Glasgow Math. J. **51** (2009) 467–471. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089509005102. Printed in the United Kingdom

ON THE RESTRICTION OF CHARACTERS OF STEINBERG–TITS TRIALITY GROUP ${}^3D_4(q)$ ON UNIPOTENT CLASSES

VAHID DABBAGHIAN

The IRMACS Centre, Simon Fraser University, Burnaby, British Columbia, V5A 1S6, Canada
e-mail: vdabbagh@sfu.ca

(Received 29 May 2008; revised 23 October 2008; accepted 17 December 2008)

Abstract. Let G be a finite Steinberg–Tits triality group ${}^3D_4(q)$, and let H be a maximal unipotent subgroup of G. In this paper we classify irreducible characters χ of G such that χ_H has a linear constituent with multiplicity one.

2000 *Mathematics Subject Classification.* Primary 20C15; secondary 20C33.

1. Introduction. R. Steinberg [**13**, Theorem 49] asserts that for any finite Chevalley group G, the Gelfand–Graev representation of G is multiplicity-free. By Frobenius reciprocity, this means that the non-degenerate linear characters of a maximal unipotent subgroup of G appear with multiplicity at most 1 in the restriction of every irreducible character of G. This general result follows an earlier work by I. M. Gelfand and M. I. Graev [**7**] on groups $\mathrm{SL}(n, q)$ for arbitrary n with a particular attention to the case $n = 3$.

Subsequently, A. V. Zelevinsky [**15**] proved that if χ is an irreducible complex character of the general linear group $G = \mathrm{GL}(n, q)$, then χ_H contains a linear constituent of a maximal unipotent subgroup H of G with multiplicity 1. Zelevinsky's work was extended by Z. Ohmori in [**11**] to a family of irreducible characters of the general unitary group $\mathrm{GU}(n, q)$. Recently in the case that G is a symplectic group $\mathrm{Sp}(4, q)$, a Chevalley group $\mathrm{G}_2(q)$, a Suzuki group $\mathrm{Sz}(q)$ or a Ree group $\mathrm{Re}(q)$ of characteristic 3, the author has classified all irreducible characters of G which their restriction on a maximal unipotent subgroup of G contain a linear constituent with multiplicity one (see [**2**]).

The classification of such irreducible characters is of interest for several reasons such as computing primitive idempotent elements [**9**], calculating Clifford classes [**14**] and computing matrix representations of finite groups [5, pp. 105–112].

In this paper we classify irreducible characters of another class of finite groups of Lie type, namely Steinberg–Tits triality groups $G = {}^3D_4(q)$. In fact we prove the following theorem:

THEOREM 1. *Let G be a Steinberg–Tits triality group ${}^3D_4(q)$. Let H be a maximal unipotent subgroup and χ be an irreducible character of G. Then χ_H has a linear constituent with multiplicity one if and only if χ does not belong to the following charaters:*

$$ {}^3D_4[-1],\ {}^3D_4[1],\ \chi_{3,1},\ \chi_{4,qs},\ \chi_{5,1},\ \chi_{9,qs'}. $$

Table 1. Linear combinations of restricted characters of ${}^3D_4(q)$ on H

Characters	q Odd	q Even
$\mathrm{St}_H =$	$g+4e+2c+b+a-2d-\mathbf{1}$	$g+2f+2e+b+a-\mathbf{1}$
$(\chi_{3,1})_H =$	$d+c+a+\mathbf{1}$	$d+c+a+\mathbf{1}$
$(\chi_{3,\mathrm{St}})_H =$	$g+4e+3c+2b+a-d-\mathbf{1}$	$g+2f+2e+d+c+2b+a-\mathbf{1}$
$(\chi_{4,1})_H =$	$e+2c-a-d+\mathbf{1}$	$f+c-a+\mathbf{1}$
$(\chi_{4,qs})_H =$	$2d+b+a-c-e$	$d+b+a-f$
$(\chi_{4,\mathrm{St}})_H =$	$g+5e+4c+a-3d-\mathbf{1}$	$g+3f+2e+c+a-\mathbf{1}$
$(\chi_{5,1})_H =$	$d+c+b+\mathbf{1}$	$d+c+b+\mathbf{1}$
$(\chi_{5,\mathrm{St}})_H =$	$g+4e+3c+b+2a-d-\mathbf{1}$	$g+2f+2e+d+c+b+2a-\mathbf{1}$
$(\chi_6)_H =$	$g+4e+4c+2b+2a$	$g+2f+2e+2d+2c+2b+2a$
$(\chi_{7,1})_H =$	$2e+c+a-d-\mathbf{1}$	$f+e+a-\mathbf{1}$
$(\chi_{7,\mathrm{St}})_H =$	$g+2e+c+a-d-\mathbf{1}$	$g+f+e+a-\mathbf{1}$
$(\chi_8)_H =$	$g+4e+2c+2a-2d-2\cdot\mathbf{1}$	$g+2f+2e+2a-2\cdot\mathbf{1}$
$(\chi_{9,1})_H =$	$e+d-a-\mathbf{1}$	$e+d-a-\mathbf{1}$
$(\chi_{9,qs'})_H =$	$e+c+b+a-2d$	$f+b+a-d$
$(\chi_{9,\mathrm{St}})_H =$	$g+3e+2c+2b+a-3d-\mathbf{1}$	$g+2f+e+2b+a-d-\mathbf{1}$
$(\chi_{10,1})_H =$	$2e+c+b-d-\mathbf{1}$	$f+e+b-\mathbf{1}$
$(\chi_{10,\mathrm{St}})_H =$	$g+2e+c+b-d-\mathbf{1}$	$g+f+e+b-\mathbf{1}$
$(\chi_{11})_H =$	$g+4e+2c+2b-2d-2\cdot\mathbf{1}$	$g+2f+2e+2b-2\cdot\mathbf{1}$
$(\chi_{12})_H =$	$g+7e+7c-a-b-6d$	$g+5f+2e+2c-a-b-d$
$(\chi_{13})_H =$	$g+3e+3c+3b+3a-6d$	$g+3f+3b+3a-3d$
$(\chi_{14})_H =$	$g+5e+c+b+a-2d$	$g+2f+3e+b+a-c$
${}^3D_4[1]_H =$	$e+c-d$	
$(\chi_{2,1})_H =$	$d-e+\mathbf{1}$	
$(\chi_{2,\mathrm{St}})_H =$	$e+c+a$	
$(\chi_{2,\mathrm{St}'})_H =$	$e+c+b$	
$(\chi_{2,\mathrm{St},\mathrm{St}'})_H =$	$g+3e+2c+b+a-d-\mathbf{1}$	

NOTATION. For a group H and a character θ, $\mathrm{Lin}(H)$ and $\mathrm{Lin}(\theta)$ denote the set of all linear characters of H and the set of all non-principal linear constituents of θ, respectively.

2. Restriction of characters. Suppose that G is a Steinberg–Tits triality group ${}^3D_4(q)$ of characteristic p with n conjugacy classes. Let t of the conjugacy classes of G be unipotent classes. Then $t = 7$ and 8 when q is odd and even, respectively. Consider the $n \times n$ matrix X constructed from the character table of G and the $n \times t$ submatrix P whose columns correspond to the unipotent classes. Since X is invertible, the columns of P are linearly independent, and so P is rank t. Thus there exist t irreducible characters $\theta_1, \ldots, \theta_t$, say, of G such that for every irreducible character χ of G the restriction χ_H is a linear combination of the restrictions $(\theta_1)_H, \ldots, (\theta_t)_H$. What we shall see below (Table 1) that we can choose the characters $\theta_1, \ldots, \theta_t$ in such a way that every χ_H is an *integral* linear combination of $(\theta_1)_H, \ldots, (\theta_t)_H$. This is analogous to the theory of π-partial characters of solvable groups developed by I. M. Isaacs, where π is a set of prime divisors of the order of group (see **[8]**). In fact he proves that if G is a solvable group and H is a π-subgroup, where π is a set of prime divisors of $|G|$, then there is a set $\{(\theta_1)_H, \ldots, (\theta_t)_H\}$ of class functions of H such that $\theta_1, \ldots, \theta_t$ are irreducible characters of G and

$$\chi_H = \sum_{i=1}^{t} m_i(\theta_i)_H$$

with non-negative integer coefficients m_i, for each irreducible character χ of G.

https://doi.org/10.1017/S0017089509005102 Published online by Cambridge University Press

This property has already been investigated by the author in [**1**]–[**3**] for some classes of finite groups of Lie type in the case that $\pi = \{p\}$ is the defining characteristic and m_i are integers. These classes are special linear groups $\mathrm{SL}(l, q)$ for $l = 2$ and 3, special unitary groups $\mathrm{SU}(3, q)$, symplectic groups $\mathrm{Sp}(4, q)$, Chevalley groups $\mathrm{G}_2(q)$, Suzuki groups $\mathrm{Sz}(q)$ and Ree groups $\mathrm{Re}(q)$ of characteristic 3. These outcomes and the result obtained in this paper support the following conjecture, which is related to a conjecture by N. Kawanaka [**10**, 3.3.1, pp. 175–206].

Conjecture 2. Let G be a finite group of Lie type, and let H be a maximal unipotent subgroup of G. Let t be the number of conjugacy classes of unipotent elements of G. Then there exist irreducible characters $\theta_1, \ldots, \theta_t$ of G such that χ_H is an integral linear combination of $(\theta_1)_H, \ldots, (\theta_t)_H$ for each irreducible character χ of G.

3. Proof of the theorem. Let $G = {}^3D_4(q)$ be a Steinberg–Tits triality group in which q is a power of a prime p. Then G is a simple group of order $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. Values of eight unipotent irreducible characters of G have been computed in [**12**]. This work is continued by computing the values of non-unipotent irreducible characters of G as a linear combination of Deligne–Lusztig characters in [**4**]. Throughout this paper all notations concerning conjugacy classes and irreducible characters are referred from [**12**] and [**4**].

Let $a = [\varepsilon_1]_H, b = [\varepsilon_2]_H, c = (\rho_1)_H, d = (\rho_2)_H, e = {}^3D_4[-1]_H, g = (\chi_{15})_H$, and let $\mathbf{1}$ be the principal character of H. Let $f = {}^3D_4[1]_H$ for q even. Then using the CHEVIE computer algebra system [**6**] we compute χ_H as an *integral* linear combination of a, b, c, d, e, f, g and $\mathbf{1}$ for each irreducible character χ of G. These linear combinations are listed in Table 1.

Since G has the Steinberg character St whose restriction to a maximal unipotent subgroup H is the regular character ρ on H; thus we can write ρ as an integral linear combination of a, b, c, d, e, f, g and $\mathbf{1}$. In the next two lemmas, using the fact that $\langle \rho, \varphi \rangle = 1$ for each $\varphi \in \mathrm{Lin}(H)$, we obtain information about the multiplicities of the linear constituents of each of these characters. Then we use this and ad hoc arguments to determine which of the χ_H have linear constituents of multiplicity one.

One observation is frequently used. The degrees of the irreducible characters of H are all powers of p. Therefore, for each χ, the sum of the multiplicities of the linear constituents of χ_H must be congruent to $\chi(1) \pmod p$.

Using [**12**], if B is a Borel subgroup of G, then

$$\mathbf{1}_B^G = \mathbf{1}_G + [\varepsilon_1] + [\varepsilon_2] + 2[\rho_1] + 2[\rho_2] + St.$$

Now by considering the fact that $\langle \chi_H, \mathbf{1} \rangle = \langle \chi, \mathbf{1}^G \rangle = \langle \chi, \mathbf{1}_B^G \rangle$ we have

$$\langle a, \mathbf{1} \rangle = \langle b, \mathbf{1} \rangle = 1 \text{ and } \langle c, \mathbf{1} \rangle = \langle d, \mathbf{1} \rangle = 2.$$

Lemma 3. *Let q be odd:*
1. $\langle e, \varphi \rangle = 0$ *for all* $\varphi \in \mathrm{Lin}(H)$*;*
2. $\langle c, \varphi \rangle = \langle d, \varphi \rangle$ *for all* $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$*;*
3. $\langle a + b + g, \varphi \rangle = 2$ *if* $\varphi = \mathbf{1}$ *and 1 otherwise.*

Proof. Considering Table 1 we have $St_H = g + 4e + 2c + b + a - 2d - \mathbf{1}$. Since St_H is the regular character of H, $\langle St_H, \mathbf{1} \rangle = 1$. Now using $\langle a, \mathbf{1} \rangle = \langle b, \mathbf{1} \rangle = 1$ and

https://doi.org/10.1017/S0017089509005102 Published online by Cambridge University Press

$\langle c, \mathbf{1}\rangle = \langle d, \mathbf{1}\rangle = 2$ we have $\langle e, \mathbf{1}\rangle = \langle g, \mathbf{1}\rangle = 0$. Suppose $\langle e, \varphi\rangle = m$ and $\langle g + 2c + b + a, \varphi\rangle = l$ for $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$. Since $\langle St_H, \varphi\rangle = 1$ we have $4m + l - 2\langle d, \varphi\rangle = 1$. Thus $\langle d, \varphi\rangle = 2m + (l-1)/2$. On the other hand $\langle g + 3c + 3b + 3a, \varphi\rangle \leqslant 3\langle g + c + b + a, \varphi\rangle \leqslant 3\langle g + 2c + b + a, \varphi\rangle = 3l$; so $0 \leqslant \langle (\chi_{13})_H, \varphi\rangle \leqslant -9m + 3$, and we get $m = 0$. This proves $\langle e, \varphi\rangle = 0$ for all $\varphi \in \mathrm{Lin}(H)$.

Using ${}^3D_4[1]_H$ and the fact that St_H is the regular character of H we have

$$\langle d, \varphi\rangle = \langle c, \varphi\rangle \ \text{ for all } \ \mathbf{1} \neq \varphi \in \mathrm{Lin}(H).$$

Furthermore it is easy to see that $\langle a + b + g, \varphi\rangle = 1$ and φ is a constituent of one and only one of g, b and a for each $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$. □

Now we prove a similar lemma for q even.

LEMMA 4. *Let q be even:*
1. $\langle e, \varphi\rangle = \langle f, \varphi\rangle = 0$ *for all* $\varphi \in \mathrm{Lin}(H)$*;*
2. $\langle a + b + g, \varphi\rangle = 2$ *if* $\varphi = \mathbf{1}$ *and 1 otherwise.*

Proof. Since $St_H = g + 2f + 2e + b + a - \mathbf{1}$ is the regular character of H, $\langle e, \varphi\rangle = \langle f, \varphi\rangle = 0$ for all $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$. Also $\langle e, \mathbf{1}\rangle = \langle f, \mathbf{1}\rangle = 0$, since $2 \mid e(1)$ and $2 \mid f(1)$; $\langle St_H, \mathbf{1}\rangle = 1$ proves part (2). □

Proof of Theorem 1. Case I. q odd: Since $\langle a, \mathbf{1}\rangle = \langle b, \mathbf{1}\rangle = 1$, $\langle g, \mathbf{1}\rangle = 0$ and $p \mid a(1)$, $p \mid b(1)$, $p \nmid g(1)$ there exist non-principal characters $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Lin}(H)$ such that $\langle a, \varphi_1\rangle = \langle b, \varphi_2\rangle = \langle g, \varphi_3\rangle = 1$ and $\varphi_i \neq \varphi_j$ for $i \neq j$.

Suppose $\langle g, \psi\rangle = 1$ for a non-principal character $\psi \in \mathrm{Lin}(H)$; then using $(\chi_{13})_H$ we have $\langle c, \psi\rangle = 0$ (and so $\langle d, \psi\rangle = 0$). This proves the theorem for

$$\chi \in \{St, \chi_{2,St,St'}, \chi_{3,St}, \chi_{4,St}, \chi_{5,St}, \chi_6, \chi_{7,St}, \chi_8, \chi_{9,St}, \chi_{10,St}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}\}.$$

Since $\langle c, \varphi\rangle = \langle d, \varphi\rangle$ and $\langle e, \varphi\rangle = 0$ for all $\varphi \in \mathrm{Lin}(H)$ the theorem holds for $\chi \in \{\chi_{7,1}, \chi_{10,1}\}$. Suppose $\langle a, \varphi\rangle = 1$ for a non-principal character $\varphi \in \mathrm{Lin}(H)$; then by $(\chi_{4,1})_H$ we have $\langle c, \varphi\rangle \neq 0$ (and so $\langle d, \varphi\rangle \neq 0$). Also using $(\chi_{9,St})_H$ we get $\langle d, \varphi\rangle = \langle c, \varphi\rangle = 1$. This holds the theorem for $\chi \in \{\rho_1, \rho_2, \chi_{2,1}\}$.

Using $(\chi_{12})_H$, $(\chi_{9,qs'})_H$ and the fact that $\langle c, \varphi\rangle = \langle d, \varphi\rangle$ for all $\varphi \in \mathrm{Lin}(H)$ we have $\langle c, \varphi\rangle = 1$ if and only if either $\langle a, \varphi\rangle = 1$ and $\langle b, \varphi\rangle = 0$ or $\langle a, \varphi\rangle = 0$ and $\langle b, \varphi\rangle = 1$. This proves the theorem for $\chi \in \{\chi_{2,St}, \chi_{2,St'}, \chi_{4,1}, \chi_{9,1}\}$. Furthermore it shows that χ_H has no linear constituents of multiplicity one for $\chi \in \{{}^3D_4[-1], {}^3D_4[1], \chi_{3,1}, \chi_{4,qs}, \chi_{5,1}, \chi_{9,qs'}\}$.

Case II. q even: Considering St_H, $\langle a, \mathbf{1}\rangle = \langle b, \mathbf{1}\rangle = 1$, $\langle g, \mathbf{1}\rangle = 0$ and the fact that $2 \mid a(1)$, $2 \mid b(1)$ and $2 \nmid g(1)$, there exist non-principal characters $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Lin}(H)$ such that $\langle a, \varphi_1\rangle = \langle b, \varphi_2\rangle = \langle g, \varphi_3\rangle = 1$ and $\varphi_i \neq \varphi_j$ for $i \neq j$. This shows that the theorem holds for $\chi \in \{\chi_{7,1}, \chi_{10,1}\}$.

The character $(\chi_{14})_H$ shows all non-principal linear constituents of c have multiplicity one. On the other hand $(\chi_{4,1})_H$ implies if $\langle a, \varphi\rangle = 1$, then $\langle c, \varphi\rangle = 1$ for all $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$. Similar arguments hold for the character d by considering the characters $(\chi_{9,1})_H$ and $(\chi_{9,qs'})_H$. Since $\langle d, \mathbf{1}\rangle = 2$ and $2 \mid d(1)$, $|\mathrm{Lin}(d)|$ is even. On the other hand $\langle a, \mathbf{1}\rangle = 1$ and $2 \mid a(1)$; so $|\mathrm{Lin}(a)|$ is odd. This shows that there exists $\mathbf{1} \neq \psi \in \mathrm{Lin}(H)$ such that $\langle d, \psi\rangle = 1$ and $\langle a, \psi\rangle = 0$. Thus the theorem holds for $\chi_{9,1}$.

Similarly from $\langle c, \mathbf{1}\rangle = 2$ and $2 \mid c(1)$ we obtain $|\mathrm{Lin}(c)|$ is even. Now since $|\mathrm{Lin}(H) - \{\mathbf{1}\}|$ is odd there exists a non-principal linear character φ of H such that

$\langle g, \varphi \rangle = 1$ and $\langle c, \varphi \rangle = 0$. A similar argument holds for the character d, using the character $(\chi_{13})_H$. These show the theorem holds for $\chi \in \{St, \chi_{3,St}, \chi_{4,St}, \chi_{5,St}, \chi_6, \chi_{7,St}, \chi_8, \chi_{9,St}, \chi_{10,St}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}\}$.

Using $(\chi_{12})_H$ we get if $\langle a, \varphi \rangle = 1$ or $\langle b, \varphi \rangle = 1$ for some $\mathbf{1} \neq \varphi \in \mathrm{Lin}(H)$, then $\langle c, \varphi \rangle = 1$. It means that the character c contains all the non-principal linear constituents of a and b. Since a and b can not have same non-principal linear constituents $(\chi_{4,1})_H$ does not have any linear constituent of multiplicity one. A similar argument holds for the characters $\{\chi_{3,1}, \chi_{4,qs}, \chi_{5,1}, \chi_{9,qs'}\}$. This completes the proof. □

Remark. If q is even, then $t = 8$ (the number of unipotent classes) and characters a, b, c, d, e, f, g and $\mathbf{1}$ are linearly independent, while for q odd we have $t = 7$ and $f = e + c - d$. Now if in all the linear combinations in Table 1 for q odd, we substitute $e + c - d$ by f, then we obtain the linear combinations of the table for q even.

Acknowledgements. The author would like to thank the anonymous referees for the valuable comments and suggestions. He would also like to acknowledge the IRMACS centre.

REFERENCES

1. V. Dabbaghian-Abdoly, An algorithm to construct representations of finite groups, PhD thesis (School of Mathematics, Carleton University, 2003).

2. V. Dabbaghian-Abdoly, Characters of some finite groups of Lie type with a restriction containing a linear character once, *J. Algebra* **309** (2007), 543–558.

3. V. Dabbaghian-Abdoly, Constructing representations of the finite symplectic group $Sp(4, q)$, *J. Algebra* **303** (2006), 618–625.

4. D. I. Deriziotis and G. O. Michler, Character table and blocks of finite simple triality groups ${}^3D_4(q)$, *Trans. Am. Math. Soc.* **303**(1) (1987), 39–70.

5. J. D. Dixon, *Constructing representations of finite groups*, Discrete Mathematics and Theoretical Computer Science **11** (American Mathematical Society, Providence, RI 1993).

6. M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, CHEVIE: A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, *Appl. Algebra Eng. Comm. Comput.* **7** (1996), 175–210. Available at "http://www.math.rwth-aachen.de/~CHEVIE".

7. I. M. Gelfand and M. I. Graev, Construction of irreducible representations of simple algebraic groups over a finite field, *Dokl. Akad. Nauk SSSR* **147** (1962), 529–532.

8. I. M. Isaacs, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.

9. G. J. Janusz, Primitive idempotents in group algebras, *Proc. Am. Math. Soc.* **17** (1966), 520–523.

10. N. Kawanaka, *Generalized Gel'fand Graev representations and Ennola duality*, Advanced Studies in Pure Mathematics **6** (North-Holland, Amsterdam, 1985).

11. Z. Ohmori, On a Zelevinsky theorem and the Schur indices of the finite unitary groups, *J. Math. Sci. Univ. Tokyo* **4** (1997), 417–433.

12. N. Spaltenstein, Caractres unipotents de ${}^3D_4(F_q)$, *Comment. Math. Helv.* **57**(4) (1982), 676–691.

13. R. Steinberg, *Lectures on Chevalley groups* (Yale University, New Haven, 1968).

14. A. Turull, Calculating Clifford classes for characters containing a linear character once, *J. Algebra*, **254** (2002), 264–278.

15. A. V. Zelevinsky, *Representations of finite classical groups*, Lecture Notes in Mathematics **869** (Springer, New York, 1981).