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GEODESIC FLOW OF VISIBILITY MANIFOLDS

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Abstract

We prove that the conservativity of the geodesic flow is equivalent to the ergodicity of the geodesic flow with respect to the Bowen-Margulis measure on visibility manifolds.

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1. Introduction

In this paper, we study the ergodicity of the geodesic flow on a manifold with weak hyperbolic properties. The weak hyperbolic properties are to have a nonpositive sectional curvature and a geodesic line between two distinct points in the ideal boundary at infinity, which are the important properties in hyperbolic manifold $\mathbb{H}^n$. These properties were introduced by Eberlein in [4, 5] and we use the term 'visibility manifold' to describe a manifold with such properties.

We are interested in finding a measure on the unit tangent space that is invariant under the geodesic flow, and in determining the ergodicity of geodesic flow with respect to this measure. In [10], Sullivan constructed a measure on the unit tangent bundle, which is invariant with respect to the geodesic flow on a hyperbolic manifold with the constant curvature -1. The measure was called the Bowen-Margulis measure and it maximized the measure entropy for geodesic flow on the compact hyperbolic manifold. He constructed this measure using the Patterson-Sullivan measures on the ideal boundary at infinity. He showed that the conservativity of the geodesic flow is equivalent to the ergodicity of the geodesic flow, with respect to the measure. In [11], Yue extended these results to a Cartan Hadamard manifold with a sectional curvature pinched by two negative constants, using the method that Sullivan had used in [10]. Here we show that these results are still true on visibility manifolds.

Let H be an n-dimensional, complete and simply connected Riemannian manifold without conjugate points. We say that H satisfies *the visibility axiom* if, for every point $p \in H$ and every number $\epsilon > 0$, there exists $R = R(p, \epsilon) > 0$ such that for any geodesic $\gamma : \mathbb{R} \to H$ with $d(p, \gamma) \geq R$, $\angle_p(\gamma) \leq \epsilon$, where

$$\angle_p(\gamma) = \sup\{\angle_p(\gamma(t), \gamma(s)) \mid t, s \in \mathbb{R}\}.$$

Approximately speaking, the visibility axiom means that geodesics with a sufficiently large distance from a point look small. In [2], it has been shown that the visibility axiom is equivalent to the property that any two distinct points in the ideal boundary are always joined by a geodesic line in H. It should be noted that for some two distinct points in the ideal boundary of H, there may be more than one geodesic line between the two points. Two geodesic lines between the same couple of points in the ideal boundary bound a flat strip. When we say that H satisfies *the uniform visibility axiom*, we attain a constant $R = R(\epsilon)$, independent of $p \in H$ in the definition of visibility axiom. This axiom implies that H can contain only a flat strip not a half plane.

Let us define a visibility manifold in general. Let M be a Riemannian manifold without conjugate points. M is said to be a *visibility manifold* if the universal cover H of M satisfies the uniform visibility axiom.

Suppose H satisfies the uniform visibility axiom. Let ∂H be the ideal boundary at infinity. ∂H is then equivalent to the set of the geodesic rays from a fixed point in H. With the cone topology, H is diffeomorphic to an open disc D^n, and the ideal boundary ∂H of H at infinity is homeomorphic to a sphere S^{n-1} in $\mathbb{R}^n$. Let Γ be a torsion-free and discrete isometry group acting on H freely and properly discontinuously.

A limit point of Γ is a limit point of an orbit Γx for some $x \in H$. Because Γ is discrete, there is no limit point of Γx in H and the limit points for Γx and Γy are the same for any x and y in H. The set of all limit points is denoted by $L(\Gamma)$. According to Eberlein [3], $L(\Gamma)$ has a singleton, two points, or infinitely many points on a visibility manifold. From now on, we only deal with the case that $L(\Gamma)$ has infinitely many points. Such Γ is called a *Fuchsian group*.

In [8], Knieper showed that the geodesic flow is ergodic with respect to the Bowen-Margulis measure on a compact manifold of rank 1. It should be noted that he proved the results without a condition on the sectional curvature except its nonpositivity. In this paper we prove that the geodesic flow is ergodic with respect to the Bowen-Margulis measure without the compactness of M.

MAIN THEOREM. *Suppose that M is a visibility manifold with nonpositive sectional curvature. Let H be a universal cover of M and $M = \Gamma \backslash H$, where Γ is a Fuchsian* *group. If the geodesic flow is conservative with respect to the Bowen-Margulis measure on the unit tangent space of M, then it is ergodic with respect to the same measure.*

We prove the Main Theorem as Theorem 3.10 in Section 3. The converse of the Main Theorem is clearly true. The Main Theorem says that the conservativity of geodesic flow is equivalent to the ergodicity of the geodesic flow with respect to the Bowen-Margulis measure. When Sullivan proved the same result as the main theorem in Hyperbolic Manifold $\mathbb{H}^n$ with the constant sectional curvature -1 in [10], he used the Hopf's generalization of Birkhoff's ergodic theorem to prove that the geodesic flow was also ergodic under the condition that the volume of M is infinite. At that time, the asymptotic geodesic rays played an important role. In $\mathbb{H}^n$, the distance between two asymptotic rays converges to 0 as the rays go to the same boundary point at infinity. This still holds in a manifold with strictly negative curvature.

If we consider the Euclidean space $\mathbb{R}^n$ or a manifold with a flat strip, then the distance of two asymptotic geodesic rays which bound a flat strip does not converge to 0 at infinity. Our visibility manifold may contain a flat strip, and we have difficulty in using the Hopf's generalization of Birkhoff's ergodic theorem. We overcome this difficulty under the hypothesis that the geodesic flow is conservative. The conservative set can be expressed with the radial limit set. We control the distance between two asymptotic geodesic rays converging to the same radial limit point. In fact, in Theorem 3.10, we prove that the distance between two asymptotic rays converges to 0 as the rays go to the same radial limit point. If the geodesic flow is conservative, then the radial limit set has full measure of SM. Therefore, in Theorem 3.10, we prove the ergodicity of the geodesic flow using the Hopf's generalization of Birkhoff's ergodic theorem.

2. Conformal density and Bowen-Margulis measure

Let H be an n-dimensional, complete and simply connected Riemannian manifold with nonpositive sectional curvature. Let Γ be a torsion-free, discrete isometry group acting on H freely and properly discontinuously in H. Let $M = \Gamma\backslash H$ be a visibility manifold. In order to construct the Bowen-Margulis measure, we begin with the construction of the Patterson-Sullivan measures in ∂H and some notations defined in [7]. We define the family of measures μ_x, for any $x \in H$, by

$$\mu_x = \lim_{s\to\delta(\Gamma)^+} \frac{1}{g_s(y, y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}\delta_{\gamma y}, \quad s > \delta(\Gamma),$$

where $g_s(x, y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}$ for $s > 0$ and $x, y \in H$, $\delta(\Gamma)$ is the critical exponent of Γ, and $\delta_{\gamma y}$ is the Dirac mass at γy. The measure μ_x is concentrated on $L(\Gamma)$. For any other point $x' \in H$, $\mu_{x'}$ and μ_x are absolutely continuous and moreover the Radon-Nikodým derivative at $\xi \in L(\Gamma)$ is

$$\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta(\Gamma)\rho_{x,\xi}(x')}, \tag{2.1}$$

where $\rho_{x,\xi}(x')$ is a Busemann function.

For $\gamma \in \Gamma$, we have

$$\gamma^*\mu_x = \mu_{\gamma^{-1}(x)}. \tag{2.2}$$

Generally, the family $\{\mu_x\}$ of measures on $L(\Gamma)$ satisfying (2.1) and (2.2) is called a $\delta(\Gamma)$-conformal density or Patterson Sullivan measures.

Let $c > 0$. Then $O_{x_0}(x, c) = \{\eta \in \partial H \mid c_{x_0,\eta} \cap B(x, c) \neq \emptyset\}$ is a shadow of a ball $B(x, c)$ from x_0 into ∂H. We say that a point $\zeta \in \partial H$ is a *radial limit point* if, for some $c > 0$ and $x \in H$, ζ belongs to infinitely many $O_x(\gamma x, c)$ for $\gamma \in \Gamma$. The radial limit set is the set of all radial points and is denoted by $L^r(\Gamma)$. We understand that $\zeta \in L^r(\Gamma)$ means that any geodesic ray from $x \in H$ to ζ intersects some c-neighbourhood of Γx infinitely many times. Obviously, $L^r(\Gamma)$ is non-empty. The following theorem proves the uniqueness of the $\delta(\Gamma)$-conformal density on H. Its proof can be found in [7].

THEOREM 2.1. *Suppose that $\{\mu_x\}_{x\in H}$ is an α-conformal density of a Fuchsian group Γ and $\mu_x(L^r(\Gamma)) > 0$. Then*

(1) $\mu_x(L^r(\Gamma)) = \mu_x(\partial H)$;

(2) $\alpha = \delta(\Gamma)$;

(3) *$\{\mu_x\}_{x\in H}$ is the unique $\delta(\Gamma)$-conformal density of Γ and Γ is ergodic on H with respect to $\{\mu_x\}_{x\in H}$;*

(4) *Γ is of divergent type.*

Let SH and SM be the unit tangent space of H and M respectively. Consider the canonical Γ-action on $\partial H \times \partial H$ induced by the Γ-action on ∂H, which is defined by $\gamma(\eta, \xi) = (\gamma\eta, \gamma\xi)$ for all $\gamma \in \Gamma$ and $\eta, \xi \in \partial H$. Since $\Gamma\backslash H = M$ we have $\Gamma\backslash SH = SM$. If we construct a measure on SH that is invariant with respect to the geodesic flow on SH and the Γ-action on SH, then it is a measure on SM that is invariant with respect to the geodesic flow on SM.

A geodesic can be thought of as two points in the ideal boundary in a hyperbolic manifold. Each geodesic $c : \mathbb{R} \to H$ determines two end points in ∂H such that

$$(c(-\infty) = c_-, c(+\infty) = c_+) \in (\partial H \times \partial H - \{\text{diag}\}).$$

Conversely, for every two distinct points $(\eta, \xi) \in (\partial H \times \partial H - \{\text{diag}\})$, there is an infinite geodesic c on H satisfying $c(-\infty) = \eta$ and $c(+\infty) = \xi$. However, this geodesic c may not be a unique one between η and ξ on the visibility manifold, even though we think of geodesics up to the reparametrization.

Let $x \in H$ be fixed. For a geodesic line c on H, $x(c)$ is the point in c satisfying $d(x(c), x) = d(x, c)$. We define the following map:

$$\mathscr{F} : SH \to \partial H \times \partial H \times \mathbb{R}$$

by $\mathscr{F}(v) = (c_v(-\infty), c_v(+\infty), t)$, where $c_v : \mathbb{R} \to H$ is a geodesic $\dot{c}_v(0) = v$ and $t = d(c_v(0), x(c))$. This map $\mathscr{F}$ is a surjective map and $\mathscr{F}(\gamma v) = \gamma \mathscr{F}(v)$ for $\gamma \in \Gamma$ and $v \in SH$. Let g^t denote the geodesic flow on SH or SM ambiguously. If we have a Γ-invariant measure ν on $\partial H \times \partial H$, we can obtain a g^t-invariant measure $\mathscr{F}^*(\nu \times dt)$ on SM defined by $\mathscr{F}^*(\nu \times dt)(A) = (\nu \times dt)(\mathscr{F}(A))$, for all $A \subset SM$.

THEOREM 2.2. *Let μ be any $\delta(\Gamma)$-conformal density of Γ. A measure dU_x^μ on $\partial H \times \partial H \times \mathbb{R}$ is defined by*

$$dU_x^\mu(\eta, \xi, t) = e^{\delta(\Gamma)\beta_x(\eta,\xi)} d\mu_x(\eta) d\mu_x(\xi) dt,$$

where $\beta_x(\eta, \xi) = \rho_{x,\eta}(y) + \rho_{x,\xi}(y)$ for any point y on the geodesic from η to ξ. The measure dU_x^μ is therefore locally finite and invariant under the action of Γ. Furthermore, this measure dU_x^μ does not depend on the choice of the base points.

Yue proved this result on the Cartan-Hadamard manifold. We prove Theorem 2.2 on the visibility manifold with a little modification of Yue's. Since $\mathscr{F}^*(dU^\mu)$ is invariant under the Γ-action, we can canonically induce a measure on $SM = \Gamma \backslash SH$, which is also invariant under the geodesic flow g^t. The corresponding g^t-invariant measure on SM is denoted by $d\sigma^\mu$, which is called *the Bowen-Margulis measure* on SM.

3. Ergodicity of geodesic flow

In this section, we prove the ergodic property of σ^μ under the geodesic flow. Note that SM is not necessarily compact and the measure $\sigma^\mu(SM)$ might be infinite. We cannot therefore use the Birkhoff ergodic theorem. We recall Hopf's generalization of this theorem without proof. Let (X, d) be a separable and complete metric space which is equipped with a σ-finite measure μ on its Borel subsets. Let T^t be a continuous flow in X.

THEOREM 3.1. *If $f, h \in L^1(X)$, $h > 0$ and $\int_0^u h(T^s(x))\,ds \to \infty$ as $u \to \infty$ for almost all $x \in X$, then the limit*

$$\phi(x) = \lim_{u \to \infty} \frac{\int_0^u f(T^s(x))\,ds}{\int_0^u h(T^s(x))\,ds}$$

exists almost all. The function ϕ is measurable and T^t-invariant.

The flow T^t on the space X is said to be *ergodic* if, whenever A is a measurable and T^t-invariant subset of X, either $\mu(A) = 0$ or $\mu(X - A) = 0$.

THEOREM 3.2. *If T^t is ergodic and f, h satisfy the hypotheses of Theorem 3.1, then for almost all $x \in X$*

$$\lim_{u\to\infty} \frac{\int_0^u f(T^s(x))\,ds}{\int_0^u h(T^s(x))\,ds} = \frac{\int_X f\,d\mu}{\int_X h\,d\mu}.$$

Theorem 3.2 means that if the geodesic flow is ergodic then the function ϕ given in Theorem 3.1 must be constant for almost all $x \in X$. The Remark 3.3 below was proved in [9] and [6].

REMARK 3.3. (1) The converse of Theorem 3.2 is also true. In general, the converse has been used in proving the ergodicity of a flow.
(2) Theorem 3.2 and (1) of Remark 3.3 are also true if the limit

$$\phi(x) = \lim_{u\to-\infty} \frac{\int_0^u f(T^s(x))\,ds}{\int_0^u h(T^s(x))\,ds}$$

exists for almost all $x \in X$.

We now define the conservative and dissipative sets associated with a flow T^t. The conservative set plays an important role when we deal with a dynamical system on a non-compact space.

DEFINITION 3.4. Let T^t be a flow on X. A point $x \in X$ is called a *dissipative* point for T^t if for any compact $A \subset X$ there exists a $t_0 > 0$ such that $T^t x \notin A$ for all $t \geq t_0$. Otherwise x is called *conservative*. Let C^{T^t} and D^{T^t} denote the set of all conservative and dissipative points in X, respectively. The flow T^t is said to be conservative with respect to μ if $\mu(D^{T^t}) = 0$.

We return to our notation. Let Γ be a Fuchsian group acting on H and let μ be a $\delta(\Gamma)$- conformal density on ∂H that is invariant under Γ. Let g^t be the geodesic flow on SH or SM and let σ^μ be the Bowen-Margulis measure on SM.

THEOREM 3.5. *If g^t is a conservative geodesic flow on SH with respect to $d\sigma^\mu$, then $\sum_{\gamma\in\Gamma} e^{-\delta(\Gamma)d(x,\gamma x)} = \infty$.*

PROOF. By the definition of the conservativity of the g^t, the image of the conservative set of the geodesic flow g^t on SM under $\mathscr{F}$ is the Γ-quotient of

$$(\partial H \times L^r(\Gamma) - \{\text{diag}\}) \times \mathbb{R}.$$

We assume that $\sum_{\gamma\in\Gamma} e^{-\delta(\Gamma)d(x,\gamma x)} < \infty$. In [7], it was shown that if

$$\sum_{\gamma\in\Gamma} e^{-\delta(\Gamma)d(x,\gamma x)} < \infty,$$

then $\mu_x(L^r(\Gamma)) = 0$. Therefore, we have that $\sigma^\mu(C^{g^t}) = 0$ and the geodesic flow g^t on SM is dissipative. □

In order to prove the ergodicity of the geodesic flow, we have to show there is a positive integrable function h on X, satisfying that $\int_0^u h(T^s(x))\,ds \to \infty$ as $u \to \infty$ for almost all $x \in X$ (see Remark 3.3 (1)). Since h is to be σ^μ-integrable on SM, we need some estimate to control the Bowen-Margulis measure σ^μ on SM. Recall that the Bowen-Margulis measure is defined on $\partial H \times \partial H \times \mathbb{R}$ by

$$dU_x^\mu(\eta, \xi, t) = e^{\delta(\Gamma)\beta_x(\eta,\xi)}\,d\mu_x(\eta)\,d\mu_x(\xi)\,dt$$

for any two points $\eta, \xi \in \partial H$, where $\beta_x(\eta, \xi) = \rho_{x,\eta}(y) + \rho_{x,\xi}(y)$ for any point y on the geodesic from η to ξ. Since the Patterson-Sullivan measure μ_x is finite on ∂H, it suffices to estimate only β_x on $\partial H \times \partial H$.

For any two points η and $\xi \in \partial H$, consider a geodesic c from η to ξ and

$$d(x, c) = \min\{d(x, c(t)) \mid t \in \mathbb{R}\} < \infty.$$

Define a map $D_x : \partial H \times \partial H \to \mathbb{R}$ by

$$D_x(\eta, \xi) = \min\{d(x, c) \mid c \text{ is a geodesic between } \eta \text{ and } \xi\}.$$

This map is well defined because of the uniform visibility axiom. Furthermore, we can get a geodesic c between η and ξ such that $D_x(\eta, \xi) = d(x, c)$.

LEMMA 3.6. *For all $x \in H$ and $\xi, \eta \in \partial H$, $\beta_x(\xi, \eta) \le 2D_x(\xi, \eta)$.*

PROOF. We consider $\beta_x(\xi, \eta)$ geometrically. Note that $\beta_x(\xi, \eta)$ is the length of the segment on γ cut out by the horosphere which is passing through x and centered at ξ and η, where γ is a geodesic line between ξ and η. Let c be a geodesic between ξ and η. It suffices to show that $\beta_x(\xi, \eta) \le 2d(x, c)$.

We may assume that x is not on c. Let q be the point on c closest to x and let $H(\xi, q)$ and $H(\eta, q)$ be horospheres centered at ξ and η respectively that pass through q. The geodesic from q to x is orthogonal to c and hence tangent to both $H(\xi, q)$ and $H(\eta, q)$ Since horoballs are convex, x must lie outside the interiors of those horoballs bounded by $H(\xi, q)$ and $H(\eta, q)$. This allows us to define p_ξ as the point where $H(\xi, q)$

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intersects that geodesic ray from x to ξ and to define p_η as the point where $H(\eta, q)$ intersects the geodesic ray from x to p_η. It is not difficult to show that

$$d(x, p_\xi) + d(x, p_\eta) = \beta_x(\xi, \eta).$$

On the other hand,

$$d(x, p_\xi) = d(x, H(\xi, q)) \leq d(x, q) \quad \text{and}$$
$$d(x, p_\eta) = d(x, H(\eta, q)) \leq d(x, q).$$

Thus $\beta_x(\xi, \eta) = d(x, p_\xi) + d(x, p_\eta) = 2d(x, c)$. □

For all $v \in S_x M$, let $c_v : \mathbb{R} \to M$ be a geodesic with $c_v'(0) = v$ and $c_v(0) = x$.

THEOREM 3.7. *There is a positive function ρ on SM such that ρ is integrable with respect to σ^μ and for all $v, w \in SM$ with $d(c_v(0), c_w(0)) \leq 1$,*

$$\frac{(\rho(v) - \rho(w))}{\rho(w)} \leq C_1 d(c_v(0), c_w(0)),$$

where $C_1 > 0$ is a constant and d is the metric on M.

PROOF. Fix a point x in M. Define a function $\tau : SM \to [0, \infty)$ by

$$\tau(v) = d(c_v(0), x).$$

Let $B_r = \tau^{-1}[0, r]$. Let us estimate $\sigma^\mu(B_r)$. If l is a geodesic passing through $B(x, r)$, then the length of the intersection $l \cap B(x, r)$ is less than or equal to $2r$. Since the $d\sigma^\mu$ is a pullback of dU^μ and $dU^\mu = e^{\delta(\Gamma)\beta_x(\eta,\xi)} d\mu_x(\eta)\, d\mu_x(\xi)\, dt$, Lemma 3.6 implies that $\sigma^\mu(B_r) \leq Cre^{2\delta(\Gamma)r}$. Let $\epsilon > 0$. Define a function $\rho : SM \to \mathbb{R}$ by

$$\rho(v) = e^{-(2\delta(\Gamma)+\epsilon)\tau(v)}$$

for any $v \in SM$. We can show that ρ is integrable with respect to σ^μ and for all $v, w \in SM$ with $d(c_v(0), c_w(0)) \leq 1$, (see [9])

$$\frac{(\rho(v) - \rho(w))}{\rho(w)} \leq Cd(c_v(0), c_w(0)),$$

where $C > 0$ is a constant. □

The conservativity of the geodesic flow g^t means that

$$dU^\mu[(\partial H \times L^r(\Gamma) - \{\text{diag}\}) \times \mathbb{R}]$$

has full measure of SM. Therefore, it suffices to show that Remark 3.3 still holds on $\mathscr{F}^{-1}[(\partial H \times L^r(\Gamma) - \{\text{diag}\}) \times \mathbb{R}]$.

Now let $v \in SH$. Let H_v be a horosphere based at $c_v(\infty) \in \partial H$ passing through $c_v(0)$. Then we have $w \in H_v$ implying that $c_v(\infty) = c_w(\infty)$, that is, $c_v(t)$ and $c_w(t)$ are asymptotic for all $t \geq 0$. Let W^{ss} and W^{su} denote the strong stable and strong unstable foliation, respectively. For any $v \in SH$, the leaves through v of these foliations are given by

$$W^{su}(v) = \{w \in H_v \mid c_v(\infty) = c_w(\infty)\}$$
$$W^{ss}(v) = \{-w \in H_{-v} \mid c_v(-\infty) = c_w(-\infty)\}.$$

We can consider Γ as a subgroup of the isometries of H. Considering all the isometries of H, we can distinguish three distinct types, namely: hyperbolic isometry, parabolic isometry and elliptic isometry. Note that our Fuchsian group Γ has no elliptic isometry.

REMARK 3.8. (1) In a visibility manifold, there is a geodesic γ fixed by an element of Γ without any flat strip, which is called an axis. The element of Γ fixing a geodesic γ is a hyperbolic isometry in Γ. This can be proved by modifying [4, Proposition 2.3] and [1, Lemma 3.2]. For a recurrent vector $v \in SH$, the set $F(v)$ of geodesics parallel to c_v is a completely flat and totally geodesic submanifold of H without boundary. Assume that the dimension m of $F(v)$ satisfies $m \geq 2$. Then $F(v)$ contains a 2-dimensional completely flat and totally geodesic submanifold of $F(v)$ without boundary. This contradicts the uniform visibility axiom. Therefore, $F(v)$ has to be a geodesic line c_v and c_v does not contain a flat strip.
(2) Let $\eta \in \partial H$ and let γ be an axis without any flat strip in (1). A geodesic from $\gamma(\infty)$ to η does not bound a flat strip. The proof is similar to the proof of Theorem 3.1 in [1] or the proof of Lemma 3.4 in [1].

Lemma 3.9 is a very important step in proving Theorem 3.10.

LEMMA 3.9. *Let $\xi \in L^r(\Gamma)$ be any point. Suppose c_ξ is a geodesic line from $\gamma(-\infty)$ to ξ, where γ is an axis without a flat strip. Let $c'_\xi(0) = v \in SH$. Then for any $w \in W^{ss}(v)$, $d(g^t v, g^t w) \to 0$ as $t \to \infty$.*

PROOF. Since ξ is in $L^r(\Gamma) \subset L(\Gamma)$, ξ is a nonwandering point in SH. There are sequences $\{\phi_n\}$ in Γ, $\{t_n\}$ in $\mathbb{R}$ and $\{v_n\}$ in SH such that $t_n \to \infty$, $v_n \to v$ and $\{D\phi_n(g^{t_n} v_n)\}$ goes to v as $n \to \infty$. Let $w \in W^{ss}(v)$. Then $d(g^t v, g^t w)$ is monotone decreasing for $t \in \mathbb{R}$. Suppose $d(g^t v, g^t w)$ does not converge to 0. Then there exists a constant $b_1 > 0$ such that

$$b_1 \leq d(g^t v, g^t w) \leq d(v, w) \tag{3.1}$$

for all $t \geq 0$.

Since $\xi \in L^r(\Gamma)$ and $\{g^{t_n} v_n\}$ is close to $D\phi_n^{-1} v$, there is a constant $d > 0$ such that $d(\pi(g^{t_n} v_n), c_\xi) < d$ for all n, where $\pi : SH \to H$ is a canonical projection and we use the same notation $\{g^{t_n} v_n\}$ as a subsequence of $\{g^{t_n} v_n\}$ in the definition of $L^r(\Gamma)$ ambiguously. We consider the triangle $(\pi(v), g^{t_n} v, g^{t_n} v_n)$. By the visibility axiom there is a constant $C > 0$ such that $d(g^{t_n} v, g^{t_n} v_n) < C$ for all n. Then for all n,

$$d(D\phi_n(g^{t_n} v), D\phi_n(g^{t_n} v_n)) < C. \tag{3.2}$$

Since $\{D\phi_n g^{t_n} v_n\}$ is a bounded sequence, so is $\{D\phi_n g^{t_n} v\}$ by (3.2). Therefore the sequence $\{D\phi_n g^{t_n} v\}$ has a subsequence converging to $\bar{v} \in SH$ which we denote by $\{D\phi_n g^{t_n} v\}$. By (3.1), we have that for all n and $-t_n \leq s \leq \infty$

$$b_1 \leq d(D\phi_n(g^{t_n+s} v), D\phi_n(g^{t_n+s} w)) \leq d(v, w). \tag{3.3}$$

Since $\{D\phi_n g^{t_n} v\}$ is converging to $\bar{v}$, $\{D\phi_n g^{t_n} w\}$ is a bounded sequence and it has a subsequence converging to $\bar{w}$ which we denote by $\{D\phi_n g^{t_n} w\}$. When n goes to ∞ in (3.3), we have that

$$b_1 \leq d(g^s \bar{v}, g^s \bar{w}) \leq d(v, w), \tag{3.4}$$

for all $s \in \mathbb{R}$. This means that $c_{\bar{w}}$ has a flat strip, where $c_{\bar{w}}$ is a geodesic with $c'_{\bar{w}}(0) = \bar{w}$.

In order to obtain a contradiction, we prove that the geodesic $c_{\bar{w}}$ with $c'_{\bar{w}}(0) = \bar{w}$ cannot have any flat strip.

Since $d(g^t v_n, g^t v) \leq d(g^{t_n} v_n, g^{t_n} v) + d(v_n, v)$ for $0 \leq t \leq t_n$, there is a constant $C > 0$ such that for all $0 \leq t \leq t_n$,

$$d(g^t v_n, g^t w) \leq d(g^t v_n, g^t v) + d(g^t v, g^t w) \leq C + d(v, w).$$

Therefore, for all n and for $-t_n \leq s \leq 0$,

$$d(g^s D\phi_n(g^{t_n} v_n), g^s D\phi_n(g^{t_n} w)) = d(g^{t_n+s} v_n, g^{t_n+s} w) \leq C + d(v, w).$$

Hence we have that for all $-\infty < s < 0$, $d(g^s v, g^s \bar{w}) \leq C + d(v, w)$. This means that $c_{\bar{w}}(-\infty) = c_v(-\infty) = \gamma(-\infty)$ in ∂H. Since γ is an axis fixed by a hyperbolic isometry in Γ and has no flat strip, $c_{\bar{w}}$ has to have no flat strip. (See Remark 3.8.) $\square$

We now prove the Main Theorem, that is, the ergodicity of the geodesic flow on SM.

THEOREM 3.10. *If the geodesic flow is conservative with respect to σ^μ, then it is ergodic with respect to the same measure.*

PROOF. Since the geodesic flow is conservative on SM, we deduce that

$$\sum_{\gamma\in\Gamma} e^{-Dd(x,\gamma x)} = \infty.$$

Consider the integrable function ρ on SM defined in Theorem 3.7. Let $w \in SM$ be a conservative point. Then it can be proved that $\int_0^\infty \rho(g^t w)\,dt = \infty$. Since the geodesic flow g^t is conservative with respect to σ^μ, the limit

$$\lim_{T\to\infty} \frac{\int_0^T f(g^t v)\,dt}{\int_0^T \rho(g^t v)\,dt} = f_\rho(v)$$

exists for $f \in L^1(\sigma^\mu)$ and for σ^μ-almost all $v \in SM$. It is sufficient to show that f_ρ is constant for almost all $f \in L^1(d\sigma^\mu)$. Without loss of generality, we assume that f is continuous with a compact support.

Let γ be an axis on M. Let $v \in SM$ be a conservative point. Let $v' \in SM$ be chosen such that $v' \in W^{ss}(v)$ and $c_{v'}(-\infty) = \gamma(-\infty)$. By Lemma 3.9 we have $d(g_t(v), g_t(v')) \to 0$ as $t \to \infty$. Since f is continuous with a compact support and ρ has the property as in Lemma 3.6, we have

$$\begin{aligned} f_\rho(v) - f_\rho(v') &= \lim_{T\to\infty}\left[\frac{\int_0^T f(g^t(v))\,dt}{\int_0^T \rho(g^t(v))\,dt} - \frac{\int_0^T f(g^t(v'))\,dt}{\int_0^T \rho(g^t(v'))\,dt}\right] \\ &= \lim_{T\to\infty}\left[\frac{\int_0^T \left[f(g^t(v)) - f(g^t(v'))\right]dt}{\int_0^T \rho(g^t(v))\,dt}\right. \\ &\qquad \left. - \frac{\int_0^T f(g^t(v'))\,dt}{\int_0^T \rho(g^t(v'))\,dt}\,\frac{\int_0^T \frac{\rho(g^t(v))-\rho(g^t(v'))}{\rho(g^t(v))}\rho(g^t(v))\,dt}{\int_0^T \rho(g^t(v))\,dt}\right] = 0. \end{aligned} \tag{3.5}$$

If v and v' are two points in SM with $v \in W^{su}(v')$ and $c_{v'}(\infty) = c_v(\infty) = \gamma(-\infty)$, then $d(g_t(v), g_t(v')) \to 0$ as $t \to -\infty$. Using the arguments similar to those used in the above case, we also have

$$f_\rho(v) = f_\rho(v'). \tag{3.6}$$

We lift f_ρ to SH and we use the same notation f_ρ. Then f_ρ is Γ-invariant on $\partial H \times \partial H$. By (3.5) and (3.6), f_ρ is constant on $\partial H \times \xi$ and on $\xi \times \partial H$ for almost all $\xi \in L^r(\Gamma)$. We can easily show that f_ρ is constant almost all on $\partial H \times \partial H$, because the geodesic flow is conservative. We deduce that f_ρ is constant almost everywhere on SM. □

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