ON FINITE SOLUBLE GROUPS

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Let $\mathscr{P}$ be a class of finite soluble groups which is closed under epimorphic images and let $\mathscr{S}$ be a saturated formation. Then if G is a group of minimal order belonging to $\mathscr{P}$ but not to $\mathscr{S}$, $F(G)$, the Fitting subgroup of G, is the unique minimal normal subgroup of G. It is to groups with this property that the following proposition is applicable.

The notation agrees with that in [1]. All groups referred to in this note are finite and soluble. The note is based on a portion of the author's Ph.D. thesis at the University of Sydney.

PROPOSITION. *Let G be a finite soluble group with unique minimal normal subgroup $F = F(G)$. Then F yields an irreducible representation of G with kernel F. Let P be a Sylow subgroup of G containing F and suppose P has derived length r. Then if N is any normal subgroup of G, $P^{(r-1)}$ is contained in the stabilizer of any homogeneous component of F regarded as an N-module.*

PROOF. In module notation, for any $f \in F$ and $\xi_i \in P^{(i)}$, $i = 0, 1, \cdots, r-1$, we have

$$f(1-\xi_0)(1-\xi_1) \cdots (1-\xi_{r-1}) = 0.$$

Now P acts as a permutation group on the homogeneous components of F when considered as an N-module, so by the method used in the proof of lemma 5 § 3 of [1], $P^{(r-1)}$ stabilizes each of these components.

COROLLARY. *The same result is true if instead of taking the G-module F over the field of p-elements, we extend the field to a splitting field of G/F and all its subgroups and take in place of F, an irreducible component of the resulting module.*

LEMMA. *Let $F = F(G)$ be the unique minimal normal subgroup of the finite soluble group G. Let P be a Sylow p-subgroup of G containing F, say P has derived length r. Then if $A \lhd G$ and A/F is abelian, $(P^{(r-1)}, A) \leqq F$.*

PROOF. Consider the representation of G obtained on its subgroup F. The kernel of this representation is F. Regarding F as a G-module over $GF(p)$, extend the field of scalars to a field $\mathscr{F}$ which is a splitting field of G and all its subgroups. Let V be an irreducible component of the resulting module. Then by the argument on [1] page 485, V also gives a representation of G with kernel F.

Since $A \lhd G$, by Cliffords theory, $V_{|A}$ is a sum of homogeneous components. $\mathscr{F}$ is a splitting field for A so as V yields a representation with kernel F and A/F is abelian, A acts by scalar multiplication on each of the homogeneous components. By the corollary, $P^{(r-1)}$ stabilizes each of the homogeneous components so $(P^{(r-1)}, A)$ is contained in the kernel of the representation of G on V. This proves the lemma.

As an application of this result we prove the

THEOREM. *Let G be the product of two complementary Hall subgroups H and K. Suppose that H is abelian and K is nilpotent of derived length r. Then $G^{(r)} \leqq F(G)$.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimum order. By [2], G is soluble. Since the class of groups satisfying the hypothesis of the theorem is closed under epimorphic images, whilst the class satisfying the conclusion is a saturated formation, $F = F(G)$ is the unique minimal normal subgroup of G.

It follows that F is a p-group for some prime p. Since F is a normal subgroup of G, $F \leqq H$ or $F \leqq K$. Suppose $F \leqq H$. Since G is soluble $C_G(F) \leqq F$, so that as H is abelian, $F = H$. Thus $G/F \cong K$ and so $G^{(r)} \leqq F$. Thus we may assume that $F \leqq K$.

Since F is a p-group, K is nilpotent and $C_G(F) \leqq F$, K is a p-group. Taking K for P in the lemma we deduce that $K^{(r-1)}$ centralizes every abelian normal subgroup of G/F.

On the other hand since F is a p-group, $F_2(G)/F$ is a p'-group. Therefore $F_2(G)/F \leqq HF/F$ and so is abelian. Now

$$K^{(r-1)}F/F \leqq C_{G/F}(F_2(G)/F) \leqq F_2(G)/F$$

so as $K^{(r-1)}F$ is a p-group whilst $F_2(G)/F$ is a p'-group, $K^{(r-1)} \leqq F$.

By the definition of G, we may now apply the theorem to G/F to find $G^{(r-1)} \leqq F_2(G)$. Thus as $F_2(G)/F$ is abelian $G^{(r)} \leqq F$. This contradiction proves the theorem.

NOTE. The group $GL(2, 3)$ satisfies the hypothesis of the theorem with $r = 2$. However G'' is not abelian, and G' is not nilpotent.

References

[1] J. N. Ward, 'Involutory Automorphisms of Groups of Odd Order', *J. Australian Math. Soc.* 6 (1966), 480–494.

[2] H. Wielandt, 'Über Produkte von Nilpotenten Gruppen', *Illinois J. Math.* 2 (1959), 611–618.

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