Macaulay ([2]) and W is the canonical module of R/I, this sequence will split—as it does so after reduction modulo a maximal regular sequence of R/I. Therefore R/I will be a Gorenstein ring, and hence a complete intersection by Serre's criterion ([14]). □

§ 3. Residually Cohen-Macaulay ideals

We prove here the naturality of sliding depth in a theorem of Huneke ([11]) on residual intersections. We also relate (SD) to various notions of syzygetic sequences (cf. [7]).

In this section $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d with infinite residue field.

DEFINITION 3.1. Let I be an ideal of R and let $\boldsymbol{x} = \{x_1, \cdots, x_s\}$ be a sequence of elements of I satisfying:

(1) $\mathrm{ht}((\boldsymbol{x})\colon I) \geq s \geq g = \mathrm{ht}(I)$.

(2) For all primes $P \supset I$ will $\mathrm{ht}(P) \leq s$, one has

(i) $(\boldsymbol{x})_p = I_p$;

(ii) $v((\boldsymbol{x}_p) \leq \mathrm{ht}(P)$.

I is said to be *residually Cohen-Macaulay* if for any such sequence, one has:

(a) $R/(\boldsymbol{x})\colon I$ is Cohen-Macaulay of dimension $d - s$;

(b) $((\boldsymbol{x}\colon I) \cap I = (\boldsymbol{x})$;

(c) $\mathrm{ht}((\boldsymbol{x})\colon I) > \mathrm{ht}((\boldsymbol{x})\colon I)$.

Remark 3.2. Let $\boldsymbol{x} = \{x_1, \cdots, x_s\}$ I be a sequence satisfying (1) and (2) above. Then:

(a) $\mathrm{ht}(\boldsymbol{x}) = \mathrm{ht}(I)$;

(b) $v((\boldsymbol{x})_p) \leq \mathrm{ht}(P)$ for all primes $P \supset (\boldsymbol{x})$.

Proof. (a): Let P be a minimal prime of $(\boldsymbol{x})$. Suppose $I \not\subset P$; then $((\boldsymbol{x})\colon I)_p = (\boldsymbol{x})_p$. It will follow from (1) that $\mathrm{ht}(P) \geq s \geq \mathrm{ht}(I)$.

(b): If $\mathrm{ht}(P) \geq s$, the assertion is trivial; if $\mathrm{ht}(P) < s$, the proof of (a) shows that $P \supset I$ and (2) applies.

THEOREM 3.3. *If I satisfies the sliding depth condition, then I is residually Cohen-Macaulay.*

THEOREM 3.4. *Suppose $v(I) \leq \mathrm{ht}(P)$ for all primes $P \supset I$. The following conditions are equivalent*:

(a) *I satisfies the sliding depth condition.*

(b) *I is residually Cohen-Macaulay.*

(c) *I can be generated by a d-sequence $\{x_1, \cdots, x_n\}$ satisfying*: *$(x_1, \cdots, x_{i+1})/(x_1, \cdots, x_i)$ is a Cohen-Macaulay module of dimension $d - i$, for $i = 0, \cdots, n-1$.*

Remark. The ideals occurring in the filtration of (3.4c) have the following homological properties. Assume that R is a regular local ring and that I is a Cohen-Macaulay ideal of height g. Consider the sequences

$$0 \longrightarrow I_i \longrightarrow I_{i+1} \longrightarrow Q_i \longrightarrow 0$$

where $I_i = (x_1, \cdots, x_i)$. We claim that the projective dimension of $I_i = i - 1$ for each $i < n$. Suppose one inequality holds; pick j largest with $\mathrm{pd}(I_j) < j - 1$. Note that $j < n - 1$ since $I = I_n$ is assumed Cohen-Macaulay and Q_{n-1} has projective dimension $n - 1$. Localize R at an associated prime of Q_j; this implies that each $Q_{j+k} = 0$ for $k > 0$, and thus $I_{j+1} = \cdots = I_n$. Consider the (localized) sequence

$$0 \longrightarrow I_j \longrightarrow I_{j+1} \longrightarrow Q_j \longrightarrow 0;$$

since $\mathrm{pd}(Q_j) = j$ and—now—$\mathrm{pd}(I_{j+1}) = 0$ or $g - 1$, we conclude $\mathrm{pd}(I_j) = j - 1$, which is a contradiction. □

The proofs of (3.3) and (3.4) require some technical lemmata on sliding depth.

LEMMA 3.5. *Let $\{x_1, \cdots, x_k\}$ be a regular sequence in I. Let "′" denote the canonical epimorphism $R \to R/(x_1, \cdots, x_k)$. I satisfies* (SD) *if and only if I' satisfies* (SD) *(in R').*

Proof. Complete the sequence to a generating set $\mathbf{x} = \{x_1, \cdots, x_n\}$ of I. The condition follows from the fact that $\dim(R') = d - k$, and the isomorphism (see [13]):

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$$H_i(x_1, \cdots, x_n; R) = H_i(x'_{k+1}, \cdots, x'_n; R'). \qquad \square$$

LEMMA 3.6. *Suppose* $I \neq 0$, *and* $I_p = 0$ *for all minimal primes* $P \supset I$. *Then*

(a) $(0: I) \cap I = 0$;

(b) $\mathrm{ht}((0: I) + I) = 1$.

Moreover, if I *satisfies* (SD), *then so does* I^*, *and* $R/0: I$ *is Cohen-Macaulay.* (*Here* "$*$" *denotes the canonical epimorphism* $R \to R/(0: I)$.)

Proof. (a) and (b) follow directly from the Abhyankar-Hartshorne lemma ([5]).

To prove the second assertion of the lemma, we use the exact sequences

$$0 \longrightarrow L_i \longrightarrow H_i(x_1, \cdots, x_n; R) \longrightarrow H_i(x_1^*, \cdots, x_n^*; R^*) \longrightarrow 0$$

of [11], where L_i is a direct sum of copies of $0: I$.

If I satisfies (SD), then $\mathrm{depth}(0: I = Z_n) = d$. From the sequences we have

$$\mathrm{depth}\, H_i(x_1^*, \cdots, x_n^*; R) \geq d - n + i \quad \text{for} \quad i < n,$$

while by (b) $\mathrm{ht}(I^*) = 1$, and hence $H_n(x_1^*, \cdots, x_n^*; R^*) = 0$.

To see that $R/0: I$ is Cohen-Macaulay, note that $R/0: I = B_{n-1}$, where $n = v(I)$. The assertion then follows from the exact sequence

$$0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

and the fact that Z_{n-1} is Cohen-Macaulay, cf. Section 1. $\square$

LEMMA 3.7. *Suppose* I *is a generated by a proper sequence* $x = \{x_1, \cdots, x_n\}$ (*cf.* [7]). *The following conditions are equivalent*:

(a) I *satisfies* (SD).

(b) $\mathrm{depth}\, R/(x_1, \cdots, x_i) \geq d - i$, *for* $i = 0, \cdots, n$.

(c) $\mathrm{depth}\,(x_1, \cdots, x_{i+1})/(x_1, \cdots, x_i) \geq d - i$, *for* $i = 0, \cdots, n-1$.

Proof. Since x is a proper sequence, we have exact sequences

$$0 \longrightarrow H_i(x_1, \cdots, x_j) \longrightarrow H_i(x_1, \cdots, x_{j+1}) \longrightarrow H_{i-1}(x_1, \cdots, x_j) \longrightarrow 0$$

for all $i > 1$. If follows by descending induction that if x satisfies (SD), then $\mathrm{depth}\, H_1(x_1, \cdots, x_i) \geq d - i + 1$ for $i = 1, \cdots, n$. It is also clear that, conversely, this diagonal condition will imply that $\mathrm{depth}\, H_i(x_1, \cdots, x_n) \geq d - i + 1$ for $i \geq 1$. We shall use this remark further in the proof.

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Denote $M_i = ((x_1, \cdots, x_i) : x_{i+1})/(x_1, \cdots, x_i)$ and $Q_i = (x_1, \cdots, x_{i+1})/(x_1, \cdots, x_i)$. We have exact sequences:

$$(1) \qquad 0 \longrightarrow H_1(x_1, \cdots, x_i) \longrightarrow H_1(x_1, \cdots, x_{i+1}) \longrightarrow M_i \longrightarrow 0$$

$$(2) \qquad 0 \longrightarrow M_i \longrightarrow R/(x_1, \cdots, x_i) \longrightarrow Q_i \longrightarrow 0$$

and

$$(3) \qquad 0 \longrightarrow Q_i \longrightarrow R/(x_1, \cdots, x_i) \longrightarrow R/(x_1, \cdots, x_{i+1}) \longrightarrow 0 .$$

(b) $\Rightarrow$ (c): Follows from the exact sequence (3).

(c) $\Rightarrow$ (a): Using the exact sequences (1), (2), (3) and the earlier remark the assertion follows by induction on i.

(a) $\Rightarrow$ (b): We show by induction on i that depth $R/(x_1, \cdots, x_{n-i}) \geq d - n + i$. For $i = 0$ this is our assumption. Suppose the assertion has been proved for $j = n - i \leq n$, and assume that

$$\operatorname{depth} R/(x_1, \cdots, x_{j-1}) = k < d - j + 1 .$$

Now by (1) we have depth $M_{j-1} \geq d - j + 1$; hence the map

$$\alpha : \operatorname{Ext}^k(R/\mathfrak{m}, R/(x_1, \cdots, x_{j-1})) \longrightarrow \operatorname{Ext}^t(R/\mathfrak{m}, Q_{j-1})$$

induced by (2) is injective. On the other hand (3) gives rise to the mapping

$$\beta : \operatorname{Ext}^k(R/\mathfrak{m}, Q_{j-1}) \longrightarrow \operatorname{Ext}^k(R/\mathfrak{m}, R/(x_1, \cdots, x_{j-1}))$$

that is injective as well. It follows that the composite $\beta\alpha$ is injective. But this is a contradiction since $\beta\alpha$ is induced by multiplication by x_j, and is thus the null mapping. □

Proof of (3.3): Suppose I satisfies (SD), ht$(I) = g$ and $\{x_1, \cdots, x_s\}$, $s \geq q$, is a sequence satisfying (1) and (2) of (3.1). All assertions depend solely on the ideal $(x_1, \cdots, x_s)$; we may therefore switch to a different set of generators. We use the general position argument of [1] (see [11]) to obtain a system of generators $\{x_1, \cdots, x_s\}$ such that for all primes $P \supset I$ with $g \leq \operatorname{ht}(P) = k \leq s$ we have

$$(*) \qquad (x_1, \cdots, x_s)_p = (x_1, \cdots, x_k)_p$$

(see Remark (3.2b)).

We now proceed by induction on s. Let $s = g$. Since by (3.2a)

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$\text{ht}(x_1, \cdots, x_g) = \text{ht}(I) = g$, it follows that $\{x_1, \cdots, x_g\}$ is a regular sequence. Denote by "$'$" the epimorphism $R \to R/(x_1, \cdots, x_g)$. According to (3.5), I' satisfies (SD) and therefore $R'/(0: I')$ is Cohen-Macaulay of dimension $d - g$ (cf. 3.6). But $R/(x_1, \cdots, x_g): I = R'/(0: I')$, and hence condition (a) in (3.1) is realized. For the conditions (b) and (c), we have by (3.6) that $(0: I') \cap I' = 0$ and $\text{ht}((0: I') + I') > 0$, which translate as desired.

We now assume that $s > g$.

1. Case $g > 0$: This is immediate from $(*)$ and the reduction to the ring R'. I' and $\{x'_1, \cdots, x'_s\}$ satisfy all the hypotheses of the theorem. By induction the statements (a), (b) and (c) of (3.1) hold then and it is easily lifted to R.

2. *Case* $g = 0$: Let "$*$" denote the canonical epimorphism $R \to R/0: I$. By (3.6) R^* is Cohen-Macaulay of dimension d, I^* and $\{x_1^*, \cdots, x_s^*\}$ satisfy (1) of (3.1). As for (2), we only have to check that $((x_1^*, \cdots, x_s^*): I^*) = ((x_1, \cdots, x_s): I)^*$. The inclusion $\supset$ is obvious. Let a^* be an element of $(x_1^*, \cdots, x_s^*): I^*$; then $aI \subset (x_1, \cdots, x_s) + 0: I$. For x in I we can therefore write $ax = y + z$, $y \in (x_1, \cdots, x_s)$, $z \in 0: I$. It follows that $z = ax - y$ lies in $I \cap 0: I = 0$, by (3.6). Furthermore we now have $\text{ht}(x_1^*, \cdots, x_s^*) = \text{ht}(I^*) > 0$ and I^* satisfies (SD); we are then back in case 1. Therefore $\{x_1^*, \cdots, x_s^*\}$ and I^* satisfy (a), (b) and (c) of (3.1); again it is easy to lift back to R. □

Proof of (3.4): (a)$\Rightarrow$(b) is already proved more generally in (3.3).

(b)$\Rightarrow$(c): Since $v(I_p) \leq \text{ht}(P)$ for all primes $P \supset I$, we may choose generators $\{x_1, \cdots, x_n\}$ of I such that

(i) $(x_1, \cdots, x_s)_P = I_P$, for all $P \supset I$, $\text{ht}(P) \leq s$, and

(ii) $\text{ht}((x_1, \cdots, x_s): I) \geq s$.

Since I is residually Cohen-Macaulay, we then have that for $s \geq g = \text{ht}(I)$, (a), (b) and (c) of (3.1) hold.

It is clear that $\{x_1, \cdots, x_g\}$ is a regular sequence. Next we show that x_{s+1} is not a zero-divisor on $R/(x_1, \cdots, x_s): I$ for $g \leq s < n$. It will then follow that $(x_1, \cdots, x_s): I = (x_1, \cdots, x_s): x_{s+1}$. Together with condition (b) this will imply that $\{x_1, \cdots, x_n\}$ is a d-sequence.

Denote by "$'$" the canonical epimorphism $R \to R/(x_1, \cdots, x_s): I$. (a) and (c) imply the I' contains a non-zero divisor z. Suppose x'_{s+1} is a zero divisor. Let $y \in (x'_{s+1}): I'$; then $zy \in (x'_{s+1})$. This shows that $(x'_{s+1}): I'$ con-

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sists of zero-divisors. Since R' is Cohen-Macaulay, this implies that $\mathrm{ht}((x'_{s+1})\colon I') = 0$, contradicting (a). Since $(x_1, \cdots, x_{s+1})/(x_1, \cdots, x_s) = R/(x_1, \cdots, x_s); x_{s+1} = R/(x_1, \cdots, x_s)\colon I$, the implication is proved.

(c)⇒(a): Apply (3.7). □

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