Forum of Mathematics, Sigma (2025), **Vol. 13**:e140 1–38
doi:10.1017/fms.2025.10088

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Quantum K theory of Grassmannians, Wilson line operators and Schur bundles

Wei Gu[1], Leonardo Mihalcea[2], Eric Sharpe[3] and Hao Zou[4,5]

[1]Zhejiang Institute of Modern Physics, School of Physics, Zhejiang University, Hangzhou, Zhejiang, 310058, China;
E-mail: guwei2875@zju.edu.cn.
[2]Department of Mathematics, Virginia Tech University, 225 Stanger Street, McBryde Hall, Blacksburg, VA, 24061, USA;
E-mail: lmihalce@vt.edu (corresponding author).
[3]Department of Physics, Virginia Tech University, MC 0435, 850 West Campus Drive, Blacksburg, VA, 24061, USA;
E-mail: ersharpe@vt.edu.
[4]Center for Mathematics and Interdisciplinary Sciences, Fudan University, Shanghai, 200433, China;
E-mail: haozou@fudan.edu.cn.
[5]Shanghai Institute for Mathematics and Interdisciplinary Sciences, Shanghai, 200433, China.

Received: 19 February 2023; **Revised:** 15 May 2025; **Accepted:** 15 May 2025

2020 Mathematical Subject Classification: *Primary* – 14M15, 14N35, 81T60; *Secondary* – 05E05

Abstract

We prove a 'Whitney' presentation, and a 'Coulomb branch' presentation, for the torus equivariant quantum K theory of the Grassmann manifold $\mathrm{Gr}(k;n)$, inspired from physics, and stated in an earlier paper. The first presentation is obtained by quantum deforming the product of the Hirzebruch λ_y classes of the tautological bundles. In physics, the λ_y classes arise as certain Wilson line operators. The second presentation is obtained from the Coulomb branch equations involving the partial derivatives of a twisted superpotential from supersymmetric gauge theory. This is closest to a presentation obtained by Gorbounov and Korff, utilizing integrable systems techniques. Algebraically, we relate the Coulomb and Whitney presentations utilizing transition matrices from the (equivariant) Grothendieck polynomials to the (equivariant) complete homogeneous symmetric polynomials. Along the way, we calculate K-theoretic Gromov-Witten invariants of wedge powers of the tautological bundles on $\mathrm{Gr}(k;n)$, using the 'quantum=classical' statement.

Contents

1. Introduction

Based on predictions inspired by physics, in [GMSZ22], we conjectured presentations by generators and relations for the quantum K theory ring of the Grassmannian and for the Lagrangian Grassmannian. The main goal in this paper is to provide rigorous mathematical proofs of these statements in the Grassmannian case, in the more general equivariant context.

Let $\mathrm{Gr}(k;n)$ denote the Grassmann manifold parametrizing linear subspaces of dimension k in $\mathbb{C}^n$, and let $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$ be the tautological sequence, where $\mathrm{rk}(\mathcal{S}) = k$. An influential result by Witten [Wit95] states that $(\mathrm{QH}^*(\mathrm{Gr}(k;n)), \star)$, the quantum cohomology ring of the Grassmannian, is determined by the 'quantum Whitney relations':

$$c(\mathcal{S}) \star c(\mathcal{Q}) = c(\mathbb{C}^n) + (-1)^k q, \tag{1}$$

where $c(E) = 1 + c_1(E) + \ldots + c_e(E)$ is the total Chern class of the rank e bundle E. This equation leads to a presentation of $\mathrm{QH}^*(\mathrm{Gr}(k;n))$ by generators and relations:

$$\mathrm{QH}^*(\mathrm{Gr}(k;n)) = \frac{\mathbb{Z}[q][e_1(x),\ldots,e_k(x);e_1(\tilde{x}),\ldots,e_{n-k}(\tilde{x})]}{\left\langle \left(\sum_{i=0}^{k} e_i(x)\right)\left(\sum_{j=0}^{n-k} e_j(\tilde{x})\right) = 1 + (-1)^k q \right\rangle}. \tag{2}$$

Here, $e_i(x) = e_i(x_1,\ldots,x_k)$, $e_j(\tilde{x}) = e_j(\tilde{x}_1,\ldots,\tilde{x}_{n-k})$ denote the elementary symmetric polynomials, and $x_i, \tilde{x}_j$ correspond to the Chern roots of $\mathcal{S}$, respectively $\mathcal{Q}$.

Let T be the torus of invertible diagonal matrices with its usual action on $\mathrm{Gr}(k;n)$. In this paper, we generalize Witten's relations (1) from the quantum cohomology to the T-equivariant quantum K ring of $\mathrm{Gr}(k;n)$, defined by Givental and Lee [Giv00, GL03, Lee04]. Denote this ring by $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ and by $\mathrm{K}_T(\mathrm{Gr}(k;n))$ the ordinary equivariant K-theory ring.

For a vector bundle E, denote by $\lambda_y(E) = 1 + yE + \ldots + y^e \wedge^e E$ the Hirzebruch λ_y-class in the K theory ring. The K-theoretic Whitney relations are

$$\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) \quad \in \mathrm{K}_T(\mathrm{Gr}(k;n)).$$

The main geometric result in this paper is a quantum deformation of these relations.

Theorem 1.1. *The following equality holds in* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

$$\lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}) - 1) \star \det \mathcal{Q}. \tag{3}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

In the theorem, we regard $\mathbb{C}^n$ as the T-module with weight space decomposition $\mathbb{C}^n = \mathbb{C}_{t_1} \oplus \ldots \oplus \mathbb{C}_{t_n}$. If we set $T_1 := e^{t_1}, \ldots, T_n := e^{t_n}$, then $\mathrm{K}_T(pt)$ is the Laurent polynomial ring $\mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$, and $\lambda_y(\mathbb{C}^n) = \prod_{i=1}^n (1 + yT_i)$; see §2.

The equality (3) may be translated into an abstract 'quantum K-theoretic Whitney' presentation of the ring $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ by generators and relations as follows. Set $X_i, \tilde{X}_j$ the exponentials of the Chern roots of $\mathcal{S}$, respectively $\mathcal{Q}$: $X_i = e^{\sigma_i}$, $\tilde{X}_j = e^{\tilde{\sigma}_j}$; see also Equation (70) below. Translate (3) into the equality of generating functions:

$$\prod_{i=1}^{k}(1+yX_i)\prod_{j=1}^{n-k}(1+y\tilde{X}_j) = \prod_{s=1}^{n}(1+yT_i) - \frac{q}{1-q}y^{n-k}\left(\prod_{i=1}^{k}(1+yX_i) - 1\right)\prod_{j=1}^{n-k}\tilde{X}_j. \tag{4}$$

Then the quantum K ring has a presentation with generators X_i and $\tilde{X}_j$ for $1 \le i \le k$ and $1 \le j \le n-k$, over the power series ring $\mathrm{K}_T(pt)[[q]]$, with relations given by the coefficients of y^s in (4); see also Theorem 1.2(b) below. This generalizes the classical presentation of $\mathrm{K}(\mathrm{Gr}(k;n))$ by Lascoux [Las90]. After passing to K-theoretic Chern roots in (4) (as in (7) below), and taking an appropriate limit, one may also obtain Witten's presentation of $\mathrm{QH}^*(\mathrm{Gr}(k;n))$ from (2) above.

In [GMSZ22], we conjectured a second 'Coulomb branch presentation' of the ring $\mathrm{QK}_T(\mathrm{Gr}(k;n))$, and in this paper, we prove an equivariant generalization of it. Let W be a twisted superpotential [MP95, AHKT01, CK16, GMSZ22] arising in the study of supersymmetric gauge theory:

$$W = \frac{k}{2}\sum_{a=1}^{k}(\ln X_a)^2 - \frac{1}{2}\left(\sum_{a=1}^{k}\ln X_a\right)^2 + \ln\left((-1)^{k-1}q\right)\sum_{a=1}^{k}\ln X_a + \sum_{i=1}^{n}\sum_{a=1}^{k}\mathrm{Li}_2\left(X_aT_i^{-1}\right),$$

with Li_2 the dilogarithm. The superpotential is defined for more general GIT quotients, and it depends on certain parameters, which are potentially related to other quantum K theories. See equation (35) and Section 9 below for further details. In this context, the exterior powers $\wedge^i\mathcal{S}$ arise as certain Wilson line operators considered in the physics literature [JM20, JMNT20, JM19, UY20]. The Coulomb branch (or vacuum) equations for W are

$$\exp\left(\frac{\partial W}{\partial \ln X_a}\right) = 1, \quad 1 \le a \le k. \tag{5}$$

As expected from physics [NS09], the equations (5) coincide with the Bethe Ansatz equations considered by Gorbounov and Korff [GK17, equ'n (4.17)]:

$$(-1)^{k-1}q(X_a)^k\prod_{j=1}^{n}T_j = \left(\prod_{b=1}^{k}X_b\right)\cdot\prod_{i=1}^{n}(T_i - X_a). \tag{6}$$

See also [KPSZ21] for more appearances of the Bethe Ansatz equations.

Strictly speaking, (6) cannot give relations in the quantum K ring because, for fixed a, $\exp\left(\frac{\partial W}{\partial \ln X_a}\right) = 1$ is not symmetric in the X_i's. As we shall see below, a symmetrization is required.

It is convenient to work with the 'shifted Wilson line operators', or, equivalently, with the (dual) K-theoretic Chern roots

$$z_i = 1 - X_i, \quad (1 \le i \le k); \quad \zeta_j = 1 - T_j, \quad (1 \le j \le n). \tag{7}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

The Coulomb branch equations (6) show that z_a's are the roots of a 'characteristic polynomial':

$$f(\xi, z, \zeta, q) \;=\; \xi^n \;+\; \sum_{i=0}^{n-1} (-1)^{n-i} \xi^i \hat{g}_{n-i}(z, \zeta, q); \tag{8}$$

see (48) below. The polynomials $\hat{g}_j(z, \zeta, q)$, defined explicitly in (51) below, are symmetric both in z_i's and ζ_j's. Our main result in this paper is that the relations from (4) generate the ideal of relations of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ and, furthermore, that the Vieta relations satisfied by the roots of the polynomial (8) are algebraically equivalent to those from Theorem 1.1; see Theorem 10.2 below. Specifically, denote by $\tilde{z}_j = 1 - \tilde{X}_j$ the K-theoretic Chern roots of the dual of $\mathcal{Q}$.

Theorem 1.2. *The following two rings give presentations by generators and relations of* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

(a) *(Coulomb branch presentation) The ring* $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *given by*

$$\mathrm{K}_T(pt)[[q]][e_1(z), \cdots, e_k(z), e_1(\hat{z}), \cdots, e_{n-k}(\hat{z})] \,/\, \langle \sum_{i+j=\ell} e_i(z) e_j(\hat{z}) - \hat{g}_\ell(z, \zeta, q) \rangle_{1 \le \ell \le n}.$$

Here, $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_{n-k})$ *and the polynomials* $\hat{g}_\ell(z, \zeta, q)$ *are defined in (51).*

(b) *(QK-theoretic Whitney presentation) The ring* $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *given by*

$$\mathrm{K}_T(pt)[[q]][e_1(z), \cdots, e_k(z), e_1(\tilde{z}), \cdots, e_{n-k}(\tilde{z})] \,/\, \langle \sum_{i+j=\ell} e_i(z) e_j(\tilde{z}) \;-\; \tilde{g}_\ell(z, \zeta, q) \rangle_{1 \le \ell \le n},$$

and where the polynomials $\tilde{g}_\ell$ *are defined in (59).*

In each situation, $e_i(z)$ *is sent to* $\sum_{p=0}^{i} (-1)^p \binom{k-p}{i-p} \wedge^p \mathcal{S}$*, for* $1 \le i \le k$.

The relations giving $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ are obtained from those in (4) by the change of variables (7) and $\tilde{X}_j = 1 - \tilde{z}_j$.

The two presentations satisfy some remarkable algebraic properties. In both situations, the polynomials $\hat{g}_\ell$ and $\tilde{g}_\ell$ do not depend on q if $1 \le \ell \le n-k$; in other words, the first $n-k$ relations are 'classical'. Furthermore, we may use the first $n-k$ relations to eliminate the variables $\hat{z}_j, \tilde{z}_j$ for $1 \le j \le n-k$. We prove that in $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$,

$$(-1)^\ell e_\ell(\hat{z}) = G'_\ell(z, \zeta) := \sum_{i+j=\ell} (-1)^j G_i(z) e_j(\zeta),$$

where $G_j(z)$ is the Grothendieck polynomial; see Proposition 11.6. Note that the elimination of variables process led us to define $G'_\ell(z, \zeta)$, which is an equivariant deformation of $G_\ell(z)$. This is different from another equivariant deformation of $G_j(z)$, the factorial Grothendieck polynomial defined by McNamara [McN06], and which represents equivariant Schubert classes. (The latter are not symmetric in ζ's.) In analogy to the presentation of $H^*_T(\mathrm{Gr}(k;n))$, the Cauchy formulae calculating $\sum_{i+j=\ell} G'_i(z, \zeta) e_j(\zeta)$ allow us to formulate the relations in $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ as expressions involving $G'_{n-k+1}(z, \zeta), \ldots, G'_n(z, \zeta)$; see Theorem 11.8 below. We have not seen a presentation formulated naturally in terms of these polynomials, even in the ordinary non-quantum ring $\mathrm{K}_T(\mathrm{Gr}(k;n))$.

A similar description holds for the QK-Whitney presentation $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$, with the caveat that the Grothendieck polynomials $G_j(z)$, $G'_\ell(z, \zeta)$ are replaced by the complete homogeneous symmetric polynomials $h_j(z)$ and their equivariant versions $h'_\ell(z, \zeta) = \sum_{i+j=\ell} (-1)^j h_i(z) e_j(\zeta)$. Geometrically, the polynomials $h'_\ell(z, \zeta)$ represent (equivariant) K-theoretic Chern classes of the tautological quotient bundle $\mathcal{Q}$. Analyzing the precise algebraic relationship between $G'_j(z, \zeta)$ and $h'_j(z, \zeta)$ lies at the foundation of proving the isomorphism between the two presentations in Theorem 1.2; this is done in §11.

One advantage of working with K-theoretic Chern roots is that the leading terms of these presentations give presentations of the equivariant quantum cohomology ring $\mathrm{QH}^*_T(\mathrm{Gr}(k;n))$; both specialize to

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Witten's presentation (2). In physics terminology, one recovers the $2d$ limit of the theory. This is illustrated in §12.

In [GK17], Gorbounov and Korff utilized integrable systems and equivariant localization techniques to obtain another presentation of the equivariant quantum K ring $\mathrm{QK}_T(\mathrm{Gr}(k;n))$, different from those in Theorem 1.2. In order to identify the geometric ring $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ with that given from integrable systems, they require the equivariant quantum K Chevalley formula (a formula to multiply by a Schubert divisor). As proved in [BCMP18a], this formula determines the ring structure.

Aside from a finite generation statement of the 'classical' Coulomb branch presentation (i.e., modulo q), used in the proof of Theorem 11.12, our proofs are logically independent from results in [GK17]. We calculate the product $\lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q})$ from Theorem 1.1 directly, based on the 'quantum=classical' statement by Buch and one of the authors [BM11], applied to the Schur bundles. This requires the calculation of certain push-forwards of the K-theory classes of the exterior powers $\wedge^i\mathcal{S}$, $\wedge^j\mathcal{Q}$. For this, we rely on sheaf cohomology vanishing statements obtained by Kapranov [Kap84]. We obtain rather explicit calculations of KGW invariants and quantum K products involving Schur bundles, which may be interesting in their own right. To illustrate, we prove the following quantum deformation of the classical equality $\wedge^i\mathcal{S} \cdot \det(\mathcal{Q}) = \wedge^{k-i}\mathcal{S}^* \cdot \det(\mathbb{C}^n) \in \mathrm{K}_T(\mathrm{Gr}(k;n))$ (cf. Theorem 6.1, see also Corollary 6.4):

$$(\lambda_y(\mathcal{S}) - 1) \star \det(\mathcal{Q}) = (1-q)((\lambda_y(\mathcal{S}) - 1) \cdot \det(\mathcal{Q})).$$

Equivalently, for any $i > 0$,

$$\wedge^i(\mathcal{S}) \star \det(\mathcal{Q}) = (1-q) \wedge^{k-i} (\mathcal{S}^*) \cdot \det(\mathbb{C}^n).$$

Of course, the QK Whitney, the Coulomb branch and the presentation from [GK17] are isomorphic. In §12.1.2, we provide a direct isomorphism from the non-equivariant Coulomb branch presentation $\widetilde{\mathrm{QK}}(\mathrm{Gr}(k;n))$ to the one from [GK17].[1] The final step in proving Theorem 1.2 is to show that there are no other relations in the quantum K ring beyond those stated in Theorem 1.1. In the quantum cohomology case, a result going back to Siebert and Tian [ST97] (see also [FP97]) states that if one has a set of quantum relations such that the $q = 0$ specialization gives the ideal of the 'classical' relations, then these generate the ideal of quantum relations. The proof utilizes the graded Nakayama lemma. The quantum K theory ring is not graded. Still, a similar statement holds, utilizing the ordinary Nakayama lemma for the quantum K ring regarded as a module over the power series ring $\mathrm{QK}_T(pt) := \mathrm{K}_T(pt)[[q]]$, together with a finite generation statement for completed rings; cf. [Eis95, Ex. 7.4, p. 203] or Remark A.4. Such arguments are useful in studying presentations of more general quantum K rings; therefore, we took the opportunity to gather in Appendix A the relevant results about completions and filtered modules. A second appendix contains a worked out example for $\mathrm{QK}_T(\mathrm{Gr}(2;5))$.

We mention that our initial paper [GMSZ22] also gives physics inspired predictions for the quantum K theory of the Lagrangian Grassmannian $\mathrm{LG}(n;2n)$; a mathematical follow-up analyzing this situation will be considered elsewhere.

2. Preliminaries on equivariant K theory

In this section, we recall some basic facts about the equivariant K-theory of a variety with a group action. For an introduction to equivariant K theory, and more details, see [CG09].

Let X be a smooth projective variety with an action of a linear algebraic group G. The equivariant K theory ring $\mathrm{K}_G(X)$ is the Grothendieck ring generated by symbols $[E]$, where $E \to X$ is an G-equivariant vector bundle, modulo the relations $[E] = [E_1] + [E_2]$ for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles. The additive ring structure is given by direct sum, and the multiplication is given by tensor products of vector bundles. Since X is smooth,

[1] In an upcoming paper, it will be shown that a certain 'functional relation', which determines the relations from [GK17], is equivalent to the quantum K Whitney relations (3).

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

any G-linearized coherent sheaf has a finite resolution by (equivariant) vector bundles, and the ring $\mathrm{K}_G(X)$ coincides with the Grothendieck group of G-linearized coherent sheaves on X. In particular, any G-linearized coherent sheaf $\mathcal{F}$ on X determines a class $[\mathcal{F}] \in \mathrm{K}_G(X)$. An important special case is if $\Omega \subset X$ is a G-stable subscheme; then its structure sheaf determines a class $[\mathcal{O}_\Omega] \in \mathrm{K}_G(X)$.

The ring $\mathrm{K}_G(X)$ is an algebra over $\mathrm{K}_G(pt) = \mathrm{Rep}(G)$, the representation ring of G. If $G = T$ is a complex torus, then this is the Laurent polynomial ring $\mathrm{K}_T(pt) = \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$, where $T_i := e^{t_i}$ are characters corresponding to a basis of the Lie algebra of T.

The (Hirzebruch) λ_y class is defined by

$$\lambda_y(E) := 1 + y[E] + y^2[\wedge^2 E] + \ldots + y^e[\wedge^e E] \in \mathrm{K}_G(X)[y].$$

This class was introduced by Hirzebruch [Hir95] in order to help with the formalism of the Grothendieck-Riemann-Roch theorem. It may be thought as the K theoretic analogue of the (cohomological) Chern polynomial

$$c_y(E) = 1 + c_1(E)y + \ldots + c_e(E)y^e$$

of the bundle E. The λ_y class is multiplicative with respect to short exact sequences; that is. if

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

is such a sequence of vector bundle,s then

$$\lambda_y(E_2) = \lambda_y(E_1) \cdot \lambda_y(E_3);$$

cf. [Hir95]. A particular case of this construction is when V is a (complex) vector space with an action of a complex torus T, and with weight decomposition $V = \oplus_i V_{\mu_i}$, where each μ_i is a weight in the dual of the Lie algebra of T. The *character* of V is the element $\mathrm{ch}_T(V) := \sum_i \dim V_{\mu_i} e^{\mu_i}$, regarded in $\mathrm{K}_T(pt)$. The λ_y class of V is the element $\lambda_y(V) = \sum_{i \geq 0} y^i ch(\wedge^i V) \in \mathrm{K}_T(pt)[y]$. From the multiplicativity property of the λ_y class, it follows that

$$\lambda_y(V) = \prod_i (1 + ye^{\mu_i})^{\dim V_{\mu_i}};$$

see [Hir95].

Since X is proper, the push-forward to a point equals the Euler characteristic, or, equivalently, the virtual representation,

$$\chi(X, \mathcal{F}) = \int_X [\mathcal{F}] := \sum_i (-1)^i \,\mathrm{ch}_T\, H^i(X, \mathcal{F}).$$

For E, F equivariant vector bundles, this gives a pairing

$$\langle -, - \rangle : \mathrm{K}_G(X) \otimes \mathrm{K}_G(X) \to \mathrm{K}_G(pt); \quad \langle [E], [F] \rangle := \int_X E \otimes F = \chi(X, E \otimes F).$$

3. Equivariant K theory of flag manifolds

In this paper, we will utilize the torus equivariant K-theory of the partial flag manifolds $\mathrm{Fl}(i_1, i_2, \ldots, i_k; n)$, which parametrize partial flags $F_{i_1} \subset \ldots \subset F_{i_p} \subset \mathbb{C}^n$, where $\dim F_i = i$, to study the equivariant quantum K theory ring of Grassmann manifolds $\mathrm{Gr}(k; n)$. The goal of this section is to

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

review some of the basic features of the equivariant K rings of the partial flag manifolds. Of special importance is the calculations of K theoretic push forwards of Schur bundles, which will play a key role later in the paper.

3.1. Basic definitions

The flag manifold $\mathrm{Fl}(i_1, i_2, \ldots, i_k; n)$ is an algebraic variety homogeneous under the action of $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$. Let $T \subset \mathrm{GL}_n$ be the subgroup of diagonal matrices acting coordinate-wise on $\mathbb{C}^n$. Denote by $T_i \in \mathrm{K}_T(pt)$ the weights of this action.

Set W to be the symmetric group in n letters, and let $W_{i_1,\ldots,i_p} \leq W$ be the subgroup generated by simple reflections $s_i = (i, i+1)$ where $i \notin \{i_1, \ldots, i_p\}$. Denote by $\ell : W \to \mathbb{N}$ the length function, and by $W^{i_1,\ldots,i_k}$ the set of minimal length representatives of $W/W_{i_1,\ldots,i_p}$. This consists of permutations $w \in W$ which have descents at positions $i_1, \ldots, i_p$, that is, $w(i_k + 1) < \ldots < w(i_{k+1})$, for $k = 0, \ldots, p$, with the convention that $i_0 = 1, i_{p+1} = n$.

The torus fixed points $e_w \in \mathrm{Fl}(i_1, i_2, \ldots, i_k; n)$ are indexed by the permutations $w \in W^{i_1,\ldots,i_p}$. For each such point, the *Schubert cell* $X_w^\circ \subset \mathrm{Fl}(i_1, i_2, \ldots, i_k; n)$ is the orbit $B^-.e_w$ of the Borel subgroup of lower triangular matrices, and the *Schubert variety* is the closure $X_w = \overline{X^{w,\circ}}$. Our convention ensures that X_w has codimension $\ell(w)$. The Schubert varieties X^w determine classes $\mathcal{O}_w := [\mathcal{O}_{X_w}]$ in $\mathrm{K}_T(X)$. Similarly, the S-fixed points give classes denoted by $\iota_w := [\mathcal{O}_{e_w}]$. The equivariant K-theory $\mathrm{K}_T(\mathrm{Fl}(i_1, \ldots, i_p; n))$ is a free module over $\mathrm{K}_T(pt)$ with a basis given by Schubert classes $\{\mathcal{O}_w\}_{w \in W^{i_1,\ldots,i_p}}$. For the Grassmannian $\mathrm{Gr}(k; n)$, the Schubert varieties are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ included in the $k \times (n-k)$ rectangle (i.e., $n - k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0$). For Grassmanians, we denote the Schubert variety by X_λ; this has codimension $|\lambda| = \lambda_1 + \ldots + \lambda_k$. The corresponding class in $\mathrm{K}_T(\mathrm{Gr}(k; n))$ is denoted $\mathcal{O}_\lambda := [\mathcal{O}_{X_\lambda}]$.

Let $\mathbf{i} = (1 < i_1 < \ldots < i_p < n)$ and $\mathbf{j}$ obtained from $\mathbf{i}$ by removing some of the indices i_s. Denote by $\mathrm{Fl}(\mathbf{i})$ and $\mathrm{Fl}(\mathbf{j})$ the respective flag manifolds and by $\pi_{\mathbf{i},\mathbf{j}} : \mathrm{Fl}(\mathbf{i}) \to \mathrm{Fl}(\mathbf{j})$ the natural projection. We will need the following well-known fact:

Lemma 3.1. *Let $w \in W^{\mathbf{i}}$. Then for any Schubert varieties $X_w \in \mathrm{Fl}(\mathbf{i})$ and $X_v \in \mathrm{Fl}(\mathbf{j})$,*

$$(\pi_{\mathbf{i},\mathbf{j}})_* \mathcal{O}_w = \mathcal{O}_{w'} \in K_T(\mathrm{Fl}(\mathbf{j})), \text{ and } \pi_{\mathbf{i},\mathbf{j}}^* \mathcal{O}_v = \mathcal{O}_v \in K_T(\mathrm{Fl}(\mathbf{i})),$$

where $w' \in W^{\mathbf{j}}$ is the minimal length representative for $w \in W/W_{\mathbf{j}}$.

Proof. The first equality follows from [BK05, Thm. 3.3.4(a)] and the second because $\pi_{\mathbf{i},\mathbf{j}}$ is a flat morphism. □

3.2. Push-forward formulae of Schur bundles

Next we recall some results about cohomology of Schur bundles on Grassmannians. Our main reference is Kapranov's paper [Kap84]. A reference for basic definitions of Schur bundles is Weyman's book [Wey03].

Recall that if X is a T-variety, $\pi : E \to X$ is any T-equivariant vector bundle of rank e, and $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition with at most e parts, the *Schur bundle* $\mathfrak{S}_\lambda(E)$ is a T-equivariant vector bundle over X. It has the property that if $x \in X$ is a T-fixed point, the fibre $(\mathfrak{S}_\lambda(E))_x$ is the T-module with character the Schur function s_λ. For example, if $\lambda = (1^k)$, then $\mathfrak{S}_{(1^k)}(E) = \wedge^k E$, and if $\lambda = (k)$, then $\mathfrak{S}_{(k)}(E) = \mathrm{Sym}^k(E)$.

In this paper $X = \mathrm{Gr}(k; \mathbb{C}^n)$ with the T-action restricted from $\mathrm{GL}_n(\mathbb{C})$. To emphasize the $T = (\mathbb{C}^*)^n$-module structure on $\mathbb{C}^n$, we will occasionally denote by $V := \mathbb{C}^n$ and by $\mathrm{Gr}(k; V)$ the corresponding Grassmannian. Further, V will also be identified with the trivial, but not equivariantly trivial, vector bundle $\mathrm{Gr}(k; V) \times V$. The following was proved in [Kap84].

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Proposition 3.2 (Kapranov). *Consider the Grassmannian* $\mathrm{Gr}(k;V)$ *with the tautological sequence* $0 \to \mathcal{S} \to V \to \mathcal{Q} \to 0$. *For any nonempty partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$ *such that* $\lambda_1 \leq n-k$, *there are the following isomorphisms of T-modules:*

(a) *For all* $i \geq 0$, $H^i(\mathrm{Gr}(k;V), \mathfrak{S}_\lambda(\mathcal{S})) = 0$.
(b)

$$H^i(\mathrm{Gr}(k;V), \mathfrak{S}_\lambda(\mathcal{S}^*)) = \begin{cases} \mathfrak{S}_\lambda(V^*) & i = 0 \\ 0 & i > 0. \end{cases}$$

(c)

$$H^i(\mathrm{Gr}(k;V), \mathfrak{S}_\lambda(\mathcal{Q})) = \begin{cases} \mathfrak{S}_\lambda(V) & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. Parts (a) and (b) were proved in [Kap84, Prop. 2.2], as a consequence of the Borel-Weil-Bott theorem on the complete flag manifold. For part (c), observe that there is a T-equivariant isomorphism $\mathrm{Gr}(k;V) \simeq \mathrm{Gr}(\dim V - k; V^*)$ under which the T-equivariant bundle $\mathcal{Q}$ is sent to $\mathcal{S}^*$. Then part (c) follows from (b). □

We also need a relative version of Proposition 3.2. Consider an T-variety X equipped with an T-equivariant vector bundle $\mathcal{V}$ of rank n. Denote by $\pi : \mathbb{G}(k,\mathcal{V}) \to X$ the Grassmann bundle over X. It is equipped with a tautological sequence $0 \to \underline{\mathcal{S}} \to \pi^*\mathcal{V} \to \underline{\mathcal{Q}} \to 0$ over $\mathbb{G}(k,\mathcal{V})$. The following corollary follows from Proposition 3.2, using that π is a T-equivariant locally trivial fibration:

Corollary 3.3. *For any nonempty partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$ *such that* $\lambda_1 \leq n-k$, *there are the following isomorphisms of T-modules:*

(a) *For all* $i \geq 0$, *the higher direct images,* $R^i\pi_*\mathfrak{S}_\lambda(\underline{\mathcal{S}}) = 0$.
(b)

$$R^i\pi_*\mathfrak{S}_\lambda(\underline{\mathcal{S}}^*)) = \begin{cases} \mathfrak{S}_\lambda(\mathcal{V}^*) & i = 0 \\ 0 & i > 0. \end{cases}$$

(c)

$$R^i\pi_*\mathfrak{S}_\lambda(\underline{\mathcal{Q}}) = \begin{cases} \mathfrak{S}_\lambda(\mathcal{V}) & i = 0 \\ 0 & i > 0. \end{cases}$$

4. Quantum K theory and 'quantum=classical'

4.1. Definitions and notation

The quantum K ring was defined by Givental and Lee [Giv00, Lee04]. We recall the definition below, following [Giv00]. The *quantum K metric* is a deformation of the usual K-theory pairing; we recall the definition of this metric for $X = \mathrm{Gr}(k;n)$. Fix any basis (a_λ) of $\mathrm{K}_T(\mathrm{Gr}(k;n))$ over $\mathrm{K}_T(pt)$, where λ varies over partitions in the $k \times (n-k)$ rectangle. The *(small) quantum K metric* is defined by

$$(a_\lambda, a_\mu)_{sm} = \sum_{d \geq 0} q^d \langle a_\lambda, a_\mu \rangle_d \quad \in \mathrm{K}_T(pt)[[q]], \tag{9}$$

extended by $K_T(pt)[[q]]$-linearity. The elements $\langle a_\lambda, a_\mu \rangle_d \in \mathrm{K}_T(pt)$ denote 2-point (genus 0, equivariant) K-theoretic Gromov-Witten (KGW) invariants if $d > 0$; if $d = 0$, then $\langle a_\lambda, a_\mu \rangle_0 = \langle a_\lambda, a_\mu \rangle$ are given by the ordinary K-theory pairing. The (n-point, genus 0) KGW invariants $\langle a_1, \ldots, a_n \rangle_d \in \mathrm{K}_T(pt)$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

are defined by pulling back via the evaluation maps, then integrating, over the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(\mathrm{Gr}(k;n),d)$. Instead of recalling the precise definition of the KGW invariants, in Theorem 4.5 below, we give a 'quantum=classical' statement calculating 2 and 3-point KGW invariants of Grassmannians. Explicit combinatorial formulae for the 2-point KGW invariants for any homogeneous space may be found in [BM15, BCLM20].

Example 4.1. Consider the projective plane $\mathbb{P}^2$, and for simplicity, work non-equivariantly. Consider the Schubert basis $\mathcal{O}_0 = [\mathcal{O}_{\mathbb{P}^2}], \mathcal{O}_1 = [\mathcal{O}_{line}], \mathcal{O}_2 = [\mathcal{O}_{pt}]$. The classical K-theory pairing gives the matrix

$$(\langle \mathcal{O}_i, \mathcal{O}_j \rangle) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

For any $i, j \geq 0$ and $d > 0$, $\langle \mathcal{O}_i, \mathcal{O}_j \rangle_d = 1$. Then the quantum K metric is given by

$$(\mathcal{O}_i, \mathcal{O}_j)_{sm} = \langle \mathcal{O}_i, \mathcal{O}_j \rangle + q + q^2 + \ldots = \langle \mathcal{O}_i, \mathcal{O}_j \rangle + \frac{q}{1-q}.$$

Theorem 4.2 (Givental [Giv00]). *Consider the* $\mathrm{K}_T[pt][[q]]$*-module*

$$\mathrm{QK}_T(\mathrm{Gr}(k;n)) := \mathrm{K}_T(\mathrm{Gr}(k;n)) \otimes \mathrm{K}_T(pt)[[q]].$$

Define the (small) equivariant quantum K product $\star$ *on* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ *by the equality*

$$(a \star b, c)_{sm} = \sum_{d \geq 0} q^d \langle a, b, c \rangle_d, \tag{10}$$

for any $a, b, c \in \mathrm{K}_T(\mathrm{Gr}(k;n))$. *Then* $(\mathrm{QK}_T(\mathrm{Gr}(k;n)), +, \star)$ *is a commutative, associative* $\mathrm{K}_T(pt)[[q]]$*-algebra. Furthermore, the small quantum K metric gives it a structure of a Frobenius algebra (i.e.,* $(a \star b, c)_{sm} = (a, b \star c)_{sm}$*).*

Remark 4.3. It was proved in [BCMP13] that the submodule $\mathrm{K}_T(\mathrm{Gr}(k;n)) \otimes \mathrm{K}_T(pt)[q]$ is stable under the QK product $\star$. This means that the product of two Schubert classes in $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ has structure constants which are polynomials in q. Similar statements hold for any flag manifold [ACT22, Kat18]. However, working over the ring of formal power series in q is needed for proving module finite generation of the claimed presentations. It is also natural if one studies quantizations of dual bundles, such as those from Theorem 6.1 below.

We record the following immediate consequence.

Corollary 4.4. *Let* $a, b, c \in \mathrm{K}_T(\mathrm{Gr}(k;n))[[q]]$. *Then* $a \star b = c$ *if and only if for any* $\kappa \in \mathrm{K}_T(\mathrm{Gr}(k;n))$,

$$\sum_{d \geq 0} q^d \langle a, b, \kappa \rangle_d = \sum_{d \geq 0} q^d \langle c, \kappa \rangle_d.$$

4.2. Quantum=classical

The goal of this section is to recall the 'quantum = classical' statement, which relates the (3-point, genus 0) equivariant KGW invariants on Grassmannians to a 'classical' calculation in the equivariant K-theory of a two-step flag manifold. This statement was proved in [BM11], and it generalized results of Buch, Kresch and Tamvakis [BKT03] from quantum cohomology. The proofs rely on the 'kernel-span' technique introduced by Buch [Buc03]. A Lie-theoretic approach, for large degrees d, and for the larger family of cominuscule Grassmannians, was obtained in [CP11]; see also [BCMP18b] for an alternative way to 'quantum=classical' utilizing projected Richardson varieties.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

We recall next the 'quantum = classical' result, proved in [BM11], and which will be used later in this paper. To start, form the following incidence diagram:

$$\begin{array}{ccccc} Z_d := \mathrm{Fl}(k-d,k,k+d;n) & \xrightarrow{p_1'} & \mathrm{Fl}(k-d,k;n) & \xrightarrow{p} & X := \mathrm{Gr}(k;n) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} & & \\ Y_d := \mathrm{Fl}(k-d,k+d;n) & \xrightarrow{pr} & \mathrm{Gr}(k-d;n) & & \end{array} \tag{11}$$

Here, all maps are the natural projections. Denote by $p_1 : \mathrm{Fl}(k-d,k,k+d;n) \to \mathrm{Gr}(k;n)$ the composition $p_1 := p \circ p_1'$. If $d \geq k$, then we set $Y_d := \mathrm{Fl}(k+d;n)$, and if $k+d \geq n$, then we set $Y_d := \mathrm{Gr}(k-d;n)$. In particular, if $d \geq \min\{k, n-k\}$, then Y_d is a single point.

Theorem 4.5. *Let $a, b, c \in K_T(\mathrm{Gr}(k;n))$ and $d \geq 0$ a degree.*

(a) *The following equality holds in $K_T(pt)$:*

$$\langle a,b,c\rangle_d = \int_{Y_d} (p_2)_*(p_1^*(a)) \cdot (p_2)_*(p_1^*(b)) \cdot (p_2)_*(p_1^*(c)).$$

(b) *Assume that $(p_2)_*(p_1^*(a)) = pr^*(a')$ for some $a' \in K_T(\mathrm{Gr}(k-d;n)$. Then*

$$\langle a,b,c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} a' \cdot (q)_*(p^*(b)) \cdot (q)_*(p^*(c)).$$

Observe that part (b) follows from (a) and the fact that the left diagram is a fibre square; for details, see [BM11]. We will often use the tautological sequence on $\mathrm{Gr}(k;n)$:

$$0 \to \mathcal{S}_k \to \mathbb{C}^n \to \mathcal{Q}_{n-k} := \mathbb{C}^n/\mathcal{S}_k \to 0.$$

To lighten notation, we will denote by the same letters the bundles on various flag manifolds from the diagram (11), but we will indicate in the subscript the rank of the bundle in question. To illustrate in a situation used below, observe that $\mathrm{Fl}(k-d,k;n)$ equals to the Grassmann bundle $\mathbb{G}(d,\mathbb{C}^n/\mathcal{S}_{k-d}) \to \mathrm{Gr}(k-d;n)$, with tautological sequence

$$0 \to \mathcal{S}_k/\mathcal{S}_{k-d} \to \mathbb{C}^n/\mathcal{S}_{k-d} \to \mathbb{C}^n/\mathcal{S}_k \simeq p^*(\mathbb{C}^n/\mathcal{S}_k) \to 0;$$

all these are bundles on $\mathrm{Fl}(k-d,k;n)$, with $\mathcal{S}_k$ is pulled back from $\mathrm{Gr}(k;n)$, and $\mathcal{S}_{k-d}$ is pulled back from $\mathrm{Gr}(k-d;n)$.

5. Cohomological calculations from 'quantum = classical' diagrams

In this section, we calculate some push-forwards of tautological bundles, which will be utilized in the proof of our main result. For a vector bundle E of rank e, we denote by $\lambda(E)_{\geq s}$, respectively $\lambda(E)_{\leq s}$ and so on, the truncations $y^s[\wedge^s E] + \ldots + y^e[\wedge^e E]$, respectively $1 + y[E] + \ldots + [\wedge^s E]$ etc. We use the notation from the diagram (11).

Proposition 5.1. *Let $d \geq 1$. Then the following hold:*

(a) *For any $j \geq 0$, $(p_2)_* p_1^*(\wedge^j \mathcal{S}_k) = \wedge^j \mathcal{S}_{k-d}$ in $\mathrm{K}_T(Y_d)$. In particular,*

$$(p_2)_*(p_1^* \lambda_y(\mathcal{S}_k)) = \lambda_y(\mathcal{S}_{k-d}). \tag{12}$$

(b) *The following holds in $\mathrm{K}_T(Y_d)$:*

$$(p_2)_* p_1^*(\lambda_y(\mathcal{Q}_{n-k})) = \lambda_y(\mathbb{C}^n/\mathcal{S}_{k+d}) \cdot \lambda_y(\mathcal{S}_{k+d}/\mathcal{S}_{k-d})_{\leq d}. \tag{13}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

(c) *The following holds in* $\mathrm{K}_T(Y_d)$*:*

$$(p_2)_* p_1^*(\lambda_y(\mathcal{S}_k^*)) = \lambda_y(\mathcal{S}_{k-d}^*) \cdot \lambda_y((\mathcal{S}_{k+d}/\mathcal{S}_{k-d})^*)_{\leq d}. \tag{14}$$

Proof. Consider the exact sequence on $\mathrm{Fl}(k-d, k, k+d)$ given by

$$0 \to \mathcal{S}_{k-d} \to S_k = p_1^*\mathcal{S} \to \mathcal{S}_k/\mathcal{S}_{k-d} \to 0.$$

It follows that $\lambda_y(\mathcal{S}_k) = \lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathcal{S}_k/\mathcal{S}_{k-d})$. By the projection formula,

$$(p_2)_*(\lambda_y(\mathcal{S}_k)) = \lambda_y(\mathcal{S}_{k-d}) \cdot (p_2)_*(\lambda_y(\mathcal{S}_k/\mathcal{S}_{k-d})). \tag{15}$$

Observe now that $p_2 : \mathrm{Fl}(k-d, k, k+d; n) \to \mathrm{Fl}(k-d, k+d; n)$ may be identified with the Grassmann bundle $\mathbb{G}(d, \mathcal{S}_{k+d}/\mathcal{S}_{k-d})$, with tautological subbundle $\mathcal{S}_k/\mathcal{S}_{k-d}$. Then part (a) follows from the projection formula and Corollary 3.3, which shows that

$$(p_2)_*(\lambda_y(\mathcal{S}_k/\mathcal{S}_{k-d})) = [\mathcal{O}_{\mathrm{Fl}(k-d,k+d;n)}].$$

For (b), consider the short exact sequence on $\mathrm{Fl}(k-d, k, k+d; n)$,

$$0 \to \mathcal{S}_{k+d}/\mathcal{S}_k \to \mathbb{C}^n/\mathcal{S}_k = p_1^*\mathcal{Q}_{n-k} \to \mathbb{C}^n/\mathcal{S}_{k+d} \to 0.$$

Then $\lambda_y(p_1^*\mathcal{Q}_{n-k}) = \lambda_y(\mathcal{S}_{k+d}/\mathcal{S}_k) \cdot (p_2)^*\lambda_y(\mathbb{C}^n/\mathcal{S}_{k+d})$, and by the projection formula,

$$(p_2)_*(\lambda_y(p_1^*\mathcal{Q}_{n-k})) = (p_2)_*(\lambda_y(\mathcal{S}_{k+d}/\mathcal{S}_k)) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k+d}).$$

The claim follows again by Corollary 3.3, using that $\mathcal{S}_{k+d}/\mathcal{S}_k$ is the tautological quotient bundle of the Grassmann bundle $\mathbb{G}(d, \mathcal{S}_{k+d}/\mathcal{S}_{k-d})$; thus,

$$\pi_*\lambda_y(\mathcal{S}_{k+d}/\mathcal{S}_k) = 1 + y[\mathcal{S}_{k+d}/\mathcal{S}_{k-d}] + \ldots + y^d \wedge^d [\mathcal{S}_{k+d}/\mathcal{S}_{k-d}].$$

Part (c) follows similarly to (b), utilizing the exact sequence $0 \to (\mathcal{S}_k/\mathcal{S}_{k-d})^* \to \mathcal{S}_k^* = p_1^*(\mathcal{S}_k^*) \to \mathcal{S}_{k-d}^* \to 0$ and Corollary 3.3. □

We need the analogues of some of the previous equations in the equivariant K theory of $\mathrm{Gr}(k-d; n)$.

Corollary 5.2. *(a) The following equations hold in* $\mathrm{K}_T(\mathrm{Gr}(k-d; n))$*:*

(a) $q_*(p^*\lambda_y(\mathcal{S}_k)) = \lambda_y(\mathcal{S}_{k-d})$.
(b) $q_*p^*(\lambda_y(\mathcal{S}_k^*)) = \lambda_y(\mathcal{S}_{k-d}^*) \cdot \lambda_y((\mathbb{C}^n/\mathcal{S}_{k-d})^*)_{\leq d}$.
(c) $q_*p^*(\lambda_y(\mathcal{Q}_{n-k})) = \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\leq n-k}$.

Proof. Parts (a) and (b) follow similarly to Proposition 5.1 (a). For (a), we utilize the short exact sequence $0 \to \mathcal{S}_{k-d} \to S_k = p_1^*\mathcal{S}_k \to \mathcal{S}_k/\mathcal{S}_{k-d} \to 0$ on $\mathrm{Fl}(k-d, k; n)$, and that the projection $q : \mathrm{Fl}(k-d, k; n) \to \mathrm{Gr}(k-d; n)$ is identified to the Grassmann bundle $\mathbb{G}(d; \mathbb{C}^n/\mathcal{S}_{k-d}) \to \mathrm{Gr}(k-d; n)$ with tautological subbundle $\mathcal{S}_k/\mathcal{S}_{k-d}$. For (b), one takes the dual of this sequence. Then by Corollary 3.3,

$$q_*(\lambda_y(\mathcal{S}_k/\mathcal{S}_{k-d})) = [\mathcal{O}_{\mathrm{Gr}(k-d;n)}] \text{ and } q_*(\lambda_y((\mathcal{S}_k/\mathcal{S}_{k-d})^*)) = \lambda_y((\mathbb{C}^n/\mathcal{S}_{k-d})^*)_{\leq d}.$$

Then (a) and (b) follow from this and the projection formula.

For part (c), observe that $\mathrm{Fl}(k-d, k; n) \to \mathrm{Gr}(k-d; n)$ is the Grassmann bundle $\mathbb{G}(d, \mathbb{C}^n/\mathcal{S}_{k-d})$ with tautological quotient bundle $\mathbb{C}^n/\mathcal{S}_k = p^*\mathcal{Q}_{n-k}$. Then the claim follows again by Corollary 3.3. □

Observe that $pr^*(\lambda_y(\mathcal{S}_{k-d})) = \lambda_y(\mathcal{S}_{k-d})$ in $\mathrm{K}_T(Y_d)$. Utilizing this, and combining Proposition 5.1, Corollary 5.2 and part (b) from Theorem 4.5, we obtain the following useful corollary:

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Corollary 5.3. *Fix arbitrary $b, c \in \mathrm{K}_T(\mathrm{Gr}(k;n))$ and any degree $d \geq 0$. Then the equivariant KGW invariant $\langle \lambda_y(\mathcal{S}_k), b, c\rangle_d$ satisfies*

$$\langle \lambda_y(\mathcal{S}_k), b, c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} \lambda_y(\mathcal{S}_{k-d}) \cdot q_* p^*(b) \cdot q_* p^*(c).$$

In particular, the 2-point KGW invariant $\langle b, c\rangle_d$ satisfies

$$\langle b, c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} q_* p^*(b) \cdot q_* p^*(c).$$

6. Quantum duals

The main goal of this section is to prove the next identity.

Theorem 6.1. *The following holds in $\mathrm{QK}_T(\mathrm{Gr}(k;n))$:*

$$(\lambda_y(\mathcal{S}_k) - 1) \star \det(\mathcal{Q}_{n-k}) = (1-q)((\lambda_y(\mathcal{S}_k) - 1) \cdot \det(\mathcal{Q}_{n-k})). \tag{16}$$

Equivalently, for any $i > 0$,

$$\wedge^i(\mathcal{S}_k) \star \det(\mathcal{Q}_{n-k}) = (1-q) \wedge^{k-i} (\mathcal{S}_k^*) \cdot \det(\mathbb{C}^n). \tag{17}$$

An equivalent formulation of the identity (16) is in terms of (equivariant) KGW invariants: for any degree $d \geq 0$ and any $a \in \mathrm{K}_T(\mathrm{Gr}(k;n))$,

$$\langle \lambda_y(\mathcal{S}_k) - 1, \det(\mathcal{Q}_{n-k}), a\rangle_d = \langle (\lambda_y(\mathcal{S}_k) - 1) \cdot \det(\mathcal{Q}_{n-k}), a\rangle_d - \langle (\lambda_y(\mathcal{S}_k) - 1) \cdot \det(\mathcal{Q}_{n-k}), a\rangle_{d-1}. \tag{18}$$

By Corollary 5.3 and Corollary 5.2, the left-hand side of (18) may be calculated as

$$\langle \lambda_y(\mathcal{S}_k) - 1, \det(\mathcal{Q}_{n-k}), a\rangle_d = \int_{\mathrm{Gr}(k-d;n)} (\lambda_y(\mathcal{S}_{k-d}) - 1) \cdot \wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d}) \cdot q_* p^*(a). \tag{19}$$

The next lemma calculates the push-forwards relevant for the right-hand side of (18).

Lemma 6.2. *The following equality holds in $\mathrm{K}_T(\mathrm{Gr}(k-d;n))$:*

$$y^{n-k} q_*(p^*(\lambda_y(\mathcal{S}_k) \cdot \det(\mathcal{Q}_{n-k}))) = \lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k}.$$

Proof. To start, observe that for any (equivariant) vector bundle E,

$$\wedge^i E \otimes \det(E^*) = \wedge^{rk(E)-i} E^* \quad \text{and} \quad y^{rk(E)} \lambda_{y^{-1}}(E^*) = \lambda_y(E) \cdot \det(E^*).$$

Utilizing this and that $\det(\mathbb{C}^n/\mathcal{S}_k) = \det \mathcal{S}_k^* \otimes \det \mathbb{C}^n$, we obtain that

$$\begin{aligned} & y^{n-k} q_*(p^*(\lambda_y(\mathcal{S}_k) \cdot \det(\mathbb{C}^n/\mathcal{S}_k))) \\ &= y^{n-k} q_*(p^*(\lambda_y(\mathcal{S}_k) \cdot \det \mathcal{S}_k^* \cdot \det \mathbb{C}^n)) \\ &= y^{n-k} y^k q_*(\lambda_{y^{-1}}(\mathcal{S}_k^*) \cdot \det \mathbb{C}^n)) \\ &= y^n q_*(\lambda_{y^{-1}}(\mathcal{S}_{k-d}^*) \cdot \lambda_{y^{-1}}(\mathcal{S}_k/\mathcal{S}_{k-d})^*) \cdot \det \mathbb{C}^n \\ &= y^n \lambda_{y^{-1}}(\mathcal{S}_{k-d}^*) \cdot q_*(\lambda_{y^{-1}}(\mathcal{S}_k/\mathcal{S}_{k-d})^*) \cdot \det \mathbb{C}^n. \end{aligned} \tag{20}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

By Corollary 3.3,

$$q_*(\lambda_y((\mathcal{S}_k/\mathcal{S}_{k-d})^*)) = \lambda_y((\mathbb{C}^n/\mathcal{S}_{k-d})^*)_{\leq d}; \tag{21}$$

therefore,

$$\begin{aligned} y^d q_*(\lambda_{y^{-1}}(\mathcal{S}_k/\mathcal{S}_{k-d})^*) &= y^d(1 + y^{-1}(\mathbb{C}^n/\mathcal{S}_{k-d})^* + \ldots + y^{-d} \wedge^d (\mathbb{C}^n/\mathcal{S}_{k-d})^*) \\ &= \det(\mathbb{C}^n/\mathcal{S}_{k-d})^* \cdot (y^d \wedge^{n-k+d} (\mathbb{C}^n/\mathcal{S}_{k-d}) \\ &\quad + y^{d-1} \wedge^{n-k+d-1} (\mathbb{C}^n/\mathcal{S}_{k-d}) + \ldots + \wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d})) \\ &= y^{-(n-k)} \det(\mathbb{C}^n/\mathcal{S}_{k-d})^* \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k}. \end{aligned} \tag{22}$$

Combining (20), (21) and (22) above, we obtain

$$\begin{aligned} &y^{n-k} q_*(p^*(\lambda_y(\mathcal{S}_k) \cdot \det(\mathbb{C}^n/\mathcal{S}_k)) \\ &= y^n \lambda_{y^{-1}}(\mathcal{S}^*_{k-d}) \cdot q_*(\lambda_{y^{-1}}(\mathcal{S}_k/\mathcal{S}_{k-d})^*) \cdot \det \mathbb{C}^n \\ &= y^{n-d} \lambda_{y^{-1}}(\mathcal{S}^*_{k-d}) \cdot y^{-(n-k)} \det(\mathbb{C}^n/\mathcal{S}_{k-d})^* \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k} \cdot \det \mathbb{C}^n \\ &= y^{k-d} \lambda_{y^{-1}}(\mathcal{S}^*_{k-d}) \cdot \det(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k} \\ &= \lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k}. \end{aligned}$$

The last expression is the one from the claim. □

Consider the following projection maps:

$$\begin{array}{ccccc} & & \mathrm{Fl}(k-d, k-d+1, k; n) & \xrightarrow{p^{k-d}} & \mathrm{Gr}(k; n) \\ & \swarrow^{q^{k-d+1}} & & \searrow^{q^{k-d}} & \\ \mathrm{Gr}(k-d+1; n) & & & & \mathrm{Gr}(k-d; n) \end{array} \tag{23}$$

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. A standard diagram chase and the equalities from Equation (19) and Lemma 6.2 imply that in order to prove Equation (18), it suffices to show that in $\mathrm{K}_T(\mathrm{Fl}(k-d, k-d+1, k; n)$,

$$\begin{aligned} &y^{n-k} p_*^{k-d}((\lambda_y(\mathcal{S}_{k-d}) - 1) \cdot \wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d})) \\ &= p_*^{k-d}\Big(\lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\geq n-k} - \lambda_y(\mathcal{S}_{k-d+1}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}\Big) \\ &\quad - y^{n-k}\Big(\wedge^{n-k} (\mathbb{C}^n/\mathcal{S}_{k-d}) - \wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})\Big). \end{aligned}$$

After expanding and canceling the like terms, this amounts to showing that

$$\begin{aligned} p_*^{k-d}(\lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k}) = p_*^{k-d}(\lambda_y(\mathcal{S}_{k-d+1}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}) \\ - y^{n-k} p_*^{k-d}(\wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})). \end{aligned} \tag{24}$$

Since this identity will be used again in the proof of relations in the quantum K ring, we will prove it in the lemma below, thus finishing the proof of Theorem 6.1. □

Lemma 6.3. *Equation (24) holds.*

Proof. Consider the projection $p' : \mathrm{Fl}(k-d, k-d+1, k; n) \to \mathrm{Fl}(k-d+1, k; n)$; this is the Grassmann bundle $\mathbb{G}(k-d; \mathcal{S}_{k-d+1})$ over $\mathrm{Fl}(k-d+1, k; n)$ with tautological subbundle $\mathcal{S}_k$ and quotient bundle $\mathcal{L} := \mathcal{S}_{k-d+1}/\mathcal{S}_{k-d}$. In order to show the equality 24, it suffices to replace p^{k-d} by p'; that is,

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

$$p'_*(\lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k}) = p'_*(\lambda_y(\mathcal{S}_{k-d+1}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}) - y^{n-k} p'_*(\wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})). \tag{25}$$

To calculate the left-hand side, consider the short exact sequences

$$0 \to \mathcal{S}_{k-d} \to \mathcal{S}_{k-d+1} \to \mathcal{L} \to 0; \quad 0 \to \mathcal{L} \to \mathbb{C}^n/\mathcal{S}_{k-d} \to \mathbb{C}^n/\mathcal{S}_{k-d+1} \to 0.$$

From this, it follows that $\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d}) = (1+y\mathcal{L})\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})$; thus,

$$\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k} = \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{>n-k} + y\mathcal{L} \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}.$$

Observe that $p'_*(\lambda_y(\mathcal{S}_{k-d})) = 1$, by Corollary 3.3. Using this and the projection formula, we calculate the left-hand side of Equation (25):

$$\begin{aligned}
&p'_*(\lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k}) \\
&= p'_*(\lambda_y(\mathcal{S}_{k-d}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{>n-k}) \\
&\quad + p'_*(\lambda_y(\mathcal{S}_{k-d}) \cdot (\lambda_y(\mathcal{L})-1) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}) \\
&= \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{>n-k}) + p'_*((\lambda_y(\mathcal{S}_{k-d+1}) - \lambda_y(\mathcal{S}_{k-d})) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k}) \\
&= \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{>n-k}) + (\lambda_y(\mathcal{S}_{k-d+1}) - 1) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k} \\
&= \lambda_y(\mathcal{S}_{k-d+1}) \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\geq n-k} - y^{n-k} \wedge^{n-k} (\mathbb{C}^n/\mathcal{S}_{k-d+1}).
\end{aligned}$$

The last expression is the right-hand side of Equation (25), again by projection formula. □

Corollary 6.4. *Let $i > 0$. The following equalities hold* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

(a) $\wedge^i \mathcal{Q}^*_{n-k} \star \det \mathcal{S}^*_k = (1-q) \wedge^{n-k-i} \mathcal{Q}_{n-k} \cdot \det(\mathbb{C}^n)^*$;
(b) $\wedge^i \mathcal{S}_k \star \det \mathcal{S}^*_k = (1-q) \wedge^{k-i} \mathcal{S}^*_k$;
(c) $\wedge^{n-k-i} \mathcal{Q}_{n-k} \star \det \mathcal{S}_k = \wedge^i \mathcal{Q}^*_{n-k} \cdot \det(\mathbb{C}^n)$;
(d) $\wedge^{n-k-i} \mathcal{Q}_{n-k} \star \det \mathcal{Q}^*_{n-k} = \wedge^i \mathcal{Q}^*_{n-k}$.

Proof. All are consequences of Theorem 6.1. Part (a) applies Equation (17) to the dual Grassmannian $\mathrm{Gr}(n-k,(\mathbb{C}^n)^*)$; we utilize that there is a T-equivariant isomorphism $\mathrm{Gr}(k;\mathbb{C}^n) \simeq \mathrm{Gr}(n-k,(\mathbb{C}^n)^*)$ under which the T-equivariant bundle $\mathcal{Q}_{n-k}$ is sent to $\mathcal{S}^*_{n-k}$. Part (b) follows because as elements of $\mathrm{K}_T(\mathrm{Gr}(k;n))$, $\det \mathcal{Q}_{n-k} = \det \mathcal{S}^*_k \cdot \det \mathbb{C}^n$, then $\wedge^i \mathcal{S}_k \star \det \mathcal{S}^*_k = (\det \mathbb{C}^n)^* \wedge^i \mathcal{S}_k \star \det \mathcal{Q}_{n-k} = (1-q) \wedge^{k-i} \mathcal{S}^*_k$. Part (c) follows by multiplying (a) by $\det \mathcal{S}_k$, then using (b), and (d) follows from (c). □

7. Relations in quantum K theory

Recall the tautological sequence $0 \to \mathcal{S}_k \to \mathbb{C}^n \to \mathcal{Q}_{n-k} \to 0$ on $X = \mathrm{Gr}(k;n)$. The goal of this section is to prove the following theorem.

Theorem 7.1. *The following equalities hold in* $\mathrm{QK}_T(X)$*:*

$$\begin{aligned}
\lambda_y(\mathcal{S}_k) \star \lambda_y(\mathcal{Q}_{n-k}) &= \lambda_y(\mathbb{C}^n) - qy^{n-k}\big((\lambda_y(\mathcal{S}_k)-1) \otimes \det \mathcal{Q}_{n-k}\big) \\
&= \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}_k)-1) \star \det \mathcal{Q}_{n-k}.
\end{aligned} \tag{26}$$

The second equality follows from Theorem 6.1; therefore, we focus on the first. Our strategy is to utilize again the 'quantum=classical' identity in order to show that for any $a \in \mathrm{K}_T(\mathrm{Gr}(k;n))$, and for any $d \geq 0$,

$$\langle \lambda_y(\mathcal{S}_k), \lambda_y(\mathcal{Q}_{n-k}), a \rangle_d = \lambda_y(\mathbb{C}^n)\langle 1, a\rangle_d - y^{n-k}\langle (\lambda_y(\mathcal{S}_k)-1) \otimes \det(\mathcal{Q}_{n-k}), a\rangle_{d-1}. \tag{27}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

By Corollary 4.4 this will imply the desired identities. The KGW invariant from the left-hand side may be calculated using the next lemma.

Lemma 7.2. *The following equality holds in* $\mathrm{K}_T(\mathrm{Gr}(k-d;n))$*:*

$$q_*p^*(\lambda_y(\mathcal{S}_k))\cdot q_*p^*(\lambda_y(\mathcal{Q}_{n-k})) = \lambda_y(\mathbb{C}^n) - \lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k}.$$

Proof. This follows from Corollary 5.2, parts (a) and (c), working in the Grassmannian $\mathrm{Gr}(k-d;n)$ with tautological sequence $0\to\mathcal{S}_{k-d}\to\mathbb{C}^n\to\mathbb{C}^n/\mathcal{S}_{k-d}\to 0$:

$$\begin{aligned} q_*p^*(\lambda_y(\mathcal{S}_k))\cdot q_*p^*(\lambda_y(\mathcal{Q}_{n-k})) &= \lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{\le n-k}\\ &= \lambda_y(\mathcal{S}_{k-d})\cdot(\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d}) - \lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k})\\ &= \lambda_y(\mathbb{C}^n) - \lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k}. \end{aligned}$$

□

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. We need to verify Equation (27), and for that, we utilize the 'quantum= classical' statement in Theorem 4.5. From Lemma 7.2, the left-hand side of Equation (27) equals

$$\int_{\mathrm{Gr}(k-d;n)}(\lambda_y(\mathbb{C}^n) - \lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k})\cdot q_*p^*(a).$$

From Lemma 6.2, the right-hand side of (27) is equal to

$$\begin{aligned} &\int_{\mathrm{Gr}(k-d;n)}\lambda_y(\mathbb{C}^n)\cdot q_*p^*(a) -\\ &\int_{\mathrm{Gr}(k-d+1;n)}\left(\lambda_y(\mathcal{S}_{k-d+1})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\ge n-k} + y^{n-k}\wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})\right)\cdot\bar{q}_*\bar{p}^*(a). \end{aligned}$$

Here, $\bar{p}:\mathrm{Fl}(k-d+1,k;n)\to\mathrm{Gr}(k;n)$ and $\bar{q}:\mathrm{Fl}(k-d+1,k;n)\to\mathrm{Gr}(k-d+1;n)$ are the natural projections. After cancelling the like terms, this amounts to proving the equality

$$\begin{aligned} &\int_{\mathrm{Gr}(k-d;n)}(\lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k})\cdot q_*p^*(a)\\ &= \int_{\mathrm{Gr}(k-d+1;n)}\left(\lambda_y(\mathcal{S}_{k-d+1})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\ge n-k} - y^{n-k}\wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})\right)\cdot q_*p^*(a). \end{aligned} \tag{28}$$

Recall the projection $p^{k-d}:\mathrm{Fl}(k-d,k-d+1,k;n)\to\mathrm{Gr}(k;n)$. As in the proof of Theorem 6.1, by diagram chasing and projection formula, one shows the equality (28) by demonstrating the following:

$$\begin{aligned} &p^{k-d}_*(\lambda_y(\mathcal{S}_{k-d})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d})_{>n-k})\\ &= p^{k-d}_*\left(\lambda_y(\mathcal{S}_{k-d+1})\cdot\lambda_y(\mathbb{C}^n/\mathcal{S}_{k-d+1})_{\ge n-k} - y^{n-k}\wedge^{n-k}(\mathbb{C}^n/\mathcal{S}_{k-d+1})\right). \end{aligned}$$

This is Equation (24), proved in Lemma 6.3. □

8. The QK Whitney presentation

In this section, we prove that the relations found in Theorem 7.1 give a presentation of the algebra $\mathrm{QK}_T(\mathrm{Gr}(k;n))$. The proof strategy is similar to that employed in [ST97, FP97] from quantum cohomology: one first proves that in the classical limit, these generate the full ideal of relations, and then one uses Nakayama-type arguments to upgrade to the quantum situation. Unlike the quantum cohomology ring, the quantum K theory is not a graded ring. One may still use the usual Nakayama lemma for

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

modules over completed rings to prove that the same phenomenon holds. The necessary statements are collected in the Appendix A below.

As usual, we consider the Grassmannian $\mathrm{Gr}(k;n)$ equipped with the tautological sequence $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$, but from now, we omit the ranks in the subscripts. Let $X = (X_1, \ldots, X_k)$ and $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_{n-k})$ denote formal variables. The elementary symmetric polynomials $e_i(X) = e_i(X_1, \ldots, X_k)$ and $e_j(\tilde{X}) = e_j(\tilde{X}_1, \ldots, \tilde{X}_{n-k})$ are algebraically independent. Geometrically, in $\mathrm{K}_T(\mathrm{Gr}(k;n))$,

$$\lambda_y(\mathcal{S}) = \prod_{i=1}^{k}(1 + yX_i); \quad \lambda_y(\mathcal{Q}) = \prod_{j=1}^{n-k}(1 + y\tilde{X}_j).$$

Define by $I \subset \mathrm{K}_T(pt)[e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]$ the ideal determined by the Whitney relations in $\mathrm{K}_T(\mathrm{Gr}(k;n))$; that is, I is generated by the coefficients of the powers of y in

$$\prod_{i=1}^{k}(1 + yX_i)\prod_{i=1}^{n-k}(1 + y\tilde{X}_i) = \prod_{i=1}^{n}(1 + yT_i). \tag{29}$$

A non-equivariant variant of the following proposition appears in [Las90, §7].

Proposition 8.1. *There is an isomorphism of* $\mathrm{K}_T(pt)$*-algebras*

$$\Psi : \mathrm{K}_T(pt)[e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]/I \to \mathrm{K}_T(\mathrm{Gr}(k;n)),$$

sending $e_i(X) \mapsto \wedge^i \mathcal{S}$ *and* $e_j(\tilde{X}) \mapsto \wedge^j \mathcal{Q}$.

Proof. Denote the ring on the left by A. Since $\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n)$, the homomorphism $\Psi : A \to \mathrm{K}_T(\mathrm{Gr}(k;n))$ is well defined. Consider the polynomial ring

$$A' := \mathbb{Z}[T_1, \ldots, T_n][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})].$$

Note that I is also an ideal in A'. Observe that A'/I is a free $\mathbb{Z}[T_1, \ldots, T_n]$-module of rank $\binom{n}{k}$. (There are several proofs. For instance, (temporarily) identify $\mathbb{Z}[T_1, \ldots, T_n]$ to $H_T^*(pt)$. Then from the Whitney relations $c^T(\mathcal{S}) \cdot c^T(\mathcal{Q}) = c^T(\mathbb{C}^n)$ in $H_T^*(\mathrm{Gr}(k;n))$, there is an isomorphism of $H_T^*(pt)$-algebras $A'/I \simeq H_T^*(\mathrm{Gr}(k;n))$, sending $e_i(X) \mapsto c_i^T(\mathcal{S})$ and $e_j(\tilde{X}) \mapsto c_j^T(\mathcal{Q})$.) If we regard the Laurent polynomial ring $\mathrm{K}_T(pt) = \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ as a $\mathbb{Z}[T_1, \ldots, T_n]$-algebra (under the natural inclusion of polynomials into Laurent polynomials), we obtain that

$$A = (A'/I) \otimes_{\mathbb{Z}[T_1, \ldots, T_n]} \mathrm{K}_T(pt)$$

is a free $\mathrm{K}_T(pt)$-module of rank $\binom{n}{k}$. We utilize this to show that Ψ induces an isomorphism between the associated graded rings. To calculate $\mathrm{gr}(A)$, we make the change of variables $z_i = 1 - X_i$ $(1 \le i \le k)$, $\tilde{z}_j = 1 - \tilde{X}_j$ $(1 \le j \le n-k)$, and $\zeta_s = 1 - T_s$ $(1 \le s \le n)$. Each of these variables has degree 1. Under this change, A becomes

$$\frac{\mathrm{K}_T(pt)[e_1(z), \ldots, e_k(z); e_1(\tilde{z}), \ldots, e_{n-k}(\tilde{z})]}{\langle \sum_{i+j=\ell} e_i(z_1, \ldots, z_k) e_j(\tilde{z}_1, \ldots, \tilde{z}_{n-k}) - e_\ell(\zeta_1, \ldots, \zeta_n) \rangle_{1 \le \ell \le n}};$$

see also (56) below (with $q = 0$). The variables $z_i = 1 - X_i$ are sent to the K-theoretic Chern roots of $\mathcal{S}^*$, and similarly, $\tilde{z}_j$ to the K-theoretic Chern roots of $\mathcal{Q}^*$. We deduce that the initial term of $\Psi(e_i(z))$, and of $\Psi(e_j(\tilde{z}))$, equals to $c_i^T(\mathcal{S}^*)$, respectively $c_j^T(\mathcal{Q}^*)$ in $H_T^*(\mathrm{Gr}(k;n))$. Thus, when taking the associated graded rings, Ψ recovers the usual presentation of the equivariant cohomology ring. This implies that Ψ is injective.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

The surjectivity follows from the theory of factorial Grothendieck polynomials. More precisely, from [Buc02, Thm. 2.1] or [Oet21, Thm. 1.2], see also [FL94, McN06, IN13], for each partition λ, the equivariant Schubert class $\mathcal{O}_\lambda$ is a symmetric polynomial in the K-theoretic Chern roots $1-X_1, \ldots, 1-X_k$ of $\mathcal{S}$ with coefficients in $\mathrm{K}_T(pt)$. By (58) below, this is a $\mathrm{K}_T(pt)$-linear combination of the (images of) $e_i(X)$. Then Ψ is also surjective, and this finishes the proof. □

Recall from Theorem 7.1 that in $\mathrm{QK}_T(\mathrm{Gr}(k;n))$,

$$\lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}) - 1) \star \det \mathcal{Q}. \tag{30}$$

Motivated by this, define the ideal

$$I_q \subset \mathrm{K}_T[pt][[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]$$

generated by polynomials obtained by equating the powers of y in the equality:

$$\begin{aligned} &\prod_{i=1}^{k}(1+yX_i) \times \prod_{j=1}^{n-k}(1+y\tilde{X}_i) \\ &= \prod_{i=1}^{n}(1+yT_i) - \frac{q}{1-q} y^{n-k} \tilde{X}_1 \cdot \ldots \cdot \tilde{X}_{n-k} \Big(\prod_{i=1}^{k}(1+yX_i) - 1 \Big). \end{aligned} \tag{31}$$

Theorem 8.2. *There is an isomorphism of* $\mathrm{K}_T[pt][[q]]$*-algebras*

$$\Psi : \mathrm{K}_T[pt][[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]/I_q \to \mathrm{QK}_T(\mathrm{Gr}(k;n)),$$

sending $e_i(X) \mapsto \wedge^i \mathcal{S}$ *and* $e_j(\tilde{X}) \mapsto \wedge^j \mathcal{Q}$.

Proof. There exists a ring homomorphism

$$\widetilde{\Psi} : \mathrm{K}_T[pt][[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})] \to \mathrm{QK}_T(\mathrm{Gr}(k;n)),$$

sending $e_i(X) \mapsto \wedge^i \mathcal{S}$ and $e_j(\tilde{X}) \mapsto \wedge^j \mathcal{Q}$. It follows from Equation (30) that $\widetilde{\Psi}(I_q) = 0$; therefore, this induces the homomorphism Ψ from the claim. In order to prove that Ψ is an isomorphism, we will use the Nakayama-type result from Proposition A.3, applied to the case when

$$M = \mathrm{K}_T[pt][[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]/I_q$$

is the claimed presentation, regarded as a module over the $\langle q \rangle$-adically complete ring $R = \mathrm{K}_T[pt][[q]]$, and the free R-module $N = \mathrm{QK}_T(\mathrm{Gr}(k;n))$. Proposition 8.1 implies that if one takes the quotient by $\langle q \rangle$, Ψ becomes an isomorphism. The fact that M is a finite R-module follows from [Eis95, Ex. 7.4, p. 203] (cf. Remark A.4), applied to the ideal $\mathfrak{m} = \langle q \rangle$, and then noting that M is finite over

$$S = \mathrm{K}_T[pt][[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})].$$

Note that R and S are Noetherian rings (e.g., by [AM69, Thm. 10.26]) and that $\langle q \rangle$ is included in the Jacobson radical of R because $1 - aq$ is invertible in R, for any $a \in \mathrm{K}_T(pt)$.[2] □

[2]See also [GMS+23, §3 and Appendix A] for more details about this argument and how it applies to more general partial flag varieties.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

9. Physics, Wilson lines and Jacobian ring presentations

In this section, we will outline the approach giving the Jacobian relations (42) derived from a holomorphic function W called the (twisted) superpotential in the physics literature. These relations will be utilized in the next section to obtain the Coulomb branch presentation of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$. See e.g. [JM20, JM19, JMNT20, GMSZ22, UY20] for references.

Briefly, in the special case that a space can be realized as $V//G$ for V a complex vector space V and G a reductive algebraic group, the quantum K theory of $V//G$ arises from a three-dimensional 'supersymmetric gauge theory', which is ultimately defined by G, a representation ρ defining the G action on V, and a matrix of real numbers $\tilde{k}$ (of the same rank as G), whose values we will give momentarily. The three-dimensional theory lives on a three-manifold, which is taken to be $\Sigma \times S^1$ for a Riemann surface Σ, partly as a result of which the theory can be described as a two-dimensional theory on Σ. Correlation functions in the two-dimensional theory include 'Wilson lines' (see, for example, [CK16, section 2])

$$\mathrm{Tr}_\rho P \exp\left(\int_{S^1} A\right) \tag{32}$$

(for ρ a representation of G, A a connection on a principal G bundle, P a path-ordering symbol) on S^1 over a fixed point in Σ, and those Wilson lines correspond to K theory elements on $V//G$. More precisely, we will identify them with Schur functors on certain vector bundles on $V//G$ associated to representations of G; see Remark 9.3 below. Quantum K theory of spaces described as critical loci of holomorphic functions on noncompact symplectic quotients can also be described physically, but in this section, we focus on the simpler case of spaces that are themselves symplectic quotients of vector spaces. Given $V//G$, the quantum K theory relations arise as derivatives of a holomorphic function known as the superpotential and conventionally denoted W; cf. Equation (36) below.

Remark 9.1. At least in some cases, physics makes predictions for the quantum K theory ring of a space realized as the critical locus of a holomorphic function on $V//G$ (such as a hypersurface, or a complete intersection), but the details are beyond the scope of this paper. See, for example, [GMSZ22] and references therein.

For simplicity, we specialize to the case that $G = \mathrm{GL}_k$ for a positive integer k. Then, define $\tilde{k}_1$, $\tilde{k}_2 \in \mathbb{R}$ as follows. Write ρ as a sum of irreducible representations

$$\rho = \rho_1 \oplus \cdots \oplus \rho_\ell. \tag{33}$$

Define

$$\tilde{k}_1 = -\frac{1}{2}\sum_{\gamma=1}^{\ell}\left(\mathrm{Cas}_1(\rho_\gamma)\right)^2, \quad \tilde{k}_2 = k - \frac{1}{2}\sum_{\gamma=1}^{\ell}\frac{\dim \rho_\gamma}{\dim GL(k)}\mathrm{Cas}_2(\rho_\gamma), \tag{34}$$

where Cas denotes eigenvalues of Casimir operators as in, for example, [Iac06, chapter 7].

Remark 9.2. The choice of $\tilde{k}_1$ and $\tilde{k}_2$ from (34) generates the standard quantum K theory. However, physics considerations suggest there are other choices, conjecturally related to the quantum K theory with level structure considered in [RZ18]. See [GPZ21] for further physics discussions.

Then, the quantum K theory ring relations are given physically as the critical locus equations of a function W known as the superpotential, which is given as [CK16, equ'n (2.33)], [GMSZ22, equ'n (2.1)]

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

$$W = \frac{\tilde{k}_2}{2}\sum_{a=1}^{k}(\ln X_a)^2 \; + \; \frac{\tilde{k}_1 - \tilde{k}_2}{2k}\left(\sum_{a=1}^{k}\ln X_a\right)^2 + \left(\ln(-1)^{k-1}q\right)\sum_{a=1}^{k}\ln X_a + \sum_{\alpha}\left[\mathrm{Li}_2(\exp(\tilde{\rho}_\alpha, \ln X)) \; + \; \frac{1}{4}(\tilde{\rho}_\alpha, \ln X)^2\right], \tag{35}$$

where we use $\ln X$ to denote a vector with components $(\ln X_a)$. Here, (X_a) is a point in $(\mathfrak{R}\otimes_{\mathbb{Z}}\mathbb{C}-\Delta)/\mathcal{W} \cong (\mathbb{C}^k - \Delta)/\mathcal{W}$ for $\mathfrak{R}$ the root lattice of GL_k,

$$\Delta \; = \; \coprod_{a<b}\{X_a = X_b\} \coprod_{a}\{X_a = 1\} \coprod_{a}\{X_a = 0\}, \tag{36}$$

$\mathcal{W} = S_k$ the Weyl group of $U(k)$, $(\cdot,\cdot)$ denotes a natural pairing between root and weight lattice vectors, and $\{\tilde{\rho}_\alpha\}$ are the weight vectors of the representation ρ. It can be shown that the superpotential is invariant under the action of the Weyl group.

For the case of a Grassmannian $\mathrm{Gr}(k;n)$, described as the GIT quotient $V//\mathrm{GL}_k$ for $V = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, with ρ given by a sum of n copies of the fundamental representation,

$$\tilde{k}_1 \; = \; -n/2, \quad \tilde{k}_2 \; = \; k - n/2, \tag{37}$$

and the superpotential specializes to

$$W = \frac{k}{2}\sum_{a=1}^{k}(\ln X_a)^2 \; - \; \frac{1}{2}\left(\sum_{a=1}^{k}\ln X_a\right)^2 + \ln\left((-1)^{k-1}q\right)\sum_{a=1}^{k}\ln X_a \; + \; n\sum_{a=1}^{k}\mathrm{Li}_2(X_a). \tag{38}$$

Remark 9.3. A Wilson line in representation ϕ of $U(k)$ is the Chern character of the Schur functor $\mathfrak{S}_\phi\mathcal{S}$ for $\mathcal{S}$ the universal subbundle, where the X_a are exponentials of Chern roots. We give a few simple examples below:

Representation	Schur functor	Wilson line
$\square$	$\mathcal{S}$	$e_1(X)$
$\square\square$	$\mathrm{Sym}^2\mathcal{S}$	$h_2(X)$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\wedge^2\mathcal{S}$	$e_2(X)$

Remark 9.4. There exists an analogous superpotential W whose derivatives encode quantum K theory relations for more general $V//G$. Write G as the complexification of a compact Lie group G', and decompose the Lie algebra of G', $\mathfrak{g}$, as a sum of central pieces and simple factors. Very schematically, there is one q_i for each central copy of Lie $U(1)$ in $\mathfrak{g}$, and the superpotential is a Weyl-invariant function determined by the weight vectors of the representation ρ and the 'Chern-Simons levels', here $\tilde{k}_1$, $\tilde{k}_2$, which are determined by slight generalizations of (34).

We claim that the quantum K theory ring is determined by an analogue of the Jacobian ring of W (involving exponentials of derivatives rather than just derivatives). For the moment, we compute the relations generated by W, and then later we will observe that the resulting ring is the quantum K theory ring of $\mathrm{Gr}(k;n)$ as presented by [GK17].

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Returning to the Grassmannian $\mathrm{Gr}(k;n)$, using the possibly obscure fact that

$$y\frac{\partial}{\partial y}\mathrm{Li}_2(y) \;=\; -\ln(1-y), \tag{39}$$

we find for each $1 \le a \le k$ that

$$\exp\left(\frac{\partial W}{\partial \ln X_a}\right) \;=\; 1 \tag{40}$$

implies that

$$(-1)^{k-1}q(X_a)^k \;=\; \left(\prod_{b=1}^{k} X_b\right)(1-X_a)^n. \tag{41}$$

There is also an equivariant version of these identities. Let $T_i \in \mathrm{K}_T(pt)$ denote equivariant parameters. These appear in the pertinent physical theories as exponentials of 'twisted masses'. Concretely, in cases with twisted masses, the superpotential (38) for $\mathrm{Gr}(k;n)$ generalizes to [UY20]

$$W = \frac{k}{2}\sum_{a=1}^{k}(\ln X_a)^2 \;-\; \frac{1}{2}\left(\sum_{a=1}^{k}\ln X_a\right)^2 \\ +\; \ln\Big((-1)^{k-1}q\Big)\sum_{a=1}^{k}\ln X_a \;+\; \sum_{i=1}^{n}\sum_{a=1}^{k}\mathrm{Li}_2\Big(X_aT_i^{-1}\Big).$$

Simplifying

$$\exp\left(\frac{\partial W}{\partial \ln X_a}\right) \;=\; 1 \tag{42}$$

for each $1 \le a \le k$, we find

$$(-1)^{k-1}q(X_a)^k\prod_{j=1}^{n}T_j = \left(\prod_{b=1}^{k}X_b\right)\cdot\prod_{i=1}^{n}(T_i - X_a). \tag{43}$$

In the next section, we will symmetrize these relations to obtain the Coulomb branch presentation of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$. This will be proved to be isomorphic to the QK Whitney ring from Theorem 8.2. We also note that the equations (43) are the same as the Bethe Ansatz equations from [GK17, equ'n (4.17)]. In *loc.cit.*, the authors utilize an approach based on algebraic properties of integrable systems and of equivariant localization, to obtain a distinct presentation of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$. See also §12.1.2 below for a comparison.

Observe that in the non-equivariant specialization (i.e., when $T_i = 1$), the equations (43) specialize to (41).

Remark 9.5. If one considers the specialization $q \mapsto 1$ in the finite difference operator $\mathcal{H}$ annihilating the I-function considered by Givental and Yan in [GY21, p. 21] (see also [Wen19]), one recovers again the vacuum (or the Bethe Ansatz) equations from (43).[3] The I-function from [GY21] corresponds to the 'abelianization' of the Grassmannian $\mathrm{Gr}(k;n)$; cf. *loc.cit.* From this perspective, our procedure below may be interpreted as a symmetrization of the specialization $\mathcal{H}_{q\mapsto 1}$. This further suggests that the abelian-nonabelian correspondence proved for quantum cohomology in [CFKS08] may extend to

[3]More precisely, in *loc. cit.*, one needs to take the T-equivariant version of all objects involved, and their q is the 'loop parameter', which is different from ours.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

quantum K theory; see [GW22]. See also [IMT15] for a related method to obtain relations in quantum K theory by identifying difference operators which annihilate the appropriate J-function.

Example 9.6. In the case of $\mathrm{Gr}(2;5)$, the superpotential is given by

$$W = \frac{1}{2}(\ln X_1)^2 + \frac{1}{2}(\ln X_2)^2 - (\ln X_1)(\ln X_2) + \ln(-q)\sum_{a=1}^{2}\ln X_a + \sum_{i=1}^{5}\sum_{a=1}^{2}\mathrm{Li}_2\left(X_a T_i^{-1}\right), \tag{44}$$

and the chiral ring relations (43) are, for $a \in \{1,2\}$,

$$\prod_{i=1}^{5}(T_i - X_a) = (-q)\frac{X_a^2}{X_1 X_2}\prod_{j=1}^{5} T_j. \tag{45}$$

In the nonequivariant case, we take $T_i = 1$, then the chiral ring relations become

$$-qX_1 = X_2(1-X_1)^5, \quad -qX_2 = X_1(1-X_2)^5, \tag{46}$$

in agreement with (41). We show in the next section how the symmetrization of this leads the quantum K relations; see also Appendix B below.

10. Coulomb branch and quantum Whitney presentations

The goal of this section is to obtain the Coulomb branch presentation we denoted by $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$, and predicted by physics. We will prove in Theorem 10.2 that this is equivalent to the presentation of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ from Theorem 8.2.

The idea of obtaining the Coulomb branch presentation was already used in the authors' previous work [GMSZ22]. First, one rewrites the 'vacuum equations' (43) in terms of the 'shifted variables' from (47) below. Since $\mathrm{Gr}(k;n) = Hom(\mathbb{C}^k, \mathbb{C}^n)//\mathrm{GL}_k$, any presentation has to satisfy a symmetry with respect to S_k, the Weyl group of GL_k. While the ideal generated by equations (43) is symmetric under permutations in S_k, the individual generators are not. To rectify this, we write down a 'characteristic polynomial' (cf. Equation (52) below) where all the coefficients satisfy the required symmetry. Then we utilize the Vieta equations for this polynomial to obtain a set of polynomial equations. These are $S_k \times S_{n-k}$ symmetric, and they give a presentation of the equivariant quantum K ring. To start, define

$$\zeta_i = 1 - T_i, \quad z_a = 1 - X_a \quad (1 \le i \le n; \quad 1 \le a \le k), \tag{47}$$

so that for any a, equation (43) becomes

$$\left(\prod_{i=1}^{n}(z_a - \zeta_i)\right)\left(\prod_{b\neq a}(1-z_b)\right) + (-1)^k q(1-z_a)^{k-1}\prod_{i=1}^{n}(1-\zeta_i) = 0. \tag{48}$$

A key observation is that we may rewrite this in the form

$$(z_a)^n + \sum_{i=0}^{n-1}(-1)^{n-i}(z_a)^i \hat{g}_{n-i}(z,\zeta,q) = 0. \tag{49}$$

where the $\hat{g}_i(z,\zeta,q)$ are *symmetric polynomials* in $z_1,\ldots,z_k$ and $\zeta_1,\ldots,\zeta_n$. To do this, it suffices to take $a = 1$. The only nontrivial part is the symmetry in the variables z. This follows from repeated application of the identity

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

$$e_j(z_2,\dots,z_k) = e_j(z_1,\dots,z_k) - z_1 e_{j-1}(z_2,\dots,z_k)$$

to the factor $\prod_{j=2}(1-z_j) = \sum_{i=0}^{k-1}(-1)^i e_i(z_2,\dots,z_k)$, then collecting the resulting powers of z_1.

To state the formula for the polynomial $\hat{g}_\ell(z,\zeta,q)$, we fix some notation. Set

$$c^z = \prod_{i=1}^{k}(1-z_i) = \sum_{i\geq 0}(-1)^i e_i(z);\ c^z_{\leq j} = \sum_{i=0}^{j}(-1)^i e_i(z);\ c^z_{\geq j} = (-1)^j(c(z) - c_{\leq j-1}(z)).$$

Informally, these are truncations of the Chern polynomial in $z = (z_1,\dots,z_k)$. One defines similarly $c^\zeta, c^\zeta_{\leq j}, c^\zeta_{\geq j}$; these are polynomials in $\zeta = (\zeta_1,\dots,\zeta_n)$. Set

$$c'_{\geq \ell}(z,\zeta) = e_\ell(\zeta) + e_{\ell-1}(\zeta)c^z_{\geq 2} + e_{\ell-2}(\zeta)c^z_{\geq 3} + \ldots + e_{\ell-k+1}(\zeta)c^z_{\geq k}. \tag{50}$$

If clear from the context, we will drop the variables z,ζ from the notation. (As usual, $e_i(z) = e_i(\zeta) = 0$ for $i<0$ and $e_0(z) = e_0(\zeta) = 1$.) Define the matrices

$$E = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -e_1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k-1} & -e_{k-2} & \dots & -1 \end{pmatrix};$$

$$C^\zeta_{\geq n-k+2} = \begin{pmatrix} c^\zeta_{\geq n-k+2} \\ \vdots \\ c^\zeta_{\geq n} \\ 0 \end{pmatrix};\quad C^{z,\zeta}_{\geq n-k+1} = \begin{pmatrix} c'_{\geq n-k+1} \\ c'_{\geq n-k+2} \\ \vdots \\ c'_{\geq n} \end{pmatrix}.$$

Besides their usefulness in the lemma below, these matrices will appear naturally in §11 below, in relation to an equivariant generalization of Grothendieck polynomials.

Lemma 10.1. *The polynomial coefficients $\hat{g}_\ell(z,\zeta,q)$ from (49) are given by*

$$\begin{cases} c'_{\geq \ell}(z,\zeta) & \text{if } 1\leq \ell \leq n-k \\ c'_{\geq \ell}(z,\zeta) + \left(E\cdot C^\zeta_{\geq n-k+2}\right)_\ell + (-1)^{n+k} q \binom{k-1}{n-\ell} c^\zeta & \text{if } n-k+1 \leq \ell \leq n. \end{cases} \tag{51}$$

Here, M_ℓ denotes the $\ell - n + k$-th component of the k-component column matrix M.

Proof. This is a tedious but rather standard algebraic manipulation, which we leave to the reader. □

Define a 'characteristic polynomial' $f(\xi,z,\zeta,q)$ by

$$f(\xi,z,\zeta,q) \;=\; \xi^n \;+\; \sum_{i=0}^{n-1}(-1)^{n-i}\xi^i \hat{g}_{n-i}(z,\zeta,q). \tag{52}$$

From (49), we deduce that $f(\xi,z,\zeta,q) = 0$ whenever $\xi = z_a$ for some a. Since f is a degree n polynomial in ξ, the equation $f(\xi,z,\zeta,q) = 0$ has n roots in some appropriate field extension, which by construction include $z_1,\dots,z_k$. Let $\{z,\hat{z}\} = \{z_1,\cdots,z_k;\hat{z}_{k+1},\cdots,\hat{z}_n\}$ denote the n roots of (52). From Vieta's formula,

$$\sum_{i+j=\ell} e_i(z)e_j(\hat{z}) \;=\; \hat{g}_\ell(z,\zeta,q). \tag{53}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

This determines the 'Coulomb branch ring':

$$\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n)) = \mathrm{K}_T(pt)[[q]][e_1(z),\cdots,e_k(z),e_1(\hat{z}),\cdots,e_{n-k}(\hat{z})]/\hat{J}, \tag{54}$$

where $\hat{J}$ is the ideal generated by the polynomials

$$\sum_{i+j=\ell} e_i(z)e_j(\hat{z}) - \hat{g}_\ell(z,\zeta,q); \quad 1 \le \ell \le n. \tag{55}$$

Our goal is to demonstrate that the relations above are equivalent to those from Theorem 8.2 arising from the QK-Whitney relations. To this aim, apply the same change of variables (47) to the relations (31), denoting in addition $\tilde{z}_i = 1 - \tilde{X}_i$. Then the presentation in Theorem 8.2 can be written as

$$\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n)) = \mathrm{K}_T(pt)[[q]][e_1(z),\ldots,e_k(z),e_1(\tilde{z}),\ldots,e_{n-k}(\tilde{z})]/\tilde{J}, \tag{56}$$

where the ideal $\tilde{J}$ is generated by $\sum_{i+j=\ell} e_i(z)e_j(\tilde{z}) - \tilde{g}_\ell(z,\zeta,q)$ for

$$\tilde{g}_\ell(z,\zeta,q) = e_\ell(\zeta) - \frac{q}{1-q}\sum_{s=n-k+1}^{\ell}(-1)^s\binom{n-s}{\ell-s}\Delta_{s+k-n}, \tag{57}$$

for $1 \le \ell \le n$, and $\Delta_i = e_i(1-z)e_{n-k}(1-\tilde{z})$. Note that $\tilde{g}$ also depends on $\tilde{z}$, although this is not included in the notation. (In fact, in the proof of Theorem 10.2, we will eliminate the dependence on $\tilde{z}$; see Theorem 11.8.) To get a more explicit formula, observe that

$$e_i(1-x_1,\ldots,1-x_n) = \sum_{s=0}^{i}(-1)^s\binom{n-s}{i-s}e_s(x_1,\ldots,x_n). \tag{58}$$

An easy algebra manipulation based on this formula shows that for $1 \le \ell \le n$,

$$\sum_{s=n-k+1}^{\ell}(-1)^s\binom{n-s}{\ell-s}e_{s+k-n}(1-z) = (-1)^{n-k+1}\left(\binom{k}{\ell+k-n} - e_{\ell+k-n}(z)\right);$$

therefore, (57) may be rewritten as

$$\tilde{g}_\ell(z,\zeta,q) = e_\ell(\zeta) + (-1)^{n-k}\frac{q}{1-q}e_{n-k}(1-\tilde{z})\left(\binom{k}{\ell+k-n} - e_{\ell+k-n}(z)\right). \tag{59}$$

Theorem 10.2. *The following three rings are isomorphic to one another and to the algebra* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

(a) *The ring* $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *from (54);*
(b) *The ring* $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *from (56);*
(c) *The ring* $\mathrm{K}_T(pt)[[q]][e_1(X),\ldots,e_k(X),e_1(\tilde{X}),\ldots,e_{n-k}(\tilde{X})]/I_q$*, where* I_q *is the ideal defined in (31).*

Note that the ring in (c) was proved in Theorem 8.2 to be isomorphic to the 'geometric' ring $\mathrm{QK}_T(\mathrm{Gr}(k;n))$. We already proved that the isomorphism of the rings in (b) and (c) follows from the change of variables

$$z_i = 1 - X_i \quad (1 \le i \le k); \quad \tilde{z}_j = 1 - \tilde{X}_j \quad (1 \le j \le n-k).$$

The isomorphism between (a) and (b) is proved in the next section. In the process, we will reformulate these presentations in terms of (equivariant) Grothendieck polynomials and (equivariant) complete

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

homogeneous symmetric functions; see Theorem 11.8 below. An example for $\mathrm{QK}_T(\mathrm{Gr}(2;5))$ is given in Appendix B.

11. An isomorphism between the Whitney and Coulomb branch presentations

The goal of this section is to prove Theorem 11.12, thus finishing the proof of Theorem 10.2. We utilize the notation from §10.

11.1. Grothendieck polynomials

We start by recording some algebraic identities about the Grothendieck polynomials $G_j(z) = G_j(z_1,\dots,z_k)$, indexed by single row partitions. As usual, $z=(z_1,\dots,z_k)$, and $e_i = e_i(z)$, $h_i = h_i(z)$ denote the elementary symmetric function, respectively the complete homogeneous symmetric function. It was proved in [Len00, p.80] that

$$G_j(z) = \sum_{a,b\geq 0; a+b\leq k} (-1)^b h_{j+a} e_b = h_j + (h_{j+1} - h_j e_1) + (h_{j+2} - h_{j+1}e_1 + h_j e_2) + \dots. \tag{60}$$

An equivalent formulation proved in [Len00, Thm. 2.2]) is

$$G_j(z) = h_j(z) - \sum_{a=2}^{k} (-1)^a s_{(j,1^{a-1})}(z), \tag{61}$$

where $s_\mu(z)$ denotes the Schur polynomial, and $(j,1^{a-1})$ is the partition $(j,1,\dots,1)$ with $a-1$ 1's. We refer to [Len00] for more about these polynomials. We record a Cauchy-type identity for Grothendieck polynomials. It is likely well known, but we could not find a reference.

Lemma 11.1. *If $\ell \geq 1$ and $z=(z_1,\dots,z_k)$, then*

$$\sum_{i,j\geq 0; i+j=\ell} (-1)^j e_i(z) G_j(z) = e_{\ell+1}(z) - e_{\ell+2}(z) + \dots. \tag{62}$$

In particular, for $\ell \geq k$, $\sum_{i,j\geq 0; i+j=\ell} (-1)^j e_i(z) G_j(z) = 0$.

Proof. From (61), we obtain

$$\begin{aligned}\sum_{i+j=\ell} (-1)^j e_i(z) G_j(z) &= e_\ell(z) + \sum_{j=1}^{\ell}\sum_{a=1}^{k} (-1)^{j+a-1} e_{\ell-j}(z) s_{(j,1^{a-1})}(z) \\ &= e_\ell(z) + \sum_{a=1}^{k} (-1)^{a-1} \left[\sum_{j=1}^{\ell} (-1)^j e_{\ell-j}(z) s_{(j,1^{a-1})}(z)\right].\end{aligned}$$

To prove the lemma, it suffices to show that

$$\sum_{j=1}^{\ell} (-1)^j e_{\ell-j}(z) s_{(j,1^{a-1})}(z) = -e_{\ell+a-1}(z).$$

The case when $a=1$ follows from $\sum_{i+j=\ell}(-1)^j e_i(z) h_j(z) = 0$ (i.e., the usual Cauchy formula) for $\ell \geq 1$. For $a>1$, we utilize the Pieri formula to multiply a Schur function by $e_i(z)$ (cf. e.g., [Ful97]). To start, only Schur functions s_μ with $\mu_1 \leq \ell+1$ and $1 \leq \mu_2 \leq 2$ may appear. Furthermore, if $\mu = (\mu_1,\dots,\mu_s)$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

is such a partition with $1 < \mu_1 \leq \ell + 1$, then the Schur function s_μ appears in the left-hand side twice, in the expansion of the terms

$$(-1)^{\mu_1} e_{\ell-\mu_1}(z) s_{\mu_1,1^{a-1}}(z) + (-1)^{\mu_1-1} e_{\ell-\mu_1+1}(z) s_{\mu_1-1,1^{a-1}}(z).$$

Since the coefficients of s_μ appearing in the Pieri rule equal to 1, and since the terms above have opposite signs, it follows that the corresponding s_μ's cancel. We are left with the situation when $\mu_1 = 1$. Then necessarily $j = 1$ (i.e., we consider the terms $-e_{\ell-1}(z)e_a(z)$). Again by Pieri formula, the only multiple of s_μ with $\mu_1 = 1$ is $-e_{\ell+a-1}(z)$. This finishes the proof. □

Define the column matrices:

$$H = \begin{pmatrix} (-1)^{n-k+1} h_{n-k+1} \\ (-1)^{n-k+2} h_{n-k+2} \\ \vdots \\ (-1)^n h_n \end{pmatrix}; \quad G = \begin{pmatrix} (-1)^{n-k+1} G_{n-k+1} \\ (-1)^{n-k+2} G_{n-k+2} \\ \vdots \\ (-1)^n G_n \end{pmatrix}.$$

Define also the $k \times k$ matrix

$$A = \begin{pmatrix} c_{\leq k-1} & -c_{\leq k-2} & \cdots & \cdots & \cdots & \cdots & (-1)^{k-1} \\ (-1)^{k-2} c_{\geq k} & c_{\leq k-2} & \cdots & \cdots & \cdots & \cdots & (-1)^{k-2} \\ \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ (-1)^{k-i} c_{\geq k} & \cdots & (-1)^{k-i} c_{\geq k-i+2} & c_{\leq k-i} & -c_{\leq k-i-1} & \cdots & (-1)^{k-i} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ -c_{\geq k} & \cdots & \cdots & \cdots & \cdots & c_{\leq 1} & -1 \\ c_{\geq k} & \cdots & \cdots & \cdots & \cdots & c_{\geq 2} & 1 \end{pmatrix}.$$

Lemma 11.2. *The following equality holds: $G = AH$.*

Proof. This follows from (60) together with the Cauchy formula $\sum_{a+b=j} (-1)^a h_a e_b = 0$ for $j \geq 1$. □

11.2. Equivariant deformations

Motivated by Propositions 11.6 and 11.7 below, define the polynomials $h'_j, G'_j \in \mathbb{C}[z, \zeta]$ by

$$h'_j(z, \zeta) = \sum_{a+b=j} (-1)^a e_a(\zeta) h_b(z); \quad G'_j(z, \zeta) = \sum_{a+b=j} (-1)^a e_a(\zeta) G_b(z).$$

Note that both are symmetric polynomials in variables z and ζ. The polynomial $h'_j(z, \zeta)$ is a specialization of the factorial complete homogeneous polynomial (see, for example, [Mih08]), but the polynomial $G'_j(z, \zeta)$ is far from being the factorial Grothendieck polynomial [McN06], which is much more complicated.[4]

Lemma 11.3. *The following Cauchy-type formulae hold:*

(a) *If $\ell \geq 1$, then*

$$\sum_{a+b=\ell} (-1)^a h'_a(z, \zeta) e_b(z) = e_\ell(\zeta).$$

[4]The factorial Grothendieck polynomials, which represent the equivariant Schubert classes in $\mathrm{K}_T(\mathrm{Gr}(k; n))$, are not symmetric in ζ.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

(b) *If $\ell \geq 1$, then*

$$\sum_{a+b=\ell} (-1)^a G'_a(z,\zeta) e_b(z) = c'_{\geq \ell}(z,\zeta),$$

where $c'_{\geq \ell}(z,\zeta)$ is defined in (50).

Proof. For (a), we collect the coefficients of $e_i(\zeta)$ from both sides. On the left-hand side this coefficient equals to

$$\sum_{i \leq a} (-1)^a e_{\ell-a}(z)(-1)^i h_{a-i}(z) = \sum_{i \leq a} (-1)^{a-i} e_{\ell-a}(z) h_{a-i}(z).$$

By Cauchy formula, this equals to 0 if $i < \ell$, and it equals to 1 if $i = \ell$, thus proving part (a). We apply the same strategy for (b). The coefficient of $e_\ell(\zeta)$ on the left-hand side equals to 1, and a calculation similar to (a) shows that if $i < \ell$, the coefficient of $e_i(\zeta)$ is

$$\sum_{i \leq a} (-1)^a e_{\ell-a}(z)(-1)^i G_{a-i}(z) = e_{\ell-i+1}(z) - e_{\ell-i+2}(z) + \ldots = c^z_{\geq \ell-i+1}.$$

Here, the first equality follows from the Cauchy-type formula (62). This finishes the proof of (b). □

Our next task is to write down the analogue of Lemma 11.2 relating the polynomials G'_ℓ and h'_ℓ. To this aim, in analogy to (60), define the polynomial

$$G^\dagger_j(z,\zeta) = \sum (-1)^b h'_{j+a}(z,\zeta) e_b(z) = h'_j + (h'_{j+1} - h'_j e_1) + (h'_{j+2} - h'_{j+1} e_1 + h'_j e_2) + \ldots, \tag{63}$$

where the sum is over $a, b \geq 0$ and $a + b \leq k$.

Lemma 11.4. *For any $\ell \geq 1$, the following holds:*

$$G^\dagger_\ell(z,\zeta) - G'_\ell(z,\zeta) = (-1)^{\ell+1}(e_{\ell+1}(\zeta) - e_{\ell+2}(\zeta) + \ldots + (-1)^{k-1} e_{\ell+k}(\zeta)).$$

Proof. This follows from a direct calculation, by identifying the coefficients of $e_i(\zeta)$ in both sides. We leave the details to the reader. □

Define the column matrices:

$$H' = \begin{pmatrix} (-1)^{n-k+1} h'_{n-k+1} \\ (-1)^{n-k+2} h_{n-k+2} \\ \vdots \\ (-1)^n h'_n \end{pmatrix}; \quad G' = \begin{pmatrix} (-1)^{n-k+1} G'_{n-k+1} \\ (-1)^{n-k+2} G'_{n-k+2} \\ \vdots \\ (-1)^n G'_n \end{pmatrix}.$$

Corollary 11.5. *The following holds: $G' = AH' + C^\zeta_{\geq n-k+2}$.*

Proof. Let $G^\dagger$ be the column matrix with entries $(-1)^{n-k+i} G^\dagger_{n-k+i}(z,\zeta)$. As in the proof of Lemma 11.2, $G^\dagger = AH'$. Then the claim follows from the formula in Lemma 11.4. □

11.3. Elimination of variables

Our strategy is to utilize the Vieta relations in order to eliminate the variables $\hat{z}_i, \tilde{z}_i$ from the quantum rings $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ and $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ respectively, and then prove that the resulting rings are isomorphic. It turns out that the elimination process naturally leads to the study of variations of the Grothendieck, respectively the complete homogeneous polynomials.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Proposition 11.6. *Consider the Coulomb branch ring* $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *from (54). Then the following holds for* $1 \le \ell \le n-k$*:*

$$e_\ell(\hat{z}) \equiv \sum_{i+j=\ell} (-1)^j e_i(\zeta) G_j(z) = (-1)^\ell G'_\ell(z,\zeta) \mod \hat{J}.$$

Proof. We proceed by induction on $\ell \ge 1$. If $\ell = 1$, the claim follows from the definition of $g_1(z,\zeta,q)$ and the fact that $G_1(z) = \sum_{i=1}^k (-1)^i e_i(z)$ from Equation (61). By induction and the relations (53), we obtain that for $\ell \ge 2$,

$$\begin{aligned} e_\ell(\hat{z}) &\equiv g_\ell(z,\zeta,q) - \sum_{s=0}^{\ell-1} e_{\ell-s}(z) \left(\sum_{i+j=s} (-1)^j e_i(\zeta) G_j(z) \right) \\ &= g_\ell(z,\zeta,q) - \sum_{s=0}^{\ell-1} e_{\ell-s}(z)(-1)^s G'_s(z,\zeta) \\ &= g_\ell(z,\zeta,q) + (-1)^\ell G'_\ell(z,\zeta) - c'_{\ge \ell}(z,\zeta) \\ &= (-1)^\ell G'_\ell(z,\zeta). \end{aligned} \tag{64}$$

Here, the third equality follows from Lemma 11.3(b), and the fourth follows from the definition of $\hat{g}_\ell$ from Lemma 10.1. □

Proposition 11.7. *Consider the quantum* K *Whitney ring* $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ *from (56). Then the following holds for* $1 \le \ell \le n-k$*:*

$$e_\ell(\tilde{z}) \equiv \sum_{i+j=\ell} (-1)^j e_i(\zeta) h_j(z) = (-1)^\ell h'_\ell(z,\zeta) \mod \tilde{J}.$$

Proof. The proof is similar to that for Proposition 11.6, utilizing the Cauchy formula in Lemma 11.3(a), and that in this case, $\tilde{g}_\ell(z,\zeta) = e_\ell(\zeta)$ by Equation (59). □

Recall the notation from (50) and the surrounding paragraphs. Define the column matrices

$$\tilde{R} = \begin{pmatrix} \tilde{g}_{n-k+1} \\ \tilde{g}_{n-k+2} \\ \vdots \\ \tilde{g}_n \end{pmatrix}; \quad \hat{R} = \begin{pmatrix} \hat{g}_{n-k+1} \\ \hat{g}_{n-k+2} \\ \vdots \\ \hat{g}_n \end{pmatrix}; \quad E^\zeta_{n-k+1} = \begin{pmatrix} e_{n-k+1}(\zeta) \\ \vdots \\ \vdots \\ e_n(\zeta) \end{pmatrix}. \tag{65}$$

Theorem 11.8. (a) *The ring* $\widetilde{\mathrm{QK}}(\mathrm{Gr}(k;n))$ *is isomorphic to* $\mathrm{K}_T(pt)[\![q]\!][e_1,\dots,e_k]/\widetilde{I}$*, where the ideal* $\widetilde{I}$ *is defined by* $\tilde{R} = EH' + E^\zeta_{n-k+1}$*; that is,*

$$\begin{pmatrix} \tilde{g}_{n-k+1} \\ \tilde{g}_{n-k+2} \\ \vdots \\ \tilde{g}_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -e_1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k-1} & -e_{k-2} & \dots & -1 \end{pmatrix} \begin{pmatrix} (-1)^{n-k+1} h'_{n-k+1} \\ (-1)^{n-k+2} h'_{n-k+2} \\ \vdots \\ (-1)^n h'_n \end{pmatrix} + \begin{pmatrix} e_{n-k+1}(\zeta) \\ \vdots \\ \vdots \\ e_n(\zeta) \end{pmatrix}.$$

(b) *The ring* $\widehat{\mathrm{QK}}(\mathrm{Gr}(k;n))$ *is isomorphic to* $\mathrm{K}_T(pt)[\![q]\!][e_1,\dots,e_k]/\widehat{I}$*, where the ideal* $\widehat{I}_q$ *is defined by* $\hat{R} = EG' + C^{z,\zeta}_{\ge n-k+1}$*; that is,*

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

$$\begin{pmatrix} \hat{g}_{n-k+1} \\ \hat{g}_{n-k+2} \\ \vdots \\ \hat{g}_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -e_1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k-1} & -e_{k-2} & \dots & -1 \end{pmatrix} \begin{pmatrix} (-1)^{n-k+1} G'_{n-k+1} \\ (-1)^{n-k+2} G'_{n-k+2} \\ \vdots \\ (-1)^n G'_n \end{pmatrix} + \begin{pmatrix} c'_{\geq n-k+1} \\ \vdots \\ \vdots \\ c'_{\geq n} \end{pmatrix}.$$

Proof. The isomorphisms are obtained by eliminating the variables $\tilde{z}_i$, respectively $\hat{z}_i$ $(1 \leq i \leq n-k)$ from the first $n-k$ relations in $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ respectively $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$ (cf. (54) respectively (56)). We indicate the main steps to obtain these formulae.

Consider first $\widetilde{\mathrm{QK}}_T(\mathrm{Gr}(k;n))$. By Proposition 11.7, $e_j(\tilde{z}) \equiv (-1)^j h'_j(z,\zeta)$ modulo $\widetilde{I}$; we utilize this to eliminate the first $n-k$ relations from $\tilde{J}$ to show that $\widetilde{I}$ is generated by

$$\sum_{a+b=\ell; b\leq n-k} (-1)^b e_a(z) h'_b(z,\zeta) - \tilde{g}_\ell(z,\zeta), \quad n-k+1 \leq \ell \leq n.$$

From the Cauchy formula in Lemma 11.3(a), it follows that for $n-k+1 \leq \ell \leq n$,

$$\sum_{a+b=\ell; b\leq n-k} (-1)^b e_a h'_b = e_\ell(\zeta) - \sum_{b=n-k+1}^{\ell} (-1)^b e_{\ell-b} h'_b \equiv \tilde{g}_\ell \mod \widetilde{I}.$$

Writing these expressions in matrix form proves the claim for $\widetilde{I}$.

The same proof works for $\widehat{I}$, after utilizing that $e_j(\hat{z}) \equiv (-1)^j G'_j(z,\zeta)$ modulo $\widehat{I}$ by Proposition 11.6, and the Cauchy-type formula from Lemma 11.3(b); the matrix $C^{z,\zeta}_{\geq n-k+1}$ accounts for the right-hand side of the Cauchy formula. □

11.4. *Proof of Theorem 10.2*

We need the following algebraic identities.

Lemma 11.9. (a) *Let A' be the antidiagonal transpose of A from (11.1). Then*

$$A' \begin{pmatrix} \binom{k}{1} - e_1(z) \\ \binom{k}{2} - e_2(z) \\ \vdots \\ \binom{k}{k} - e_k(z) \end{pmatrix} = \prod_{i=1}^{k} (1 - z_i) \begin{pmatrix} \binom{k-1}{k-1} \\ \binom{k-1}{k-2} \\ \vdots \\ \binom{k-1}{0} \end{pmatrix}.$$

(b) $\hat{R} - C^{z,\zeta}_{\geq n-k+1} - E \cdot C^{\zeta}_{\geq n-k+2} = (-1)^{n+k} q \prod_{i=1}^{n} (1-\zeta_i) \begin{pmatrix} \binom{k-1}{k-1} \\ \binom{k-1}{k-2} \\ \vdots \\ \binom{k-1}{0} \end{pmatrix}.$

Proof. Part (a) follows, for example, from induction on i; we leave the details to the reader. Part (b) is equivalent to Lemma 10.1. □

Next, we state the key result which relates the two presentations.

Lemma 11.10. *The following equality holds:*

$$\hat{R} - C^{z,\zeta}_{\geq n-k+1} - E \cdot C^{\zeta}_{\geq n-k+2} \equiv A'(\tilde{R} - E^{\zeta}_{n-k+1}) \mod \widetilde{I},$$

where A' is the antidiagonal transpose of A.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Proof. From part (b) of Lemma 11.9, we need to show that

$$A'(\tilde{R} - E^{\zeta}_{n-k+1}) \equiv (-1)^{n+k} q \prod_{i=1}^{n} (1-\zeta_i) \begin{pmatrix} \binom{k-1}{k-1} \\ \binom{k-1}{k-2} \\ \vdots \\ \binom{k-1}{0} \end{pmatrix}.$$

From the definition of $\tilde{g}_\ell$ from (59), it follows that for $\ell \geq n-k+1$,

$$\tilde{g}_\ell - e_\ell(\zeta) \equiv (-1)^{n-k} \frac{q}{1-q} e_{n-k}(1-\tilde{z}) \left(\binom{k}{\ell+k-n} - e_{\ell+k-n}(z) \right).$$

By Lemma 11.9(a), for $n-k+1 \leq \ell \leq n$,

$$\begin{aligned} A'(\tilde{R} - E^{\zeta}_{n-k+1}) &\equiv (-1)^{n-k} \frac{q}{1-q} e_{n-k}(1-\tilde{z}) A' \cdot \left(\binom{k}{\ell+k-n} - e_{\ell+k-n}(z) \right)_{\ell+k-n} \\ &= (-1)^{n-k} \frac{q}{1-q} e_{n-k}(1-\tilde{z}) e_k(1-z) \begin{pmatrix} \binom{k-1}{k-1} \\ \binom{k-1}{k-2} \\ \vdots \\ \binom{k-1}{0} \end{pmatrix}. \end{aligned}$$

If one writes $e_{n-k}(1-\tilde{z})e_k(1-z)$ in the presentation from Theorem 10.2(c), it follows that in $\widetilde{QK}_T(\mathrm{Gr}(k;n))$,

$$e_{n-k}(1-\tilde{z})e_k(1-z) = e_{n-k}(\tilde{X})e_k(X) \equiv (1-q) \prod_{i=1}^{n} (1-\zeta_i) \mod \widetilde{I}.$$

Then the claim follows by combining the previous equalities. □

Corollary 11.11. *The ideal* $\widehat{I} \subset \widetilde{I}$.

Proof. We need to show that $\hat{R} - EG' - C^{z,\zeta}_{\geq n-k+1} \equiv 0$ modulo $\widetilde{I}$. From definitions, Lemma 11.10 and Corollary 11.5, we have

$$\begin{aligned} \hat{R} - EG' - C^{z,\zeta}_{\geq n-k+1} &= \hat{R} - E(AH' + C^{\zeta}_{\geq n-k+2}) - C^{z,\zeta}_{\geq n-k+1} \\ &= \hat{R} - EC^{\zeta}_{\geq n-k+2} - C^{z,\zeta}_{\geq n-k+1} - EAH' \\ &\equiv A'(\tilde{R} - E^{\zeta}_{n-k+1}) - EAH' \\ &\equiv AEH' - EAH'. \end{aligned}$$

Then the claim follows because $EA = A'E$ as matrices with polynomial coefficients. □

The next theorem finishes the proof of Theorem 10.2.

Theorem 11.12. *There is* $\mathrm{K}_T(pt)[[q]]$*-algebra isomorphism*

$$\mathrm{K}_T(pt)[[q]][e_1, \ldots, e_k]/\widehat{I} \to \mathrm{K}_T(pt)[[q]][e_1, \ldots, e_k]/\widetilde{I}$$

sending $e_i(z) \mapsto e_i(z)$.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Proof. By Corollary 11.11, there is a surjective ring homomorphism of $\mathrm{K}_T(pt)[[q]]$-algebras

$$\Phi : \mathrm{K}_T(pt)[[q]][e_1(z),\dots,e_k(z)]/\widehat{I} \to \mathrm{K}_T(pt)[[q]][e_1(z),\dots,e_k(z)]/\widetilde{I},$$

sending $e_i(z) \mapsto e_i(z)$. To prove Φ is an isomorphism, we follow the same approach as in the proof of Theorem 8.2, relying on Proposition A.3. We need to check that the Coulomb branch presentation $\mathrm{K}_T(pt)[[q]][e_1,\dots,e_k]/\widehat{I}$ is a finite module over $\mathrm{K}_T(pt)[[q]]$, and that we obtain an isomorphism after taking the quotient by $\langle q\rangle$. To start, note that from the definition of the ideal $\hat{J}$ from (55), the variables $z_1,\dots,z_k$ are solutions of the Bethe Ansatz equations (48). Then the same happens for the variables $z_1,\dots,z_k$ appearing in $\mathrm{K}_T(pt)[[q]][e_1,\dots,e_k]/\widehat{I}$. From this, and from the special case $q=0$ of [GK17, Lemma 4.7], we obtain that the quotient $\mathrm{K}_T(pt)[[q]][e_1,\dots,e_k]/(\langle q\rangle+\widehat{I})$ is $\mathrm{K}_T(pt)$-module-generated by the factorial Grothendieck polynomials $G_\lambda(z;\xi)$ from [McN06], where λ is included in the $k\times(n-k)$ rectangle. (This result uses a determinantal formula for factorial Grothendieck polynomials from [GK17, eq. 2.14], in turn attributed to [IN13].) Since Φ is surjective, Proposition 8.1 and Lemma A.2 imply that the images $\Phi(G_\lambda(z;\xi))$ modulo q form a basis in $\mathrm{K}_T(\mathrm{Gr}(k;n)) \simeq \mathrm{K}_T(pt)[[q]][e_1,\dots,e_k]/(\langle q\rangle+\widetilde{I})$. This implies that Φ must be an isomorphism modulo $\langle q\rangle$. Then the same argument as in Theorem 8.2, based on [Eis95, Ex. 7.4, p. 203] (cf. Remark A.4), implies that $\mathrm{K}_T(pt)[[q]][e_1,\dots,e_k]/\widehat{I}$ is a finite module; thus, Φ is an isomorphism, by Proposition A.3. □

12. Examples: Non-equivariant and quantum cohomology specializations

In this section, we illustrate the non-equivariant specializations of the Coulomb QK Whitney presentations of $\mathrm{QK}_T(\mathrm{Gr}(k;n))$ (i.e., when $\zeta_i = 0$ for $1 \le i \le n$). The second part is dedicated to the quantum cohomology limit. In particular, we show that both presentations specialize to Witten's presentation of $\mathrm{QH}^*_T(\mathrm{Gr}(k;n))$.

12.1. Non-equivariant presentations

We abuse notation and denote by the same symbols the non-equivariant specializations of relations, ideals, etc. For $\ell \ge 1$, the polynomials $\tilde{g}_\ell, \hat{g}_\ell$ are given by

$$\tilde{g}_\ell(z,q) = (-1)^{n-k}\frac{q}{1-q}e_{n-k}(1-\tilde{z})\left(\binom{k}{\ell+k-n} - e_{\ell+k-n}(z)\right);$$

$$\hat{g}_\ell(z,q) = c^z_{\ge \ell+1} + (-1)^{n+k}q\binom{k-1}{n-\ell}.$$

In particular, if $1 \le \ell \le n-k$, $\tilde{g}_\ell = 0$ and $\hat{g}_\ell$ does not depend on q. The non-equivariant versions of propositions 11.7 and 11.6 state that for any $1 \le j \le n-k$, the following identities hold:

$$e_j(\tilde{z}) = (-1)^j h_j(z) \mod \tilde{J}; \quad e_j(\hat{z}) = (-1)^j G_j(z) \mod \hat{J}. \tag{66}$$

Remark 12.1. It is well known that the Grothendieck polynomials $G_j(z)$ represent Schubert classes $\mathcal{O}_j \in K(\mathrm{Gr}(k;n))$ [Las90]. Nevertheless, the geometric interpretation $e_j(\hat{z}) = (-1)^j\mathcal{O}_j$ fails in the equivariant case, already for $j=1$. The Schubert divisor class may be calculated from the exact sequence:

$$0 \to \wedge^k\mathcal{S}\otimes\mathbb{C}_{-t_1-\dots-t_k} \to \mathcal{O}_{\mathrm{Gr}(k;n)} \to \mathcal{O}_1 \to 0.$$

Then, from the geometric interpretation of the variables z_i, ζ_i,

$$\mathcal{O}_1 = 1 - \frac{(1-z_1)\cdot\ldots\cdot(1-z_k)}{(1-\zeta_1)\cdot\ldots\cdot(1-\zeta_k)} \quad \in \mathrm{K}_T(\mathrm{Gr}(k;n)). \tag{67}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

However, from Proposition 11.6,

$$e_1(\hat{z}) = -G_1'(z,\zeta) = -G_1(z) + e_1(\zeta),$$

which is different from the expression in Equation (67).

Corollary 12.2. *The rings* $\widetilde{\mathrm{QK}}(\mathrm{Gr}(k;n))$ *and* $\widehat{\mathrm{QK}}(\mathrm{Gr}(k;n))$ *are isomorphic to*

$$\widetilde{\mathrm{QK}}(\mathrm{Gr}(k;n)) = \mathbb{Z}[[q]][e_1,\dots,e_k]/\widetilde{I}; \quad \widehat{\mathrm{QK}}(\mathrm{Gr}(k;n)) = \mathbb{Z}[[q]][e_1,\dots,e_k]/\widehat{I},$$

where the ideal $\widetilde{I}$ *is defined by* $\tilde{R} = EH$*; that is,*

$$\begin{pmatrix} \tilde{g}_{n-k+1} \\ \tilde{g}_{n-k+2} \\ \vdots \\ \tilde{g}_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -e_1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k-1} & -e_{k-2} & \dots & -1 \end{pmatrix} \begin{pmatrix} (-1)^{n-k+1} h_{n-k+1} \\ (-1)^{n-k+2} h_{n-k+2} \\ \vdots \\ (-1)^n h_n \end{pmatrix},$$

and the ideal $\widehat{I}$ *is defined by* $\hat{R} = EG + C^z_{\geq n-k+2}$*; that is,*

$$\begin{pmatrix} \hat{g}_{n-k+1} \\ \hat{g}_{n-k+2} \\ \vdots \\ \hat{g}_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -e_1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -e_{k-1} & -e_{k-2} & \dots & -1 \end{pmatrix} \begin{pmatrix} (-1)^{n-k+1} G_{n-k+1} \\ (-1)^{n-k+2} G_{n-k+2} \\ \vdots \\ (-1)^n G_n \end{pmatrix} + \begin{pmatrix} c^z_{\geq n-k+2} \\ \vdots \\ c^z_{\geq n} \\ 0 \end{pmatrix}.$$

12.1.1. Projective spaces

To illustrate both presentations, we take two 'opposite' examples: for the projective space $\mathrm{Gr}(1;n)$ and its dual $\mathrm{Gr}(n-1;n)$. These are isomorphic manifolds, and their quantum K rings are also isomorphic. But the isomorphism is highly nontrivial; this is expected, given that the Grassmannians are realized as different GIT quotients. A worked-out example for $\mathrm{QK}_T(\mathrm{Gr}(2;5)$ is included in Appendix B.

If $k = 1$, then $G_n(z) = h_n(z) = z_1^n$. The presentations from Corollary 12.2 are

$$\widehat{\mathrm{QK}}(\mathrm{Gr}(1;n)) = \mathbb{Z}[[q]][z]/\langle z^n - q\rangle = \widetilde{\mathrm{QK}}(\mathrm{Gr}(1;n)).$$

Consider now $k = n-1$. From (61), it follows that the Grothendieck polynomials G_j are significantly more complicated than the polynomials h_j. To illustrate, consider $\mathrm{QK}(\mathrm{Gr}(3;4))$. The QK Whitney presentation is

$$\widetilde{\mathrm{QK}}(\mathrm{Gr}(3;4)) = \frac{\mathbb{Z}[[q]][e_1(z_1,z_2,z_3), e_2(z_1,z_2,z_3), e_3(z_1,z_2,z_3)]}{\langle \tilde{g}_2 + h_2, \tilde{g}_3 + e_1 h_2 - h_3, \tilde{g}_4 + e_2 h_2 - e_1 h_3 + h_4\rangle},$$

where

$$\tilde{g}_2 = \frac{q}{q-1}(1+e_1)(3-e_1); \quad \tilde{g}_3 = \frac{q}{q-1}(1+e_1)(3-e_2); \quad \tilde{g}_4 = \frac{q}{q-1}(1+e_1)(1-e_3).$$

The Coulomb presentation is

$$\widehat{\mathrm{QK}}(\mathrm{Gr}(3;4)) = \frac{\mathbb{Z}[[q]][e_1(z_1,z_2,z_3), e_2(z_1,z_2,z_3), e_3(z_1,z_2,z_3)]}{\langle \hat{g}_2 + G_2, \hat{g}_3 + e_1 G_2 - G_3 - e_3, \hat{g}_4 + e_2 G_2 - e_1 G_3 + G_4\rangle},$$

where $\hat{g}_2 = e_3 - q, \hat{g}_3 = -2q, \hat{g}_4 = -q$ and

$$G_2 = h_2 - s_{2,1} + s_{2,1,1}; \quad G_3 = h_3 - s_{3,1} + s_{3,1,1}; \quad G_4 = h_4 - s_{4,1} + s_{4,1,1}.$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

12.1.2. Relation to the Gorbounov and Korff presentation

In this section, we recall the description of the non-equivariant quantum K-theory ring $\mathrm{QK}(\mathrm{Gr}(k;n))$ from Gorbounov and Korff's paper [GK17], based on the Bethe Ansatz equations. Then we indicate an isomorphism between this and our non-equivariant Coulomb presentation $\widehat{\mathrm{QK}}(\mathrm{Gr}(k;n))$.

For a variable ξ, denote by $\ominus\xi = \frac{-\xi}{1-\xi}$. Consider a sequence of indeterminates $E_1,\dots,E_k,H_1,\dots,H_{n-k}$, and extend this sequence by requiring $E_0 = H_0 = 1$ and $H_r = 0$ for $r > n-k$ and $E_r = 0$ for $r > k$. Consider the generating series

$$H(\xi) = \sum_{r=0}^{n-k}(H_r - H_{r+1})\xi^{n-k-r}; \quad E(\xi) = \sum_{r=0}^{k}(E_r - E_{r+1})\xi^{k-r}.$$

The following is stated in [GK17, Theorem 1.1].

Theorem 12.3 (Gorbounov-Korff). *The non-equivariant quantum K theory ring of* $\mathrm{Gr}(k;n)$ *is generated by* $H_1,\dots,H_{n-k}$, $E_1,\dots,E_k$ *with relations given by the coefficients of* ξ *in the expansion of*

$$H(\xi)E(\ominus\xi) = \left(\prod_{i=1}^{k}\ominus\xi\right)\xi^{n-k}(1-H_1) + q. \tag{68}$$

Denote by $\mathrm{QK}^{GK}(\mathrm{Gr}(k;n))$ the Gorbounov and Korff's presentation. A direct computation gives that the coefficient of ξ^ℓ in (68) is equal to

$$(-1)^k(H_\ell - H_{\ell+1}) + \sum_{j=1}^{\ell}(H_{\ell-j} - H_{\ell+1-j})\left[(-1)^{k-j}\sum_{s=j}^{k}\binom{s-1}{s-j}E_s\right]$$
$$= \begin{cases} 0 & 1 \le \ell \le n-k-1, \\ (-1)^{n-\ell}q\binom{k}{n-\ell} & n-k \le \ell \le n. \end{cases} \tag{69}$$

We provide next an algebra isomorphism between the (non-equivariant) Coulomb branch presentation $\widehat{\mathrm{QK}}(\mathrm{Gr}(k;n))$ and the presentation $\mathrm{QK}^{GK}(\mathrm{Gr}(k;n))$.

Proposition 12.4. *Consider the map* $\Xi : \mathrm{QK}^{GK}(\mathrm{Gr}(k;n)) \to \widehat{\mathrm{QK}}(\mathrm{Gr}(k;n))$ *defined by*

$$\Xi(H_j) = G_j(z) \text{ for } 1 \le j \le n-k; \quad \Xi(E_j) = G_{1^j}(z) \text{ for } 1 \le j \le k.$$

Then Ξ *is an algebra isomorphism.*

Proof. The main argument is showing that Ξ is well defined. To calculate the image of the left-hand side of (69), one utilizes the polynomial identity $e_\ell(z) = \sum_{j=\ell}^{k}\binom{j-1}{j-\ell}G_{1^j}(z)$ (see, for example, [Len00, Thm. 2.2]), together with the (non-equivariant) Cauchy identity from Lemma 11.3(b). We leave the details to the reader. □

12.2. Quantum cohomology as a limit

Next, we discuss how the quantum K theory reduces to the quantum cohomology. Mathematically, this is achieved by taking leading terms or, equivalently, taking the associated graded rings. In physics this corresponds to 'taking a two-dimensional limit', which we recall next; see [GMSZ22]. If we assume the theory is defined on a 3-manifold $S^1 \times \Sigma$ for some Riemann surface Σ, where S^1 has diameter L, then

$$q = L^n q_{2d}, \quad X_a = \exp(L\sigma_a) = 1 + L\sigma_a + \frac{L^2}{2}{\sigma_a}^2 + \cdots,$$
$$T_i = \exp(Lt_i) = 1 + Lt_i + \frac{L^2}{2}{t_i}^2 + \cdots \tag{70}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Here, q_{2d} is the quantum parameter in the $2d$ theory and t_i are elements in $H_T^*(pt)$ (in physics terminology, the twisted masses). The term $-Lt_i$ corresponds to the cohomological parameter $t_i \in H_T^*(pt)$. We have

$$z_a = 1 - X_a = -L\sigma_a - \frac{L^2}{2}{\sigma_a}^2 - \ldots; \quad \zeta_i = 1 - T_i = -Lt_i - \frac{L^2}{2}{t_i}^2 - \ldots.$$

In the two-dimensional limit, $L \to 0$ and the leading terms will dominate. We now take the $2d$ limit in Equation (43) arising from the Coulomb branch. After taking the leading terms, (43) becomes

$$\prod_{i=1}^{n}(\sigma_a - t_i) = (-1)^{k-1}q_{2d}, \quad a = 1, \ldots, k.$$

These are the generators which determine the chiral rings of the $2d$ gauge linear sigma model (GLSM) for $\mathrm{Gr}(k;n)$ when twisted masses t_i are turned on. The $2d$ limits of the polynomials $\hat{g}_\ell(z,\zeta,q)$ are $\hat{g}_\ell(z,\zeta,q) \to L^\ell g_\ell^{2d}(\sigma, m, q_{2d})$, with $\hat{g}_\ell^{2d}(z,t,q_{2d})$ given by

$$\hat{g}_\ell^{2d}(\sigma,t,q_{2d}) = e_\ell(t) + (-1)^{n+k}q_{2d}\delta_{\ell,n}, \quad \ell = 1, \ldots, n,$$

where $\delta_{\ell,n}$ is the Kronecker delta. Then the $2d$ limit of the Coulomb branch presentation in (54) is

$$\sum_{i+j=\ell} e_i(\sigma)e_j(\hat{\sigma}) = \hat{g}_\ell^{2d}(\sigma,t,q_{2d}), \quad \ell = 1, \ldots, n. \tag{71}$$

We may identify $\{-\sigma_a\}$ with Chern roots of $\mathcal{S}$ and $\{-\hat{\sigma}_a\}$ with Chern roots of $\mathcal{Q}$. This recovers the equivariant quantum Whitney relations from (1):

$$c^T(\mathcal{S}) \star c^T(\mathcal{Q}) = c^T(\mathbb{C}^n) + (-1)^k q_{2d}.$$

As we observed earlier, the $2d$ limits of the Coulomb branch and the Whitney presentations from (54) and (56) coincide. Indeed, under the change of variables from (70) and after taking leading terms, we obtain

$$\tilde{g}_\ell^{2d}(\sigma,t,q_{2d}) = e_\ell(t) + (-1)^{n-k}q_{2d}\delta_{\ell,n} = g_\ell^{2d}(\sigma,t,q_{2d}), \quad \ell = 1, \ldots, n.$$

A. Completions of filtered modules

For the convenience of the reader, in this appendix we gather some results about I-adic completions of modules. We utilize them to deduce properties of the (equivariant) quantum K theory in terms of those of the ordinary (equivariant) K theory, but they may be of more general interest. We follow the terminology from [AM69, Ch. 10].

From now on, we will consider a commutative ring R with a descending filtration by additive groups $R = R_0 \supset R_1 \supset \ldots$ such that $R_i \cdot R_j \subset R_{i+j}$. Consider a filtered R-module M (i.e., M is equipped with a filtration by additive groups $M = M_0 \supset M_1 \supset \ldots$ such that $R_i M_j \subset M_{i+j}$).

Fix $I \subset R$ an ideal. This determines filtrations $R_n := I^n R$ and $M_n := I^n M$ of R respectively M. We denote by $\widehat{R}$, $\widehat{M}$ the I-adic completions.

If we assume in addition that R is Noetherian, then $\widehat{R}$ is Noetherian and it is equipped with an ideal $\widehat{I} \simeq I \otimes_R \widehat{R}$ included in the Jacobson radical of $\widehat{R}$ [AM69, Prop. 10.15]. For a finitely generated R-module M, there is an isomorphism

$$M \otimes_R \widehat{R} \simeq \widehat{M}; \tag{72}$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

see [AM69, Prop. 10.13]. Furthermore, $\widehat{R}$ is a flat R-algebra; in particular, the operation $M \mapsto \widehat{M}$ preserves short exact sequences. Finally, if we assume that M is equipped with another filtration compatible with I, that is, $I.M_n \subset M_{n+1}$, then (72) equips $\widehat{M}$ with a natural filtration induced from M which is compatible with respect to $\widehat{I}$.

We recall next a version of Nakayama's Lemma.

Lemma A.1. *Let R be a Noetherian ring, $I \subset R$ an ideal, and let M be a finitely generated $\widehat{R}$-module. Let $M' \subset M$ be a submodule such that*

$$M/M' = \widehat{I}.(M/M').$$

Then $M = M'$ as $\widehat{R}$-modules.

Proof. The claim follows from the ordinary Nakayama Lemma [AM69, Prop. 2.6], utilizing that $\widehat{I}$ is included in the Jacobson radical of $\widehat{R}$. □

Lemma A.2. *Let M be a free R-module of rank p where R is an integral domain. Then any set of R-module generators $m_1, \ldots, m_p \in M$ of cardinality p forms a basis.*

Proof. Let K be the fraction field of R. Then $M \otimes_R K$ is a K-vector space of dimension p. The hypothesis on the m_i's implies that there is an isomorphism of K-vector spaces $K^p \to M \otimes_R K$, and that $m_1 \otimes 1, \ldots, m_p \otimes 1$ is a basis for $M \otimes_R K$. If $c_1 m_1 + \ldots + c_p m_p = 0$ in M, then $c_1 m_1 \otimes 1 + \ldots + c_p m_p \otimes 1 = 0$, from where we get that $c_i = 0$, which finishes the proof. □

We note that a more general statement of Lemma A.2 holds, which does not use that R is an integral domain; see, for example, [Sta25, Tag 05G8].

Next is the main result of this Appendix.

Proposition A.3. *Let R be a Noetherian integral domain, and let $I \subset R$ be an ideal. Assume that R is complete in the I-adic topology. Let M, N be finitely generated R-modules.*

Assume that the R-module N, and the R/I-module N/IN, are both free modules of the same rank $p < \infty$, and that we are given an R-module homomorphism $f : M \to N$ such that the induced R/I-module map $\overline{f} : M/IM \to N/IN$ is an isomorphism of R/I-modules.

Then f is an isomorphism.

Proof. We start by observing that since R is complete in the I-adic topology, $\widehat{R} \simeq R$ and $\widehat{I} \simeq I$; cf. [AM69, Prop. 10.5 and Prop. 10.15]. Let $m_1, \ldots, m_p \in M$ be any lifts of a basis of the free R/I-module $M/IM \simeq N/IN$. By Lemma A.1 applied to the modules M and $M' = \langle m_1, \ldots, m_p \rangle$, the elements $m_1, \ldots, m_p$ generate M, and similarly the elements $f(m_1), \ldots, f(m_p)$, generate N. Since N is free of rank p, it follows from Lemma A.2 that $f(m_1), \ldots, f(m_p)$ is a basis over R. In particular, f must be surjective. To prove that f is an isomorphism, it suffices to show that $m_1, \ldots, m_p$ are linearly independent over R. If $\sum c_i m_i = 0$ with $c_i \in R$, taking the image under f and using that $\{f(m_i)\}$ form a basis implies that $c_i = 0$. This finishes the proof. □

Remark A.4. A useful criterion for finite generation of a module is provided in [Eis95, Ex. 7.4]. Assume that $R \subset S$ are Noetherian rings such that R is complete with respect to an ideal $\mathfrak{m} \subset S$, and that $\mathfrak{m}$ is contained in the Jacobson radical of S. If M is a finitely generated S-module and $M/\mathfrak{m}M$ is a finitely generated $R/\mathfrak{m}$-module, then M is a finitely generated R-module. See also [GMS+23, Appendix A] for more details.

Remark A.5. By [AM69, Prop. 10.24], the hypothesis that M, N are finitely generated may be deduced if we know that R is I-complete, and M, N are equipped with good and separated filtrations compatible with respect to I. (Recall that a filtration of M is good if the associated graded $\mathrm{gr}M = \bigoplus_i M_i/M_{i+1}$ is a finitely generated $\mathrm{gr}R$-module, and it is separated if $\bigcap_i M_i = 0$.)

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

Remark A.6. If $f : M \to N$ is a filtered homomorphism of R-modules (i.e., $f(M_i) \subset N_i$) which induces an isomorphism $\mathrm{gr}f : \mathrm{gr}M \to \mathrm{gr}N$ of graded $\mathrm{gr}R$-modules, then the induced map between the completions $\widehat{f} : \widehat{M} \to \widehat{N}$ is an isomorphism [AM69, Lemma 10.23]. This is an alternative to Proposition A.3.

Remark A.7. Instead of working with completions in Lemma A.1 and Proposition A.3, one may work with the localizations with respect to the multiplicative set $S = 1 + I$. The hypothesis that R is Noetherian is not needed in this case, and the ideal $S^{-1}I$ is included in the Jacobson radical of $S^{-1}R$; cf. [AM69, Ch. 3, Ex. 2]. If R is Noetherian, then $S^{-1}R \hookrightarrow \widehat{R}$ is a subring [AM69, Remark, p. 110].

B. Example: Equivariant quantum K theory of $\mathrm{Gr}(2;5)$

In this section, we illustrate the Whitney and Coulomb branch presentations for $\mathrm{QK}_T(\mathrm{Gr}(2;5))$. The QK Whitney relations are obtained by equating powers of y in

$$(1+yX_1)(1+yX_2)(1+y\tilde{X}_1)(1+y\tilde{X}_2)(1+y\tilde{X}_3) = \prod_{i=1}^{5}(1+T_iy) - \frac{q}{1-q}y^4\tilde{X}_1\tilde{X}_2\tilde{X}_3(X_1+X_2+yX_1X_2). \tag{73}$$

Under the changes of variable $X_i = 1-z_i, (1 \le i \le 2), \tilde{X}_j = 1-\tilde{z}_j, (1 \le j \le 3), T_s \equiv 1-\zeta_s\ (1 \le s \le 5)$, the QK Whitney relations become

$$\sum_{i+j=\ell} e_i(z)e_j(\tilde{z}) \equiv e_\ell(\zeta) - \frac{q}{1-q}\big(\delta_{\ell,4}\Delta_1 + \delta_{\ell,5}(\Delta_1-\Delta_2)\big),$$

for $1 \le \ell \le 5$, with

$$\Delta_1 = e_1(1-z_1, 1-z_2)(1-\tilde{z}_1)(1-\tilde{z}_2)(1-\tilde{z}_3);$$
$$\Delta_2 = e_2(1-z_1, 1-z_2)(1-\tilde{z}_1)(1-\tilde{z}_2)(1-\tilde{z}_3).$$

One may solve for $e_i(\tilde{z})$ in terms of $e_i(z)$ to obtain

$$e_1(\tilde{z}) \equiv e_1(\zeta) - h_1(z) = -h_1'(z,\zeta);$$
$$e_2(\tilde{z}) \equiv e_2(\zeta) - h_1(z)e_1(\zeta) + h_2(z) = -h_2'(z,\zeta);$$
$$e_3(\tilde{z}) \equiv e_3(\zeta) - h_1(z)e_2(\zeta) + h_2(z)e_1(\zeta) - h_3(z) = -h_3'(z,\zeta).$$

This allows the elimination of the variables $\tilde{z}$. The remaining two relations and the Cauchy formula from Lemma 11.3 give the generators for the ideal $\widetilde{I}$:

$$e_4(\zeta) - h_4'(z,\zeta) = -e_1(z)h_3'(z,\zeta) + e_2(z)h_2'(z,\zeta) \equiv e_4(\zeta) - \frac{q}{1-q}(1+h_1(z)+h_2(z))(2-e_1(z)).$$

$$e_5(\zeta) + h_5'(z,\zeta) - e_1(z)h_4'(z,\zeta) = -e_2(z)h_3'(z,\zeta) \equiv e_5(\zeta) - \frac{q}{1-q}(1+h_1(z)+h_2(z))(1-e_2(z)).$$

We now turn to the Coulomb branch relations for $\mathrm{QK}_T(\mathrm{Gr}(2;5))$. The Vieta relations give

$$\sum_{i+j=\ell} e_i(z)e_j(\hat{z}) \equiv \hat{g}_\ell(z,\zeta,q),$$

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

for $1 \le \ell \le 5$, where the polynomials $\hat{g}_\ell(z,\zeta,q)$ are given by

$$\begin{aligned}
\hat{g}_\ell(z,\zeta) &= e_\ell(\zeta) + e_{\ell-1}(\zeta)e_2(z), \quad \ell = 1,2,3,\\
\hat{g}_4(z,\zeta) &= e_4(\zeta) + e_3(\zeta)e_2(z) - e_5(\zeta) - q\sum_{s=0}^{5}(-1)^s e_s(\zeta),\\
\hat{g}_5(z,\zeta) &= e_5(\zeta) + e_2(z)e_4(\zeta) - e_1(z)e_5(\zeta) - q\sum_{s=0}^{5}(-1)^s e_s(\zeta).
\end{aligned}$$

One may solve for $e_i(\tilde{z})$ in terms of $e_i(z)$ to obtain

$$\begin{aligned}
&e_1(\hat{z}) = e_1(\zeta) - G_1(z) = -G_1'(z,\zeta);\\
&e_2(\hat{z}) = e_2(\zeta) - G_1(z)e_1(\zeta) + G_2(z) = G_2'(z,\zeta);\\
&e_3(\hat{z}) = e_3(\zeta) - G_1(z)e_2(\zeta) + G_2(z)e_1(\zeta) - G_3(z) = -G_3'(z,\zeta).
\end{aligned}$$

Here, $G_i(z)$ are the Grothendieck polynomials, given by

$$\begin{aligned}
&G_1(z) = h_1 - e_2 = z_1 + z_2 - z_1z_2;\\
&G_2(z) = h_2 - s_{2,1} = z_1^2 + z_1z_2 + z_2^2 - z_1^2z_2 - z_1z_2^2;\\
&G_3(z) = h_3 - s_{3,1} = z_1^3 + z_1^2z_2 + z_1z_2^2 + z_2^3 - z_1^3z_2 - z_1^2z_2^2 - z_1z_2^3.
\end{aligned}$$

After eliminating the variables $\hat{z}$, the remaining two relations and the Cauchy formula from Lemma 11.3(b) give the generators for the ideal $\widehat{I}$ of $\widehat{\mathrm{QK}}_T(\mathrm{Gr}(2;5))$:

$$\begin{aligned}
\hat{g}_4(z,\zeta,q) &\equiv -G_4'(z,\zeta) + c'_{\ge 4} = -G_4'(z,\zeta) + e_4(\zeta) + e_3(\zeta)e_2(z)\\
\hat{g}_5(z,\zeta,q) &\equiv -e_1(z)G_4'(z,\zeta) + G_5'(z,\zeta) + c'_{\ge 5}\\
&= -e_1(z)G_4'(z,\zeta) + G_5'(z,\zeta) + e_5(\zeta) + e_4(\zeta)e_2(z),
\end{aligned}$$

with the equivariant Grothendieck polynomials defined by

$$\begin{aligned}
&G_4'(z,\zeta) = G_4(z) - G_3(z)e_1(\zeta) + G_2(z)e_2(\zeta) - G_1(z)e_3(\zeta) + e_4(\zeta);\\
&G_5'(z,\zeta) = G_5(z) - G_4(z)e_1(\zeta) + G_3(z)e_2(\zeta) - G_2(z)e_3(\zeta) + G_1(z)e_4(\zeta) - e_5(\zeta),
\end{aligned}$$

and $G_4(z), G_5(z)$ are given by (61).

Acknowledgements. We would like to thank Dave Anderson, Vassily Gorbounov, Takeshi Ikeda, Christian Korff, Yaoxiong Wen and Ming Zhang for useful discussions. Special thanks are due to Prof. Satoshi Naito, whose remarks helped us fix some hypotheses about completeness in Appendix A, in an initial preprint version of this paper. Finally, all the authors are grateful to two anonymous referees for a careful reading, and useful suggestions, including catching an inaccuracy in the proof of Lemma A.2.

Throughout this project, we utilized the Maple based program `Equivariant Schubert Calculator`, written by Anders Buch.[5]

Competing interest. The authors have no competing interests to declare.

Financial support. W.G. was partially supported by NSF grant PHY-1720321. This material is based in part on work supported by the National Science Foundation under Grant No. DMS-1929284 while LM was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Combinatorial Algebraic Geometry program. LM was also partially supported by the NSF grant DMS-2152294 and a Simons Collaboration Grant. E.S. was partially supported by NSF grant PHY-2014086.

[5]The program is available at https://sites.math.rutgers.edu/~asbuch/equivcalc/

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

References

[ACT22] D. Anderson, L. Chen and H.-H. Tseng, 'On the finiteness of quantum K-theory of a homogeneous space (with an appendix by H. Iritani)', *Int. Math. Res. Not. IMRN* **2** (2022), 1313–1349.

[AHKT01] M. Aganagic, K. Hori, A. Karch and D. Tong, 'Mirror symmetry in 2 + 1 and 1 + 1 dimensions', *J. High Energy Phys.* **7** (2001), Paper 22, 30.

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969).

[BCLM20] A. S. Buch, S. Chung, C. Li, and L. C. Mihalcea, 'Euler characteristics in the quantum K-theory of flag varieties', *Selecta Math. (N.S.)* **26**(2) (2020), Paper No. 29, 11.

[BCMP13] A. S. Buch, P.-E. Chaput, L. C. 'Mihalcea and N. Perrin, Finiteness of cominuscule quantum K-theory', *Ann. Sci. Éc. Norm. Supér. (4)* **46**(3), (2013), 477–494.

[BCMP18a] A. S. Buch, P.-E. Chaput, L. C. Mihalcea and N. Perrin, 'A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties', *Algebr. Geom.* **5**(5) (2018), 568–595.

[BCMP18b] A. S. Buch, P.-E. Chaput, L. C. Mihalcea and N. Perrin, 'Projected Gromov-Witten varieties in cominuscule spaces', *Proc. Amer. Math. Soc.* **146**(9) (2018), 3647–3660.

[BK05] M. Brion and S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory (Progr. Math)* vol. 231 (Birkhäuser Boston, Inc., Boston, MA, 2005).

[BKT03] A. S. Buch, A. Kresch and H. Tamvakis, 'Gromov-Witten invariants on Grassmannians', *J. Amer. Math. Soc.* 16(4) (2003), 901– 915(electronic).

[BM11] A. S. Buch and L. C. Mihalcea, 'Quantum K-theory of Grassmannians', *Duke Math. J.* **156**(3) (2011), 501–538.

[BM15] A. S. Buch and L. C. Mihalcea, 'Curve neighborhoods of Schubert varieties', *J. Differential Geom.* **99**(2) (2015), 255–283.

[Buc02] A. S. Buch, 'Grothendieck classes of quiver varieties', *Duke Math. J.* **115**(1) (2002), 75–103.

[Buc03] A. S. Buch, 'Quantum cohomology of Grassmannians', *Compos. Math.* **137**(2) (2002), 227–235.

[CFKS08] I. Ciocan-Fontanine, B. Kim and C. Sabbah, 'The abelian/nonabelian correspondence and Frobenius manifolds', *Invent. Math.* **171**(2) (2008), 301–343.

[CG09] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry* (Springer Science & Business Media, 2009).

[CK16] C. Closset and H. Kim, 'Comments on twisted indices in 3d supersymmetric gauge theories', *JHEP* **08** (2016), 059.

[CP11] P.-E. Chaput and N. Perrin, 'Rationality of some Gromov-Witten varieties and application to quantum K-theory', *Commun. Contemp. Math.* **13**(1) (2011), 67–90.

[Eis95] D. Eisenbud, *Commutative Algebra (Graduate Texts in Mathematics)* vol. 150 (Springer-Verlag, New York, 1995). With a view toward algebraic geometry.

[FL94] W. Fulton and A. Lascoux, 'A Pieri formula in the Grothendieck ring of a flag bundle', *Duke Math. J.* **76**(3) (1994), 711–729.

[FP97] W. Fulton and R. Pandharipande, 'Notes on stable maps and quantum cohomology', in *Algebraic Geometry — Santa Cruz 1995* (Proc. Sympos. Pure Math.) vol. 62 (Amer. Math. Soc., Providence, RI, 1997), 45–96.

[Ful97] W. Fulton, *Young Tableaux (London Mathematical Society Student Texts)* vol. 35 (Cambridge University Press, Cambridge, 1997). With applications to representation theory and geometry.

[Giv00] A. Givental, 'On the WDVV equation in quantum K-theory', *Michigan Math. J.* **48** (2000), 295–304. Dedicated to William Fulton on the occasion of his 60th birthday.

[GK17] V. Gorbounov and C. Korff, 'Quantum integrability and generalised quantum Schubert calculus', *Adv. Math.* **313** (2017), 282–356.

[GL03] A. Givental and Y.-P. Lee, 'Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups', *Invent. Math.* **151**(1) (2003), 193–219.

[GMS+23] W. Gu, L. Mihalcea, E. Sharpe, W. Xu, H. Zhang and H. Zou, 'Quantum K Whitney relations for partial flag varieties', Preprint, 2023, arXiv:2310.03826.

[GMSZ22] W. Gu, L. Mihalcea, E. Sharpe and H. Zou, 'Quantum K theory of symplectic Grassmannians', *J. Geom. Phys.* **177** (2022), 7.

[GPZ21] W. Gu, D. Pei and M. Zhang, 'On phases of 3d $N = 2$ Chern-Simons-matter theories', *Nucl. Phys. B* **973** (2021), 115604.

[GW22] E. González and C. Woodward, 'Quantum Kirwan for quantum K-theory', in *Facets of Algebraic Geometry. Vol. I* (London Math. Soc. Lecture Note Ser.) vol. 472 (Cambridge Univ. Press, Cambridge, 2022), 265–332.

[GY21] A. Givental and X. Yan, 'Quantum K-theory of Grassmannians and non-abelian localization', *SIGMA* **17** (2021), Paper No. 018, 24.

[Hir95] F. Hirzebruch, *Topological Methods in Algebraic Geometry* (Classics in Mathematics) (Springer-Verlag, Berlin, 1995). Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Harada Reprint of the 1978 edition.

[Iac06] F. Iachello, *Lie Algebras and Applications*, *2nd ed.* (Springer-Verlag, Berlin, 2006).

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press

[IMT15] H. Iritani, T. Milanov and V. Tonita, 'Reconstruction and convergence in quantum K-theory via difference equations', *Int. Math. Res. Not. IMRN* **11** (2015), 2887–2937.
[IN13] T. Ikeda and H. Naruse, 'K-theoretic analogues of factorial Schur P- and Q-functions', *Adv. Math.* **243** (2013), 22–66.
[JM19] H. Jockers and P. Mayr, 'Quantum K-theory of Calabi-Yau manifolds', *JHEP* **11** (2019), 011.
[JM20] H. Jockers and P. Mayr, 'A 3d gauge theory/quantum K-theory correspondence', *Adv. Theor. Math. Phys.* **24**(2) (2021), 327–457.
[JMNT20] H. Jockers, P. Mayr, U. Ninad and A. Tabler, 'Wilson loop algebras and quantum K-theory for Grassmannians', *JHEP* **10** (2020), 036.
[Kap84] M. M. Kapranov, 'Derived category of coherent sheaves on Grassmann manifolds', *Izv. Akad. Nauk SSSR Ser. Mat.* **48**(1) (1984), 192–202.
[Kat18] S. Kato, 'Loop structure on equivariant K-theory of semi-infinite flag manifolds', Preprint, 2018, arXiv:1805.01718.
[KPSZ21] P. Koroteev, P. P. Pushkar, A. V. Smirnov and A. M. Zeitlin, 'Quantum K-theory of quiver varieties and many-body systems', *Selecta Math. (N.S.)* **27**(5) (2021), Paper No. 87, 40.
[Las90] A. Lascoux, 'Anneau de Grothendieck de la variété de drapeaux', in *The Grothendieck Festschrift*, *Vol. III* (Progr. Math.) vol. 88 (Birkhäuser Boston, Boston, MA, 1990), 1–34.
[Lee04] Y.-P. Lee. 'Quantum K-theory. I. Foundations', *Duke Math. J.* **121**(3) (2004), 389–424.
[Len00] C. Lenart, 'Combinatorial aspects of the K-theory of Grassmannians', *Ann. Comb.* **4**(1) (2000), 67–82.
[McN06] P. J. McNamara, 'Factorial Grothendieck polynomials', *Electron. J. Combin.* **13**(1) (2006), Research Paper 71, 40.
[Mih08] L. C. Mihalcea, 'Giambelli formulae for the equivariant quantum cohomology of the Grassmannian', *Trans. Amer. Math. Soc.* **360**(5) (2008), 2285–2301.
[MP95] D. R. Morrison and M. R. Plesser, 'Summing the instantons: quantum cohomology and mirror symmetry in toric varieties', *Nuclear Phys. B* **440**(1–2) (1995), 279–354.
[NS09] N. A. Nekrasov and S. L. Shatashvili, 'Supersymmetric vacua and Bethe ansatz', *Nuclear Phys. B Proc. Suppl.* **192 / 193** (2009), 91–112.
[Oet21] D. Oetjen, 'Factorial Schur and Grothendieck polynomials from Bott Samelson varieties', Preprint, 2021, arXiv:2109.10483.
[RZ18] Y. Ruan and M. Zhang, 'Verlinde/Grassmannian correspondence and rank 2 δ-wallcrossing', *Peking Math. J.* **6**(1) (2023), 217–306; MR4552643.
[ST97] BerndS. and G. Tian, 'On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator', *Asian J. Math.* **1**(4) (1997), 679–695.
[Sta25] The Stacks project authors, The stacks project, https://stacks.math.columbia.edu, 2025.
[UY20] K. Ueda and Y. Yoshida, '3d $\mathcal{N}$ = 2 Chern-Simons-matter theory, Bethe ansatz, and quantum K-theory of Grassmannians', *JHEP* **08** (2020), 157.
[Wen19] Y. Wen, 'K-theoretic I-function of $V//_\theta \mathbf{G}$ and application', Preprint, 2019, arXiv:1906.00775.
[Wey03] J. Weyman, *Cohomology of Vector Bundles and Syzygies* (Cambridge Tracts in Mathematics) vol. 149 (Cambridge University Press, Cambridge, 2003).
[Wit95] E. Witten, 'The Verlinde algebra and the cohomology of the Grassmannian', in *Geometry, Topology, & Physics* (Conf. Proc. Lecture Notes Geom. Topology, IV) (Int. Press, Cambridge, MA, 1995), 357–422.

https://doi.org/10.1017/fms.2025.10088 Published online by Cambridge University Press