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Orbifold Euler characteristics for compactified universal Jacobians over $\overline{\mathcal{M}}_{g,n}$

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Abstract

We calculate the orbifold Euler characteristics of all the degree d fine universal compactified Jacobians over the moduli space of stable curves of genus g with n marked points, as defined by Pagani and Tommasi. We show that this orbifold Euler characteristic agrees with the Euler characteristic of $\overline{\mathcal{M}}_{0,2g+n}$ up to a combinatorial factor, and in particular, is independent of the degree d and the choice of degree d fine compactified universal Jacobian.

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1. *Introduction*

1·1. *Background and motivation*

Orbifold Euler characteristics have been studied for a long time by mathematicians as a generalisation of topological Euler characteristics. Perhaps most famously, in [12], Harer and Zagier calculate the orbifold Euler characteristics of the moduli spaces of smooth genus g, n-pointed curves for all g and n such that $2g-2+n>0$, obtaining the formula:

$$\chi_{orb}(\mathcal{M}_{g,n}) = \frac{B_{2g}(-1)^n(2g-1)(2g+n-3)!}{(2g)!}.$$

They then proceed to use this calculation to compute the ordinary Euler characteristics of $\mathcal{M}_g$ for $g \geq 2$, which consequently, can be used to deduce facts about the cohomology of $\mathcal{M}_g$.

In the paper [7], Bini and Harer use the Harer–Zagier calculation to compute $\chi_{orb}(\overline{\mathcal{M}}_{g,n})$ for all g, n such that $2g-2+n>0$. More recenty, some examples of orbifold Euler characteristics of the moduli spaces of abelian differentials $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ are calculated in the paper [9] by Zachhuber, Costantini and Möller. In this paper, we extend these calculations to fine compactified universal Jacobians.

Over the moduli space of smooth curves $\mathcal{M}_{g,n}$, there is a smooth, proper morphism $\mathcal{J}^d_{g,n}\to\mathcal{M}_{g,n}$ of Deligne–Mumford (DM) stacks where the geometric points of $\mathcal{J}^d_{g,n}$ are pairs consisting of a genus g algebraic curve with n marked points and a line bundle of degree d on it. The morphism is the forgetful morphism, obtained by forgetting the data of the line bundle. The fibers over geometric points are isomorphic to abelian varieties and therefore have Euler characteristic zero. By properties of the morphism and of orbifold Euler characteristics,

$$\chi_{orb}(\mathcal{J}^d_{g,n})=\chi_{orb}(\mathcal{M}_{g,n})\chi_{orb}(F)=0,$$

for $g>0$ where F is any fiber of the morphism.

To obtain a space with non-trivial Euler characteristic, it is natural to consider an extension of this morphism to a proper morphism over the compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. Finding such an extension is a classical problem and many such extensions have been found, such as those in [8, 10, 18, 22, 23]. In the paper [17], Kass and Pagani construct fine compactified universal Jacobians $\overline{\mathcal{J}}^d_{g,n}(\phi)$, which depend on a stability condition ϕ. In the more recent paper [21], it is shown that these universal compactified Jacobians belong to the more general class of fine compactified universal Jacobians.

We compute the orbifold Euler characteristics of all fine compactified universal Jacobians over $\overline{\mathcal{M}}_{g,n}$ such that $2g-2+n>0$.

Recently, universal compactified Jacobians have been used in the exploration of tautological classes on the moduli space of stable curves, e.g. in the study of (logarithmic) double ramification cycles (see [14, 15]). This raises important questions, e.g. how to define the tautological ring of these spaces, and lift known tautological relations to this ring (extending work in [25]). Work in this direction can be seen, for example, in [4]. The work in this paper was partly motivated by the goal of obtaining a better understanding of the global geometry of these compactified Jacobians.

THEOREM 1·1. *For all $d\in\mathbb{Z}$, non-negative integers g and n such that $2g-2+n>0$ and universal fine compactified Jacobians $\overline{\mathcal{J}}^d_{g,n}$ of degree d, we have*:

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n})=\frac{1}{2^g(g!)}\chi(\overline{\mathcal{M}}_{0,2g+n}),$$

where $\chi(\overline{\mathcal{M}}_{0,2g+n})$ is the ordinary topological Euler characteristic of the variety $\overline{\mathcal{M}}_{0,2g+n}$.

This may be surprising to some readers, in light of the fact that in [17, section 6], it is shown that there exist g, n and d, along with universal polarisations ϕ_1 and ϕ_2, such that the corresponding fine compactified universal Jacobians $\overline{\mathcal{J}}^d_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}^d_{g,n}(\phi_2)$ are not isomorphic as Deligne–Mumford stacks.

1·2. *Idea of the proof*

We consider the morphisms $\overline{\mathcal{J}}^d_{g,n}\to\overline{\mathcal{M}}_{g,n}$ for a fine compactified universal degree d Jacobian over the moduli space of stable curves. Such a morphism is a proper, representable morphism of DM stacks and the fiber over a point $[(C; p_1, ..., p_n)]$ is a degree d fine, smoothable compactified Jacobian over C. Such compactified Jacobians over a nodal curve C are defined in terms of stability conditions σ and we denote the Jacobian by $\overline{J}^d_\sigma(C)$.

In particular, under the stratification $\overline{\mathcal{M}}_{g,n} = \bigsqcup_{\Gamma \in G(g,n)} \mathcal{M}^\Gamma$ by dual graph, we show that the topological Euler characteristic of the fiber over a geometric point $[(C; p_1, ..., p_n)]$ in $\mathcal{M}^\Gamma$ depends only on Γ.

1·3. *Stratifying the fibers and their Euler characteristics*

We adapt results from the paper [19], where a stratification of a compactified Jacobian $\overline{J}^d_{\sigma_\phi}(C)$ corresponding to a stability condition σ_ϕ which is defined in terms of a polarisation ϕ on C is obtained. The strata in this stratification are all isomorphic to the generalised Jacobian of some curve which is a partial normalisation of C at a subset of its nodes. We show that the same is true for any smoothable, fine compactified Jacobian $\overline{J}^d(C)$ of a nodal curve C which is necessarily defined in terms of some stability condition σ on its dual graph $\Gamma(C)$. We write $\overline{J}^d(C) = \overline{J}^d_\sigma(C)$.

If $G(g, n)$ is a set of representatives of automorphism classes of stable graphs of genus g with n marked points and $G(g,n)^0$ is the subset of $G(g, n)$ consisting of stable graphs whose vertices all have genus 0, we use our stratification of a smoothable fine compactified Jacobian of C corresponding to an arbitrary stability condition σ on $\Gamma(C)$ to obtain:

$$\chi_{top}(\overline{J}^d_\sigma(C)^{an}) = \chi_{orb}(\overline{J}^d_\sigma(C)) = \begin{cases} c(\Gamma(C)) & \text{if } \Gamma(C) \in G(g,n)^0 \\ 0 & \text{if } \Gamma(C) \in G(g,n) \setminus G(g,n)^0, \end{cases}$$

where $c(\Gamma(C))$ is the number of spanning trees of $\Gamma(C)$.

1·4. *Stratifying and obtaining the orbifold Euler characteristic of a fine compactified universal Jacobian*

We then apply Proposition 3·9 to the morphism $\overline{\mathcal{J}}^d_{g,n}|_{\mathcal{M}^\Gamma} \to \mathcal{M}^\Gamma$. Informally, this proposition states that for a morphism of DM stacks $\mathcal{M} \to \mathcal{N}$ which is a fibration on some stratification of the base such that all fibers have the same topological Euler characteristic, $\chi_{orb}(\mathcal{M}) = \chi_{top}(F^{an})\chi_{orb}(\mathcal{N})$ where F is any fiber.

Using this, we obtain:

$$\begin{aligned} \chi_{orb}(\overline{\mathcal{J}}^d_{g,n}|_{\mathcal{M}^\Gamma}) &= \chi_{orb}(\mathcal{M}^\Gamma)\chi_{top}(\overline{J}^d(C)) \\ &= \begin{cases} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma) & \text{if } \Gamma \in G(g,n)^0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where C is any genus g nodal curve with dual graph Γ.

Using the fact that orbifold Euler characteristics are additive under locally closed stratifications, we obtain the formula:

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \sum_{\Gamma \in G(g,n)^0} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma) = \sum_{\Gamma \in G(g,n)^0} \sum_{\substack{T \text{ is a} \\ \text{spanning} \\ \text{tree of } \Gamma}} \chi_{orb}(\mathcal{M}^\Gamma).$$

By combinatorial reasoning, we manipulate this double sum to obtain the formula:

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \frac{1}{2^g(g!)}\chi_{orb}(\overline{\mathcal{M}}_{0,2g+n}).$$

Since $\overline{\mathcal{M}}_{0,2g+n}$ is a variety, $\chi_{orb}(\overline{\mathcal{M}}_{0,2g+n}) = \chi(\overline{\mathcal{M}}_{0,2g+n})$.

2. *Definitions and notation*

In the following section, we recall the theory of (compactified) Jacobians following the treatment of [21]. Readers familiar with this subject are encouraged to skip to the next section and refer back to this section as needed.

Associated to a complex, nonsingular projective curve C is a *Jacobian* $J^0(C)$ which is a complex, projective abelian variety whose dimension is equal to the genus of the curve. These are moduli spaces parametrising degree 0 line bundles on C up to isomorphism. The group law on $J^0(C)$ is by tensor product. In fact, as an abstract group, it is isomorphic to $(\mathbb{R}/\mathbb{Z})^{2g}$ and is the subgroup of Pic(C) consisting of degree 0 line bundles. For $g \geq 1$, it has n-torsion isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. In particular, for every n, there is an element of order n in $J^0(C)$ that generates a subgroup of order n. If $d \in \mathbb{Z}$, there is a variety $J^d(C)$ parametrising degree d line bundles which is isomorphic to $J^0(C)$ as a variety where the isomorphism is given by tensoring with a fixed line bundle of degree d. $J^d(C)$ is no longer an algebraic group but $J^0(C)$ acts freely on $J^d(C)$ via tensor products.

Similarly, singular complex projective curves C have associated *generalised Jacobian varieties* $J^{\underline{0}}(C)$ which in general fail to be proper over $\mathbb{C}$ (see below for the definition). These Jacobian varieties are abelian group schemes and there are several approaches to compactifying these varieties.

Given a nodal curve C and its total normalisation $\nu : C^\nu \to C$, if $C_1, ..., C_t$ are the irreducible components of C^ν, then the *multidegree* of $\mathcal{L} \in$ Pic(C) is the vector $\underline{\deg(\mathcal{L})} = (\deg_{C_i}(\nu^*\mathcal{L}|_{C_i}))_i$. Its *total degree*, $\deg(\mathcal{L})$ is given by the sum of the components of the vector $\underline{\deg(\mathcal{L})}$. As a group, $J^{\underline{0}}(C)$ is the subgroup of $Pic(C)$ consisting of line bundles of multidegree $\underline{0}$ and for each $\underline{d} \in \mathbb{Z}^t$, there is a scheme $J^{\underline{d}}(C)$ parametrising line bundles of multidegree $\underline{d}$ on C. In particular, these are all isomorphic to $J^{\underline{0}}(C)$ where the isomorphism is given by tensoring with a fixed line bundle of multidegree $\underline{d}$.

Given a nodal curve C with dual graph $\Gamma(C)$, we define:

$$b_1(\Gamma(C)) = |E(\Gamma(C))| - |V(\Gamma(C))| + 1.$$

If δ is the number of nodes of C, then this quantity is $\delta - t + 1$.

There is an exact sequence of group schemes: ([2, p.89])

$$1 \to (\mathbb{C}^\times)^{b_1(\Gamma(C))} \to J^{\underline{0}}(C) \xrightarrow{\mathcal{L} \mapsto \nu^*(\mathcal{L})} J^{\underline{0}}(C^\nu) \cong \prod_{i=1}^t J^0(C_i) \to 0.$$

When all the C_i are genus 0 curves, $J^{\underline{d}}(C) \cong (\mathbb{C}^\times)^{b_1(\Gamma(C))}$.

The construction of Jacobians also works for families of smooth curves and for each g, n such that $2g - 2 + n > 0$, there exists a universal Jacobian $\mathcal{J}^d_{g,n}$ which is a DM stack along with a smooth, proper morphism $\mathcal{J}^d_{g,n} \to \mathcal{M}_{g,n}$ such that the fiber over any geometric point $[(C; p_1, ..., p_n)]$ of $\mathcal{M}_{g,n}$ is the Jacobian variety $J^d(C)$.

A coherent sheaf on a nodal curve C has *rank* 1 if the stalk at each generic point of C has length 1. It is *torsion free* if it has no embedded components and it is *singular at P* if it fails to be locally free at P. If $\mathcal{F}$ is a torsion free sheaf on C, we say it is *simple* if its automorphism group is $\mathbb{C}^{\times}$. This is equivalent to the condition that removing the singular points of $\mathcal{F}$ from C does not disconnect C.

Every rank 1, torsion free, simple sheaf $\mathcal{F}$ on a nodal curve C can be associated with a pair $(S(\mathcal{F}), \underline{\deg}(\mathcal{F}))$. Here $S(\mathcal{F})$ is the set of nodes of C at which $\mathcal{F}$ fails to be locally free. Since $\mathcal{F}$ is simple, $S(\mathcal{F})$ is a *non-disconnecting* subset of nodes of C. The multidegree $\underline{\deg}(\mathcal{F})$ is the multidegree of the maximal torsion free quotient of the pullback of $\mathcal{F}$ under the total normalisation of C. In other words, the partial normalisation C_S of C at the nodes $S(\mathcal{F})$ of C is connected.

If C is a nodal curve, a degree d *fine compactified Jacobian* $\overline{J}^d(C)$ is a nonempty, connected, proper, open subscheme of $Simp^d(C)$ where $Simp^d(C)$ is the scheme parametrising rank 1, torsion free, simple coherent sheaves on C. It is *smoothable* if there exists a regular smoothing $C \to Spec(\mathbb{C}[[t]])$ of C such that the fiber over $0 \in \mathbb{C}[[t]]$ of a morphism $U \to Spec(\mathbb{C}[[t]])$ where U is an open, proper subscheme of $Simp^d(C/\mathbb{C}[[t]])$ is $\overline{J}^d(C)$.

A degree *d fine compactified universal Jacobian* is an open substack of $Simp^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ (see [21] for a definition) that is proper over $\overline{\mathcal{M}}_{g,n}$. In particular, since $Simp^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ is representable over $\overline{\mathcal{M}}_{g,n}$ and since open immersions of DM stacks are representable, any fine compactified universal Jacobian is representable over $\overline{\mathcal{M}}_{g,n}$.

We now define stability conditions and explain their relation to fine, smoothable compactified Jacobians. If $G \subset \Gamma$ is a connected, spanning subgraph of Γ, define

$$S^d_\Gamma(G) = \left\{ \underline{d} \in \mathbb{Z}^{|V(\Gamma)|} : \sum_{v \in V(\Gamma)} d(v) = d - |E(\Gamma) \setminus E(G)| \right\} \subset \mathbb{Z}^{|V(\Gamma)|}.$$

If $\mathcal{F}$ is a rank 1, torsion free, simple sheaf on a nodal curve C, then $\underline{\deg}(\mathcal{F}) \in S^d_\Gamma(\Gamma(\mathcal{F}))$ where $\Gamma(\mathcal{F})$ is the graph obtained from $\Gamma(C)$ by removing the edges corresponding to the nodes of C at which $\mathcal{F}$ fails to be locally free.

The *twister* of a graph Γ at the vertex v is defined to be the element of $\mathbb{Z}^{|V(\Gamma)|}$ defined by:

$$\mathrm{Tw}_{\Gamma,v}(w) = \begin{cases} \#\text{ edges of } \Gamma \text{ having endpoints } v \text{ and } w, & v \neq w, \\ -\#\text{ non-loop edges of } \Gamma & \\ \text{having one of its endpoints } v = w, & v = w. \end{cases}.$$

The twister group, $\mathrm{Tw}(\Gamma)$ is the subgroup of $\mathbb{Z}^{|V(\Gamma)|}$ generated by the elements $\{\mathrm{Tw}_{\Gamma,v}\}_{v \in V(\Gamma)}$. We have the inclusions $\mathrm{Tw}(\Gamma) \subset S^0_\Gamma(\Gamma) \subset \mathbb{Z}^{|V(\Gamma)|}$ and therefore, for a connected, spanning subgraph $G \subset \Gamma$, $\mathrm{Tw}(G)$ acts on $S^d_\Gamma(G)$ by vector addition.

A *degree d stability condition* σ on Γ is a set of pairs $(G, \underline{d})$ such that $\underline{d} \in S^d_\Gamma(G)$ where $G \subset \Gamma$ is a connected, spanning subgraph. The pairs $(G, \underline{d})$ are additionally required to satisfy the following two conditions:

(1) if G is a connected, spanning subgraph of Γ, then for all edges e of $E(\Gamma) \setminus E(G)$ with endpoints v_1 and v_2, if $(G, \underline{d}) \in \sigma$, then $(G \cup \{e\}, \underline{d} + \underline{e}_{v_i})$ is in σ for $i = 1, 2$ where $\underline{e}_{v_i}$ is the standard basis vector of $\mathbb{Z}^{|V(\Gamma)|}$ corresponding to vertex v_i;

(2) for every connected, spanning subgraph G of Γ:

$$\sigma(G) = \{\underline{d} : (G, \underline{d}) \in \sigma\} \subset S^d_G(G)$$

is a minimal, complete set of representatives for the action of the twister group $\mathrm{Tw}(G)$ on $S^d_\Gamma(G)$.

For every connected, spanning subgraph G of Γ, we have $|\sigma(G)| = c(G)$ by [21, remark 4·4].

For a nodal curve C, the scheme $Simp^d(C)$ has a stratification into locally closed subsets $Simp^d(C) = \bigsqcup_{(G,\underline{d})} \overline{J}_{(G,\underline{d})}(C)$ where $\overline{J}_{(G,\underline{d})}(C)$ is the locus whose points correspond to sheaves $\mathcal{F}$ where $\Gamma(\mathcal{F}) = G$ and $\underline{\deg(\mathcal{F})} = \underline{d}$. Here the disjoint union is over all connected, spanning subgraphs G of Γ and all $\underline{d} \in \mathbb{Z}^{|V(\Gamma)|}$ such that

$$\sum_{i=1}^{|V(\Gamma)|} d_i = d - |E(\Gamma) \setminus E(G)|.$$

We view these as schemes, endowed with the reduced schematic structure.

Given a stability condition σ on the dual graph Γ of a nodal curve C, a rank 1, torsion free simple sheaf $\mathcal{F}$ is *σ-stable* if $(\Gamma(\mathcal{F}), \underline{\deg(\mathcal{F})})$ is in σ where $\Gamma(\mathcal{F})$ is the subgraph of $\Gamma(C)$ obtained by removing edges of $\Gamma(C)$ corresponding to nodes of C at which $\mathcal{F}$ fails to be locally free.

THEOREM 2·1. ([21, corollary 6·4]) *Given a degree d stability condition σ on the dual graph $\Gamma(C)$ of a nodal curve C, the subscheme $\overline{J}^d_\sigma(C)$ of $Simp^d(X)$ parametrising sheaves that are σ-stable is a smoothable, degree d fine compactified Jacobian for C.*

THEOREM 2·2. ([21, corollary 7·11]) *If C is a nodal curve, then there is a bijection between degree d fine, smoothable compactified Jacobians of C and degree d stability conditions on $\Gamma(C)$.*

This will play an important role in our calculation of the orbifold Euler characteristics of fine compactified universal Jacobians. To be precise, for every fine, smoothable compactified Jacobian $\overline{J}^d(C)$ of C, there is some unique stability condition σ on $\Gamma(C)$ such that $\overline{J}^d(C) = \overline{J}^d_\sigma(C)$.

A non-degenerate *polarisation* ϕ (see [21]) on a stable graph Γ defines a stability condition σ_ϕ on Γ. We also have the notion of a non-degenerate universal polarisation, which is a collection $\phi = (\phi_\Gamma)_{\Gamma \in G(g,n)}$ such that each ϕ_Γ is a non-degenerate polarisation on Γ and such that the ϕ_Γ are compatible with graph morphisms. Every non-degenerate polarisation on a graph Γ gives rise to a smoothable, fine compactified Jacobian on curves with dual graph Γ. Additionally, non-degenerate universal polarisations give rise to fine compactified universal Jacobians.

The fine compactified universal Jacobians $\overline{\mathcal{J}}^d_{g,n}(\phi)$ over $\overline{\mathcal{M}}_{g,n}$ corresponding to a universal polarisation $\phi = (\phi_\Gamma)_{\Gamma \in G(g,n)}$ are introduced in [17].

THEOREM 2·3. ([20, proposition 2·9]) *For every non-degenerate universal polarisation $\phi = (\phi_\Gamma)_{\Gamma \in G(g,n)}$, the moduli stack $\overline{\mathcal{J}}^d_{g,n}(\phi)$ is a fine compactified universal Jacobian.*

There are fine compactified universal Jacobians that do not arise from universal polarisations. For example, in [20, section 6], fine compactified universal Jacobians $\overline{\mathcal{J}}^d_{g,n}$, not arising from a universal stability condition are found for $n \geq 6$. In fact, it is rarely the case that all fine compactified Jacobians arise from universal stability conditions.

3. *Orbifold Euler characteristics of separated, finite type, complex DM stacks*

In this section, we recall basic properties of the orbifold Euler characteristics of finite type, separated Deligne–Mumford (DM) stacks defined over $\mathbb{C}$ and prove Proposition 3·9 which is central in our calculation.

THEOREM 3·1. *The orbifold Euler characteristic $\chi_{orb}(\mathcal{M})$ of a finite type separated DM stack $\mathcal{M}$, satisfies and is characterised by the following three properties*:

(i) if $\bigsqcup_{i=1}^{n} \mathcal{M}_i = \mathcal{M}$ is a stratification of $\mathcal{M}$ into finitely many locally closed substacks, then

$$\chi_{orb}(\mathcal{M}) = \sum_{i=1}^{n} \chi_{orb}(\mathcal{M}_i);$$

(ii) if $f : \mathcal{M} \to \mathcal{N}$ is a finite, surjective, étale morphism of finite type separated DM stacks where $\mathcal{N}$ is integral, then

$$\chi_{orb}(\mathcal{M}) = \deg(f)\chi_{orb}(\mathcal{N}),$$

where $\deg(f)$ *is defined as in* [**26**];

(iii) if $\mathcal{M}$ is a scheme, then

$$\chi_{orb}(\mathcal{M}) = \chi_{top}(\mathcal{M}^{an}),$$

where $\mathcal{M}^{an}$ is the scheme $\mathcal{M}$ viewed as a topological space endowed with the complex topology from [**13**, *appendix B*].

The existence of an orbifold Euler characteristic satisfying these properties is well known within the mathematical community. Once existence of such a definition has been shown, uniqueness follows from its properties.

Remark 3·2. In this paper, when we speak of locally closed substacks of $\mathcal{M}$, we mean locally closed subsets with respect to the underlying topology of $\mathcal{M}$ with the unique reduced structure. Moreover, the compliment of a locally closed substack is a locally closed substack in the same way.

THEOREM 3·3. ([**3**, Theorem 4·5·1]) *If $\mathcal{M}$ is a separated, finite type DM stack over a noetherian scheme, there exists a finite, surjective, generically étale morphism $Z \to \mathcal{M}$ from a scheme Z.*

LEMMA 3·4. *A DM stack $\mathcal{M}$ which is finite type and separated over $\mathbb{C}$ has a stratification by finitely many locally closed substacks $\mathcal{U}_i$ such that $\mathcal{M} = \bigsqcup_{i=0}^{n} \mathcal{U}_i$ with each of the $\mathcal{U}_i$ covered by a finite, surjective, étale morphism from a scheme.*

Proof. Separated DM stacks, of finite type over $\mathbb{C}$ are noetherian. Therefore, we may argue by noetherian induction. There is some atlas $U \to \mathcal{M}$ where U is a reduced, separated, finite type scheme over $\mathbb{C}$. This scheme U contains a dense, smooth open subscheme U_0. The

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image of U_0 under $U \to \mathcal{M}$ is a nonempty open, smooth substack $\mathcal{U}_0$ of $\mathcal{M}$. Then applying Lemma 3·4, shrinking $\mathcal{U}_0$ if necessary, we may assume $\mathcal{U}_0$ is smooth and that there is some finite, surjective, étale morphism from a scheme U to $\mathcal{U}_0$ by Theorem 3·3. We may further assume that $\mathcal{U}_0$ is integral by taking a connected component.

Let $\mathcal{V}_0 = \mathcal{M} \setminus \mathcal{U}_0$. If $\mathcal{V}_0 \neq \emptyset$, repeat the above process to find a nonempty open substack $\mathcal{U}_1$ of $\mathcal{V}_0$ with the property that $\mathcal{U}_1$ has a finite, surjective, étale cover by a scheme. Set $\mathcal{V}_1 = \mathcal{V}_0 \setminus \mathcal{U}_0$. Continue inductively until $\mathcal{V}_i = \emptyset$. This process must terminate after finitely many iterations since we obtain a chain of closed subsets $\mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \ldots$ of $\mathcal{M}$. Therefore, using the fact that $\mathcal{M}$ is noetherian, there must be some finite n such that $\mathcal{V}_n = \emptyset$. We obtain $\mathcal{M} = \bigsqcup_{i=0}^{n} \mathcal{U}_i$ where the $\mathcal{U}_i$ are locally closed, integral substacks of $\mathcal{M}$, each having a finite, surjective étale morphism from a scheme.

Remark 3·5. The smoothness at the beginning of the proof of Lemma 3·4 was only introduced so that we could simultaneously obtain openess and integrality of $\mathcal{U}_0$ in $\mathcal{M}$. The openess is needed to proceed by noetherian induction. In general, an open subset of an integral component is not open. However, for smooth DM stacks, integral components are connected components, so are open.

Remark 3·6. An explicit construction of orbifold Euler characteristics for finite type, separated DM stacks over $\mathbb{C}$ can be defined as follows. Take a locally closed stratification of $\mathcal{M} = \bigsqcup_{i=1}^{n} \mathcal{U}_i$ as in Lemma 3·4 where there is a finite, surjective, étale morphism $U_i \xrightarrow{f_i} \mathcal{U}_i$ from some scheme U_i to $\mathcal{U}_i$ for each i. We can define

$$\chi_{orb}(\mathcal{M}) = \sum_{i=1}^{n} \frac{\chi_{top}(U_i^{an})}{\deg(f_i)}.$$

This can be shown to be independent of the choice of stratification and of the f_i once such a stratification has been chosen. The properties in Theorem 3·1 can easily be seen to follow from this definition.

THEOREM 3·7. ([24, theorem 1, p. 481]) *If $p : E \to B$ is an orientable topological fibration with fiber F and B is path-connected, then $\chi_{top}(E) = \chi_{top}(F)\chi_{top}(B)$.*

THEOREM 3·8. ([27, section 1·2]) *Given a proper, surjective morphism of schemes $X \to Y$. There is a finite, locally closed stratification $Y = \bigsqcup_{i=1}^{m} Y_i$ of the base such that $X|_{Y_i} = X \times_Y Y_i \to Y_i$ is a fibration with respect to the complex topology.*

PROPOSITION 3·9. *Suppose $\mathcal{M} \to \mathcal{N}$ is a proper, surjective, representable morphism of separated, finite type DM stacks such that the topological Euler characteristic of the fiber of any geometric point of $\mathcal{N}$ is independent of the choice of geometric point. Then*

$$\chi_{orb}(\mathcal{M}) = \chi_{top}(F^{an})\chi_{orb}(\mathcal{N}),$$

where F is the fiber over any geometric point of $\mathcal{N}$.

Proof. By property (1) of Theorem 3·1 and Lemma 3·4, it suffices to consider the case where $\mathcal{N}$ is integral and has a finite, surjective, étale morphism $U \xrightarrow{f} \mathcal{N}$ from a scheme.

We obtain the following cartesian diagram

$$\begin{array}{ccc} \mathcal{M}\times_{\mathcal{N}} U & \longrightarrow & U \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{M} & \longrightarrow & \mathcal{N} \end{array}$$

We take a locally closed stratification $U = \bigsqcup_{j=1}^{n} U_j$ as in Theorem 3·8. Taking connected components, we may further assume that they are connected. Letting F_j be a fiber of $\mathcal{M}|_{U_j} \to U_j$ by Theorem 3·8, we obtain

$$\chi_{top}((\mathcal{M}\times_{\mathcal{N}} U|_{U_j})^{an}) = \chi_{top}(F_j^{an})\chi_{top}(U_j^{an}).$$

Using the fact that F_j is a fiber of $\mathcal{M} \to \mathcal{N}$ and writing F for any general fiber of this morphism,

$$\chi_{top}(F^{an}) = \chi_{top}(F_j^{an}).$$

By Lemma 3·4, we can obtain a locally closed stratification of $\mathcal{M}$ into finitely many locally closed, integral substacks. Using the fact that over each integral substack $\mathcal{M}' \subset \mathcal{M}$, $g|_{\mathcal{M}'}$ is a finite, surjective étale morphism of degree deg(f), by properties (1) and (3) of Theorem 3·1,

$$\chi_{orb}(\mathcal{M}) = \frac{\chi_{orb}(\mathcal{M}\times_{\mathcal{N}} U)}{\deg(f)} = \frac{\chi_{top}((\mathcal{M}\times_{\mathcal{N}} U)^{an})}{\deg(f)}.$$

We compute

$$\begin{aligned} \chi_{orb}(\mathcal{M}) &= \frac{\chi_{top}((\mathcal{M}\times_{\mathcal{N}} U)^{an})}{\deg(f)} \\ &= \frac{1}{\deg(f)}\sum_{j=1}^{n} \chi_{top}((\mathcal{M}\times_{\mathcal{N}} U|_{U_j})^{an}) \\ &= \frac{1}{\deg(f)}\sum_{j=1}^{n} \chi_{top}(U_j^{an})\chi_{top}(F^{an}) \\ &= \chi_{top}(F^{an})\frac{\chi_{top}(U^{an})}{\deg(f)} = \chi_{top}(F^{an})\chi_{orb}(\mathcal{N}). \end{aligned}$$

4. *Proof of the main result*

If $\overline{\mathcal{J}}^d_{g,n}$ is a fine compactified universal Jacobian over $\overline{\mathcal{M}}_{g,n}$, then the fiber $\overline{J}^d(C)$ over any point $[(C; p_1, ..., p_n)]$ of $\overline{\mathcal{M}}_{g,n}$ is a degree d, fine smoothable compactified Jacobian of C. The smoothability of the fibers $\overline{J}^d(C)$ follows from the fact that the morphism $\overline{\mathcal{J}}^d_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is smooth over the open, dense substack $\mathcal{M}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$. Therefore, every fiber $\overline{J}^d(C)$ over a point $[(C; p_1, ..., p_n)]$ is of the form $\overline{J}^d(C) = \overline{J}^d_\sigma(C)$ for some stability condition σ on $\Gamma(C)$.

LEMMA 4·1. ([1, lemma 1·5]) *A rank* 1*, torsion free, simple sheaf* $\mathcal{F}$ *of a nodal curve* C *is the direct image* $\nu_*\mathcal{F}'$ *under some partial normalisation* $\nu : C' \to C$ *at the set of nodes where* $\mathcal{F}$ *fails to be locally free for some unique line bundle* $\mathcal{F}'$ *on* C'*.*

Let C_S be the partial normalisation of C at the subset of nodes corresponding to edge set S in $E(\Gamma(C))$. For any line bundle $\mathcal{L}$ on C_S, by [19, proposition 1·14 (iii)], if S_v is the set of self-edges in S incident at vertex v and C^v is the corresponding irreducible component of C

$$\deg_{C^v}(\mathcal{L}|_{C_S^v}) = \deg_{C^v}((\nu_S)_*\mathcal{L}|_{C^v}) - |S_v|. \tag{4·1}$$

As in [19], for a nodal curve C, given a fine, smoothable compactified Jacobian $\overline{J}^d_\sigma(C)$ corresponding to a stability condition σ, we have the stratification $\overline{J}^d_\sigma(C) = \bigsqcup_{(G,\underline{d})\in\sigma} \overline{J}_{(G,\underline{d})}(C)$ where $\overline{J}_{(G,\underline{d})}(C)$ is the locus whose points correspond to sheaves $\mathcal{F}$ where $\Gamma(\mathcal{F}) = G$ and $\underline{\deg(\mathcal{F})} = \underline{d}$. These are locally closed subsets which we view as schemes endowed with the reduced schematic structure.

LEMMA 4·2. *If σ is a stability condition on a graph Γ and S is a set of edges whose removal does not result in a disconnected graph, and if Γ_S is the resulting connected graph, then we can define a stability condition σ_S on Γ_S to be*:

$$\sigma_S = \left\{ \begin{array}{c} (G, (d_v - |S_v|)_{v\in V(\Gamma)}) : G \subset \Gamma_S \textit{ is spanning and connected}, \\ (d_v)_{v\in V(\Gamma)} \in \sigma(G) \end{array} \right\}.$$

Moreover, if σ is a stability condition on Γ of degree d, then σ_S is a degree

$$d_S = d + (|S| - \sum_{v\in V(\Gamma)} |S_v|),$$

or alternatively,

$$d_S = d + \#\{\textit{edges in } S \textit{ that are not self-edges}\}.$$

Proof. We have:

$$\sigma_S \subset \{\text{connected, spanning subgraphs of } \Gamma_S\} \times \mathbb{Z}^{|V(\Gamma_S)|}.$$

So it suffices to check that the two conditions in the definition of a degree d_S stability condition hold.

(1) If G is a connected, spanning subgraph of Γ_S, then if e is an edge in $E(\Gamma_S) \setminus E(G)$, then for $(d_v)_v \in \sigma_S(G)$, we have $(d_v + |S_v|)_v \in \sigma(G)$ and therefore if v_1 and v_2 are the endpoints of e, $\underline{e}_{v_i} + (d_v + |S_v|)_v$ is in $\sigma(G \cup \{e\})$ and $\underline{e}_{v_i} + (d_v)_v$ is in $\sigma_S(G \cup \{e\})$ for $i = 1, 2$.

(2) For each connected, spanning subgraph G of Γ_S and for $\underline{d} \in \sigma_S(G)$, we have:

$$\begin{aligned} \sum_{v\in V(\Gamma)} d_v &= d - \sum_{v\in V(\Gamma)} |S_v| - |E(\Gamma) \setminus E(G)| \\ &= d + (|S| - \sum_{v\in V(\Gamma)} |S_v|) - |E(\Gamma_S) \setminus E(G)| \\ &= d_S - |E(\Gamma_S) \setminus E(G)|. \end{aligned}$$

Therefore $\sigma_S(G) \subset S^{d_S}_{\Gamma_S}(G)$. We also have $S^{d_S}_{\Gamma_S}(G) = S^d_\Gamma(G) - (|S_v|)_v$ and therefore, since $\sigma(G)$ is a minimal, complete set of representatives for the action of $\mathrm{Tw}(G)$ on $S^{d_S}_{\Gamma_S}(G)$, using the fact that $\sigma_S(G) = \sigma(G) - (|S_v|)_v$.

Given a stability condition σ on $\Gamma = \Gamma(C)$, we write:

$$\overline{J}_{\sigma,\Gamma_S}(C) = \bigsqcup_{\underline{d}\in\sigma(\Gamma_S)} \overline{J}_{(\Gamma_S,\underline{d})}(C) \subset \overline{J}^d_\sigma(C).$$

The following proof is adapted from the proof of [**19**, theorem 4·1].

THEOREM 4·3. *Given a nodal curve and a degree d stability condition, σ on $\Gamma = \Gamma(C)$, there is a morphism $(\nu_S)_* : Simp^{d_S}(C_S) \to Simp^d(C)$ induced by the normalisation $\nu_S : C_S \to C$. It gives rise to an isomorphism*

$$(\overline{J}^{d_S}_{\sigma_S}(C_S))_{sm} = \overline{J}^{d_S}_{\sigma_S,\Gamma_S}(C_S) \cong \overline{J}^d_{\sigma,\Gamma}(C).$$

Here $(\overline{J}^{d_S}_{\sigma_S}(C_S))_{sm}$ denotes the smooth locus of $\overline{J}^{d_S}_{\sigma_S}(C_S)$, or alternatively, the locus where points correspond to line bundles.

Proof. The morphism $(\nu_S)_* : Simp^{d_S}(C_S) \to Simp^d(C)$ is a monomorphism as a result of [**11**, lemma 2·4], which may be applied with families of reduced, not just integral curves (see proof of [**19**, theorem 4·1]). It states that for a given scheme T, the functor

$$\left\{\begin{array}{l} T\text{-flat rank 1, torsion free} \\ \text{sheaves on } C_S \times T \text{ which} \\ \text{have degree } d \text{ over fibers} \\ \text{of } C_S \times T \to T \end{array}\right\} \xrightarrow{(\nu_{S,T})_*} \left\{\begin{array}{l} T\text{-flat rank 1, torsion free} \\ \text{sheaves on } C \times T \text{ which} \\ \text{have degree } d \text{ over fibers} \\ \text{of } C \times T \to T \end{array}\right\},$$

where $\nu_{S,T}$ is the morphism $\nu_{S,T}: C_S \times T \to C \times T$, is a fully faithful embedding. By Equation (4·1) and Lemma 4·2, this induces a morphism:

$$(\nu_S)_* : \overline{J}^d_{\sigma_S}(C_S) \to \bigsqcup_{G\subset\Gamma_S} \overline{J}_{\sigma,G}(C) \subset \overline{J}^d_\sigma(C)$$

over $\bigsqcup_{G\subset\Gamma_S} \overline{J}_{\sigma,G}(C)$. This is a morphism from a proper scheme to a separated scheme and is therefore a proper monomorphism so is a closed immersion. By Lemma 4·1 and Equation (4·1), it induces a bijection of geometric points on the restriction $(\overline{J}^{d_S}_{\sigma_S}(C_S))_{sm} \to \overline{J}^d_{\sigma,\Gamma_S}(C)$ over $\overline{J}^d_{\sigma,\Gamma_S}(C)$. Therefore, the morphism $(\overline{J}^{d_S}_{\sigma_S}(C_S))_{sm} \to \overline{J}^d_{\sigma,\Gamma_S}(C)$ is an isomorphism.

Remark 4·4. For each $\underline{d} \in \sigma_S(\Gamma_S)$, we have $\overline{J}_{(\Gamma_S,\underline{d})} = J^{\underline{d}}(C_S)$, the degree $\underline{d}$ generalised Jacobian of C_S. Therefore, Theorem 4·3 tells us that, given a stability condition σ on $\Gamma(C)$, $\overline{J}^d_\sigma(C)$ is the disjoint union of locally closed strata, each of which is isomorphic to a generalised Jacobian of some partial normalisation of C at a subset of non-disconnecting nodes.

COROLLARY 1. *Suppose we have a fine, smoothable compactified Jacobian $\overline{J}^d_\sigma(C)$ of a nodal curve C corresponding to a stability condition σ on $\Gamma = \Gamma(C)$ and a subset $S \subset E(\Gamma(C))$ whose removal results in a connected graph. Then $\overline{J}_{\sigma,\Gamma_S}(C)$ has a stratification into $|\sigma_S(\Gamma_S)| = c(\Gamma_S)$ strata, each of which is isomorphic to a generalised Jacobian on C_S. By abuse of notation, we write*

$$\overline{J}_{\sigma,\Gamma_S}(C) = \bigsqcup_{\underline{d}\in\sigma_S(\Gamma_S)} J^{\underline{d}}(C_S),$$

and

$$\overline{J}^d_\sigma(C) = \bigsqcup_{\substack{S\subset E(\Gamma)\\ \text{non-disconnecting}}} \bigsqcup_{\underline{d}\in\sigma_S(\Gamma_S)} J^{\underline{d}}(C_S),$$

where $J^{\underline{d}}(C_S)$ *is the generalised Jacobian of* C_S *of multidegree* $\underline{d}$.

Let $G(g,n)^0 \subset G(g,n)$ be the set of representatives of isomorphism classes of stable graphs where all vertices have genus 0. We recall the notation that if C is a nodal curve:

$$b_1(\Gamma(C)) = |E(\Gamma(C))| - |V(\Gamma(C))| + 1 \geq 0.$$

LEMMA 4·5. *If* C *is any nodal curve such that in its total normalisation, there is some connected component which has genus at least* 1*, or equivalently if* $\Gamma(C) \in G(g,n) \setminus G(g,n)^0$*, then for any multidegree* $\underline{d} \in \mathbb{Z}^{|V(\Gamma(C))|}$*, it follows that* $\chi_{top}((J^{\underline{d}}(C))^{an}) = 0$.

This follows from [5, proposition 2·2]

COROLLARY 2. *If* C *is a nodal curve whose dual graph* $\Gamma(C)$ *lies in* $G(g,n) \setminus G(g,n)^0$*, then for any stability condition* σ *on* $\Gamma(C)$*, we have* $\chi_{top}((\overline{J}^d_\sigma(C))^{an})= 0$.

Proof. This follows from the fact that for finite type separated schemes, the topological Euler characteristic is additive with respect to locally closed stratifications, the stratification given in Theorem 4·3 and Lemma 4·5. For each non-disconnecting subset $S \subset E(C)$, the partial normalisation C_S has dual graph with vertices having the same genera as those of $\Gamma = \Gamma(C)$. So for each $\underline{d} \in \sigma_S$, by Lemma 4·5, $\chi_{top}((J^{\underline{d}}(C_S))^{an}) = 0$ and therefore:

$$\begin{aligned}\chi_{top}((\overline{J}^d_\sigma(C))^{an}) &= \sum_{\substack{S\subset E(\Gamma)\\ \text{non-disconnecting}}} \chi_{top}((\overline{J}_{\sigma,\Gamma_S}(C))^{an})\\ &= \sum_{\substack{S\subset E(\Gamma)\\ \text{non-disconnecting}}} \sum_{\underline{d}\in\sigma_S} \chi_{top}((J^{\underline{d}}(C_S))^{an}) = 0.\end{aligned}$$

LEMMA 4·6. *Let* C *be a nodal curve such that* $\Gamma(C) \in G(g,n)^0$*, let* $\underline{d} \in \mathbb{Z}^{|V(\Gamma(C))|}$ *be any multidegree and* $S \subset E(\Gamma(C))$*, then*:

$$\chi_{top}((J^{\underline{d}}(C_S))^{an}) = \begin{cases} 1 & \Gamma(C_S) \text{ is a tree,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We have an isomorphism $J^{\underline{d}}(C_S) \cong J^{\underline{0}}(C_S)$ given by tensoring with a fixed line bundle on C_S of appropriate multidegree. By [2, p.89], $J^{\underline{0}}(C) \cong (\mathbb{C}^\times)^{b_1(\Gamma(C))}$. For $n > 0$, $\chi_{top}((\mathbb{C}^\times)^n) = 0$. So all that remains is to consider the case when $|E(C_S)| = |V(C_S)| - 1$, or equivalently, when $b_1(\Gamma(C_S)) = 0$. This corresponds to C_S being a tree and, in this case, $J^{\underline{0}}(C_S)$ is a point so $\chi_{top}((J^{\underline{0}}(C_S))^{an}) = 1$.

COROLLARY 3. *If* $[(C, p_1, ..., p_n)]$ *is a nodal curve of genus 0 such that the corresponding stable graph* $\Gamma = \Gamma(C)$ *lies in* $G(g,n)^0$*, then*:

$$\chi_{top}((\overline{J}^d_\sigma(C))^{an}) = c(\Gamma)$$

for any stability condition σ *on* Γ.

Proof. By the additivity of the orbifold Euler characteristic with respect to locally closed stratifications:

$$\begin{aligned}\chi_{top}(\overline{J}^d_\sigma(C)) &= \sum_{S\subset E(\Gamma)} \chi_{top}(\overline{J}_{\sigma,\Gamma_S}(C)) \\ &= \sum_{S\subset E(\Gamma)} \sum_{\underline{d}\in\sigma_S(\Gamma_S)} \chi_{orb}(J^{\underline{d}}(C_S)) \\ &= \sum_{\{S\subset E(\Gamma)\,:\,\Gamma_S \text{ is a tree}\}} |\sigma_S(\Gamma_S)|.\end{aligned}$$

The first equality follows from the stratification $\overline{J}^d_\sigma(C) = \bigsqcup_{S\subset E(\Gamma)} \overline{J}_{\sigma,\Gamma_S}(C)$. The second follows from the stratification given in Corollary 1 and the third follows from Lemma 4·5.

Since $|\sigma_S(\Gamma_S)| = c(\Gamma_S) = 1$, whenever Γ_S is a tree, it follows that $\chi_{top}(\overline{J}^d_\sigma(C)) = c(\Gamma)$.

THEOREM 4·7. *For all $d\in\mathbb{Z}$, $g,n\geq 0$ such that $2g-2+n>0$ and all fine compactified universal Jacobians $\overline{\mathcal{J}}^d_{g,n}$ of degree d,*

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \sum_{\Gamma\in G(g,n)^0} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma).$$

In particular, this is independent of d and the choice of fine compactified universal Jacobian.

Proof. The stratification of $\overline{\mathcal{M}}_{g,n}$ by dual graph induces a stratification of $\overline{\mathcal{J}}^d_{g,n} = \bigsqcup_{\Gamma\in G(g,n)} \overline{\mathcal{J}}^d_{g,n}|_{\mathcal{M}^\Gamma}$. By additivity of orbifold Euler characteristics with respect to locally closed stratifications:

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \sum_{\Gamma\in G(g,n)} \chi_{orb}(\overline{\mathcal{J}}^d_{g,n}|_{\mathcal{M}^\Gamma}).$$

Suppose $\Gamma\in G(g,n)$ and $[(C^\Gamma;p_1,...,p_n)]$ is in $\mathcal{M}^\Gamma(\mathbb{C})$ such that

$\overline{J}^d_{\sigma(C^\Gamma)}(C^\Gamma)$ is the fiber over $[(C^\Gamma;p_1,...,p_n)]$ which is a smoothable, fine compactified Jacobian with $\sigma(C^\Gamma)$ being a corresponding stability condition. Then combining Proposition 3·9 with Corollary 2 and Corollary 3, we obtain:

$$\begin{aligned}\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}|_{\mathcal{M}^\Gamma}) &= \chi_{orb}(\mathcal{M}^\Gamma)\chi_{top}(\overline{J}^d_{\sigma(C^\Gamma)}(C^\Gamma)) \\ &= \begin{cases} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma) & \text{if } \Gamma\in G(g,n)^0 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

So $\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \sum_{\Gamma\in G(g,n)^0} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma)$.

THEOREM 4·8. *For all g and n such that $2g-2+n>0$,*

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \frac{1}{2^g(g!)}\chi(\overline{\mathcal{M}}_{0,2g+n}),$$

where $\chi(\overline{\mathcal{M}}_{0,2g+n})$ is the ordinary topological Euler characteristic of the variety $\overline{\mathcal{M}}_{0,2g+n}$.

Proof. Let $\widehat{G}(g,n)^0$ be a set of representatives of isomorphism classes of pairs (Γ, T) such that $[\Gamma] \in G(g,n)^0$ and T is a spanning tree of Γ. An isomorphism of (Γ, T) is an automorphism of Γ fixing T and we denote this group by $\mathrm{Aut}(\Gamma, T)$. The group $\mathrm{Aut}(\Gamma)$ acts on the set of spanning trees of Γ. The stabiliser of a fixed spanning tree T under this action is given by $\mathrm{Aut}(\Gamma, T)$. Therefore,

$$\begin{aligned}
\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) &= \sum_{\Gamma \in G(g,n)^0} c(\Gamma)\chi_{orb}(\mathcal{M}^\Gamma) \\
&= \sum_{\Gamma \in G(g,n)^0} \sum_{\substack{T \text{ is a} \\ \text{spanning} \\ \text{tree of } \Gamma}} \chi_{orb}(\mathcal{M}^\Gamma) \\
&= \sum_{(\Gamma,T) \in \widehat{G}(g,n)^0} |\mathrm{orb}(T)|\chi_{orb}(\mathcal{M}^\Gamma).
\end{aligned}$$

Additionally, by [7]:

$$\chi_{orb}(\mathcal{M}^\Gamma) = \frac{\prod_{v \in V(\Gamma)} \chi_{orb}(\mathcal{M}_{0,n(v)})}{|\mathrm{Aut}(\Gamma)|}.$$

We combine this with the fact that by the orbit stabiliser theorem, $|\mathrm{Aut}(\Gamma)| = |\mathrm{Aut}(\Gamma, T)||\mathrm{orb}(T)|$ (where T is any spanning tree of Γ) to obtain:

$$\begin{aligned}
\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) &= \sum_{(\Gamma,T) \in \widehat{G}(g,n)^0} |\mathrm{orb}(T)| \frac{\prod_{v \in V(\Gamma)} \chi_{orb}(\mathcal{M}_{0,n(v)})}{|\mathrm{Aut}(\Gamma)|} \\
&= \sum_{(\Gamma,T) \in \widehat{G}(g,n)^0} \frac{\prod_{v \in V(\Gamma)} \chi_{orb}(\mathcal{M}_{0,n(v)})}{|\mathrm{Aut}(\Gamma, T)|}.
\end{aligned}$$

We define a surjective function $\phi : G(0, 2g+n) \to \widehat{G}(g,n)^0$. Gluing pairs of markings $n+2i-1, n+2i$ for $1 \le i \le g$ of $T' \in G(g,n)^0$ forms a stable graph Γ with n legs of genus g and the images of the original edges in T' result in a spanning tree T or Γ. We define $\phi(T') = (\Gamma, T)$. We have $|\phi^{-1}([(\Gamma, T)])| = 2^g(g!)/|\mathrm{Aut}(\Gamma, T)|$ which can be seen as follows.

Fixing (Γ, T) in $\hat{G}(g,n)^0$, any T' in the preimage can be obtained by matching pairs of half-edges which form an edge in $E(\Gamma) \setminus E(T)$ to the entries of the g pairs $(n+1, n+2), ..., (2g+n-1, 2g+n)$. Then T' is obtained by cutting these edges and identifying the corresponding half-edges with the markings following the assignments previously described. There are $g!$ ways to match pairs of half-edges in $E(\Gamma) \setminus E(T)$ with pairs $(n+1, n+2), ..., (2g+n-1, 2g+n)$. Once the pairs of half-edges have been assigned, there are 2^g ways of determining the ordering of the half-edges within the pairs. Let S be the set of these $2^g(g!)$ choices. Then there is a surjective map $S \to \phi^{-1}(\Gamma, T)$ where the image can be identified with the orbit classes of an action of $\mathrm{Aut}(\Gamma, T)$ on S. This action is described as follows. If $T' \in S$ and there is an element σ of $\mathrm{Aut}(\Gamma, T)$ permuting the half-edges in the edge (1, 2), then $\sigma \cdot T'$ is the element of S obtained by swapping markings 1 and 2 of T'. This action is free so all the orbits, which partition S have size $|\mathrm{Aut}(\Gamma, T)|$ and therefore:

$$|\phi^{-1}((\Gamma, T))| = \frac{|S|}{|\mathrm{Aut}(\Gamma, T)|} = \frac{2^g(g!)}{|\mathrm{Aut}(\Gamma, T)|}.$$

Observing that $G(0, 2g+n)^0 = G(0, 2g+n)$, we obtain:

$$\chi_{orb}(\overline{\mathcal{J}}^d_{g,n}) = \sum_{(\Gamma,T)\in\widehat{G}(g,n)^0} \sum_{\{T'\in\phi^{-1}((\Gamma,T))\}} \frac{\prod_{v\in V(\Gamma)} \chi_{orb}(\mathcal{M}_{0,n(v)})}{|Aut(\Gamma,T)||\phi^{-1}(\Gamma,T)|}$$
$$= \sum_{T'\in G(0,2g+n)} \frac{\prod_{v\in V(T')} \chi_{orb}(\mathcal{M}_{0,n(v)})}{2^g(g!)} = \frac{\chi_{orb}(\overline{\mathcal{M}}_{0,2g+n})}{2^g(g!)}.$$

Where in the last two equalities, we use the fact that $\overline{\mathcal{M}}_{0,2g+n}$ has a locally closed stratification $\overline{\mathcal{M}}_{0,2g+n} = \bigsqcup_{T'\in G(0,2g+n)} \mathcal{M}^{T'}$ and the fact that for $T' \in G(0, 2g+n)$, the automorphism group of T' is trivial and therefore, the gluing morphism $\xi_{T'} : \prod_{v\in V(T')} \mathcal{M}_{0,n(v)} \to \mathcal{M}^{T'}$ is an isomorphism onto its image.

We conclude using the fact that $\overline{\mathcal{M}}_{0,2g+n}$ is a projective variety, hence its orbifold Euler characteristic is equal to its topological Euler characteristic.

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