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Hermite-Fejer interpolation at the 'practical' Chebyshev nodes

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Berman has raised the question in his work of whether Hermite-Fejér interpolation based on the so-called "practical" Chebyshev points, $t_j = \cos\frac{j\pi}{n}$, $0(1)n$, is uniformly convergent for all continuous functions on the interval $[-1, 1]$. In spite of similar negative results by Berman and Szegö, this paper shows this result is true, which is in accord with the great similarities of Lagrangian interpolation based on these points versus the points $x_j = \cos\frac{2j-1}{2n}\pi$, $1(1)n$.

1. Introduction

This paper answers some questions which naturally arise from the work of Berman [1] in his consideration of "extended" Hermite-Fejér interpolation processes. We consider the point sets $S_1 = \{t_j\}_{j=0}^{n}$, $S_2 = \{t_j\}_{j=1}^{n-1}$, and $S_3 = \{x_j\}_{j=0}^{n-1}$, where $t_j = \cos\frac{j\pi}{n}$ and $x_j = \cos\left(\frac{2j+1}{2n}\pi\right)$. In what follows, for a given function $f(x)$, $-1 \le x \le 1$, and P_m being the set of all polynomials of degree m or less, we define the following three Hermite-Fejér operators:

(1) $H_n^1(f;\, x) \in P_{2n+1}$, where $H_n^1(f;\, t_j) = f(t_j)$ and $D_x\left[H_n^1(f;\, t_j)\right] = 0$, $0 \le j \le n$;

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(2) $H_n^2(f;\ x) \in P_{2n-3}$, where $H_n^2(f;\ t_j) = f(t_j)$ and $D_x\left[H_n^2(f;\ t_j)\right] = 0$, $1 \le j \le n-1$;

(3) $L_n(f;\ x) \in P_{2n-1}$, where $L_n(f;\ x_j) = f(x_j)$ and $D_x[L_n(f;\ x_j)] = 0$, $0 \le j \le n-1$.

In 1916, Fejér [5] proved the result that $L_n(f;\ x)$ is uniformly convergent to $f(x)$, for all $f(x) \in C[-1,\ 1]$; that is, the set of continuous functions on the closed interval $[-1,\ 1]$. In 1969, Berman [1] showed that, surprisingly, this result is not true for the "extended Hermite-Fejér" interpolation process, that is, the polynomials are of degree $2n + 3$ and interpolate $f(x)$ at ± 1 , and their derivatives are zero at ± 1 , in addition to the points x_j , $0 \le j \le n-1$. In fact Berman showed this process was everywhere divergent for the function $f(x) = 1 - x^2$, $x \in (-1,\ 1)$, and also for $f(x) = x$, $x \in (-1,\ 1)$, $x \ne 0$. Szegö [9], Theorem 14.6, p. 340, shows that the Hermite-Fejér interpolation at the roots of the Jacobi polynomials, $\alpha > -1$, $\beta > -1$, is uniformly convergent to any continuous function on any closed subinterval of $[-1,\ 1]$, and shows divergence for some continuous functions at $x = 1$ when $\alpha \ge 0$.

This leaves open the question of convergence for the "practical" Chebyshev nodes, S_1 . Berman begins to answer this question by showing convergence for $f(x) = |x|$ at $x = 0$ and uniform convergence for the functions $f(x) = x$ and $f(x) = x^2$. But since the sequence of operators $\left\{H_n^1\right\}_{n=1}^{\infty}$ is not monotone (see Cheney [3], p. 67), this is not enough to insure uniform convergence for all $f \in C[-1,\ 1]$. However, we shall show that this is indeed true, by showing the norms of the operators are uniformly bounded. In all that follows $T_m(x)$ and $U_m(x)$ are the Chebyshev polynomials of degree m of the first and second kind, respectively.

2. Construction of the interpolating polynomials

It is relatively easy, though often tedious, to compute $H_n^i(T_k;\ x)$; $i = 1, 2$; for an arbitrary integer k , using elementary trigonometric identities and the following discrete orthogonality relationship (Cheney [3], p. 135):

$$(4) \qquad \frac{2}{n} \sum_{j=0}^{n}{}'' T_i(t_j) T_m(t_j) = \Delta(n,\ i,\ m) \ ,$$

where the double prime indicates the first and last terms are to be halved, and $\Delta(n,\ i,\ m)$ denotes the number of integers in the set $\left\{\frac{i+m}{2n}, \frac{i-m}{2n}\right\}$.

We will not go through the lengthy derivation of the necessary interpolating polynomials. In each case the derivation proceeds by writing each $H_n^i(T_k;\ x)$ in the form $\sum \varphi_i T_i(x)$ and using (4) and the interpolating properties to solve for the φ_i's. Calling on the uniqueness of the interpolating polynomials, we shall verify our answers in one case and leave the others to the reader, since their validity rather than their derivation is of central importance to this paper. In all that follows, we write $k = 2rn + \alpha$, with $r \geq 0$ and $-n \leq \alpha \leq n-1$, with n taken as an even integer for simplicity. Although the first case is of primary importance, we list the results for all three cases mainly in the interests of construction and error analyses of particular functions.

$$(5)\quad H_n^1(T_k;\ x) = \begin{cases} \frac{|\alpha|(2n-|\alpha|)}{4n^2} + \frac{2n-|\alpha|}{2n} T_{|\alpha|}(x) + \frac{|\alpha|}{2n} T_{2n-|\alpha|}(x) \\ \qquad - \frac{|\alpha|(2n-|\alpha|)}{4n^2} T_{2n}(x)\ , \text{ for } 2 \le |\alpha| \le n-2\ , \text{ even,} \\ \frac{4n^2-1}{4n^2} T_1(x) - \frac{2n+1}{8n^2} T_{2n-1}(x) - \frac{2n-1}{8n^2} T_{2n+1}(x)\ , \\ \qquad \text{for } |\alpha| = 1\ , \\ T_0(x)\ , \text{ for } \alpha = 0\ , \\ \frac{|\alpha|(2n-|\alpha|)}{4n^2} T_1(x) + \frac{2n-|\alpha|}{2n} T_{|\alpha|}(x) + \frac{|\alpha|}{2n} T_{2n-|\alpha|}(x) \\ \qquad - \frac{|\alpha|(2n-|\alpha|)}{8n^2} \left(T_{2n-1}(x)+T_{2n+1}(x)\right)\ , \\ \qquad \text{for } 3 \le |\alpha| \le n-1\ , \text{ odd,} \\ \frac{1}{4} T_0(x) + T_n(x) - \frac{1}{4} T_{2n}(x)\ , \text{ for } \alpha = -n\ . \end{cases}$$

The uniform convergence of H_n^1 for $f(x) = x$ and $f(x) = x^2$ on $[-1, 1]$ displayed by Berman is immediate from (5). In fact (5) yields uniform convergence of H_n^1 for any polynomial of fixed degree k. (Note that for k fixed and n sufficiently large, $k = \alpha$, and uniform convergence for each $T_k(x)$ is evident.) (5) also immediately yields the result that $H_n^1(f)$ is uniformly convergent to any function $f(x)$ which has an absolutely and uniformly convergent Fourier-Chebyshev series on $[-1, 1]$. It is easily seen from (5) that

$$(6a)\qquad H_n^2(T_k;\ x) = \frac{2n-|\alpha|}{2n} T_{|\alpha|}(x) + \frac{|\alpha|}{2n} T_{2n-|\alpha|}(x)\ ,$$

but since $H_n^2(T_k;\ x) \in P_{2n-3}$, formula (6a) is valid only when $|\alpha| \ge 3$. The cases $|\alpha| = 1$ and $|\alpha| = 2$ must be considered individually. These results are, for n even,

$$(6b)\qquad H_n^2(T_k;\ x) = \sum_{\substack{i=3 \\ \text{odd}}}^{2n-3} \left(\frac{i-2n}{2n}\right) T_i(x)\ , \text{ for } |\alpha| = 1\ ,$$

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(6c) $$H_n^2(T_k;\ x) = -\frac{1}{2} + \sum_{\substack{i=4\\ \text{even}}}^{2n-4} \left(\frac{i-2n}{2n}\right) T_i(x) \ , \text{ for } |\alpha| = 2 \ .$$

The following two formulas may be used to simplify (6b) and (6c). (They are proved by induction and well-known identities given, for example, in Snyder [8], p. 97.

(7) $$\sum_{\substack{i=2\\ \text{even}}}^{m} iT_i(x) = \frac{m}{2}\, U_m(x) - \frac{1}{2}\left(\frac{1-T_m(x)}{1-x^2}\right) \ , \quad m \ \text{ even,}$$

(8) $$\sum_{\substack{i=1\\ \text{odd}}}^{m} iT_i(x) = \frac{m+1}{2}\, U_m(x) - \frac{1}{2}\,\frac{x}{1-x^2}\,\left(1-T_{m+1}(x)\right) \ , \quad m \ \text{ odd.}$$

With the aid of (7) and (8) we can further Szegö's result, and show divergence at ± 1 for $H_n^2(T_k)$ for all k. The final formula is:

(9) $$L_n(T_k;\ x) = \begin{cases} \dfrac{(-1)^r}{2n}\left[(2n-|\alpha|)T_{|\alpha|}(x) - |\alpha| T_{2n-|\alpha|}(x)\right] \ , & \text{for } \alpha \neq 0 \ , \\[2ex] (-1)^r T_0(x) \ , \ \text{for } \alpha = 0 \ . & \end{cases}$$

We shall verify only the first line of formula (5), the other cases being similar. We shall need the following identities easily verified by elementary trigonometric formulas:

(i) $T_k(t_j) = T_{|\alpha|}(t_j)$,

(ii) $T_{2n}(t_j) = 1$,

(iii) $T_{2n-|\alpha|}(t_j) = T_{|\alpha|}(t_j)$,

(iv) $U_{2n-1}(t_j) = 0$, $1 \le j \le n-1$, and

(v) $U_{2n-|\alpha|-1}(t_j) = -U_{|\alpha|-1}(t_j)$, $1 \le j \le n-1$.

Thus, by (i), (ii), and (iii), $H_n^1(T_k;\ t_j) = T_{|\alpha|}(t_j) = T_k(t_j)$. Now,

$$D_x\left[H_n^1(T_k;\ t_j)\right] = |\alpha| \left(\frac{2n-|\alpha|}{2n}\right) U_{|\alpha|-1}(t_j) + |\alpha|\ \frac{(2n-|\alpha|)}{2n}\ U_{2n-|\alpha|-1}(t_j) \ .$$

Thus, by (iv) and (v), $D_x\left[H_n^1(T_k;\ t_j)\right] = 0$ for $1 \le j \le n-1$. Also,

$$D_x\left[H_n^1(T_k;\ 1)\right] = |\alpha|^2\left(\frac{2n-|\alpha|}{2n}\right) + |\alpha|\ \frac{(2n-|\alpha|)^2}{2n} - \frac{|\alpha|(2n-|\alpha|)}{2n}\ 2n$$
$$= \frac{|\alpha|}{2n}\ \left[2n|\alpha|-|\alpha|^2+4n^2-4n|\alpha|+|\alpha|^2-4n^2+2n|\alpha|\right] = 0$$
$$= D_x\left[H_n^1(T_k;\ -1)\right]\ .$$

3. Convergence at the "practical" nodes

We let $W(x) = \left(x^2-1\right)U_{n-1}(x)$; then the "practical" Hermite-Fejér operator can be written as

$$(10)\qquad H_n^1(f;\ x) = \sum_{j=0}^{n} f\left(t_j\right)\left[1 - \frac{\left(x-t_j\right)W''\left(t_j\right)}{W'\left(t_j\right)}\right]\left[\frac{W(x)}{\left(x-t_j\right)W'\left(t_j\right)}\right]^2 .$$

It is easily shown by trigonometric identities that $W'\left(t_j\right)^2 = n^2$, $1 \le j \le n-1$, and $W'(\pm 1)^2 = 4n^2$. Using the well-known differential equation, Davis [4], p. 366, $\left(1-x^2\right)y'' - 3xy' + \left(n^2-1\right)y = 0$, satisfied $U_{n-1}(x)$, it is easily shown that $W(x)$ satisfies

$$(11)\qquad \left(1-x^2\right)^2W''(x) + x\left(1-x^2\right)W'(x) + \left[\left(1-x^2\right)\left(n^2-1\right)+2\right]W(x) = 0\ .$$

Thus we have

$$\frac{W''\left(t_j\right)}{W'\left(t_j\right)} = \begin{cases} \dfrac{t_j}{\left(t_j^2-1\right)}\ , & \text{for } 1 \le j \le n-1\ , \\ \frac{1}{3}\left(2n^2+1\right)\ , & \text{for } t_0 = 1\ , \\ -\frac{1}{3}\left(2n^2+1\right)\ , & \text{for } t_n = -1\ . \end{cases}$$

Thus (10) becomes

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$$(13)\quad H_n^1(f;\, x) = \frac{W(x)^2}{n^2} \sum_{j=1}^{n-1} f(t_j) \left[\frac{1+xt_j-2t_j^2}{1-t_j^2}\right] \frac{1}{(x-t_j)^2}$$
$$+ \frac{W(x)^2}{4n^2} f(1) \left[1 + \frac{(1-x)(2n^2+1)}{3}\right] \frac{1}{(x-1)^2}$$
$$+ \frac{W(x)^2}{4n^2} f(-1) \left[1 + \frac{(x+1)(2n^2+1)}{3}\right] \frac{1}{(x+1)^2}$$
$$\equiv \sum_{j=1}^{n-1} f(t_j) A_j(x) + f(1)A_0(x) + f(-1)A_n(x) .$$

Now consider any $x \in (-1,\, 1)$, and let $r(x,\, t) = 1 + xt - 2t^2$. Now, $r(x,\, x) = 1 - x^2 > 0$, and thus there exists a δ_x such that if $|x-t| \le \delta_x$, then $r(x,\, t) \ge 0$. Note that δ_x is a monotonic decreasing function of x as $x \to \pm 1$ inside $[-1,\, 1]$. Thus the same δ_x makes the above valid for any other $x' \in [-x,\, x]$. Let
$J_x = \{j : |x-t_j| \le \delta_x\} \cup \{0,\, n\}$ and $K_x = \{j : |x-t_j| > \delta_x\}$,

$1 \le j \le n-1$. Now, $\sum_{j=0}^{n} A_j(x) = 1$, and $A_j(x) \ge 0$ for $j \in J_x$; thus

$$(14)\quad \sum_{j=0}^{n} |A_j(x)| = \sum_{j\in J_x} A_j(x) + \sum_{j\in K_x} |A_j(x)| \le 1 + 2 \sum_{j\in K_x} |A_j(x)| .$$

Now, $|W(x)| = \frac{1}{2} |\cos(n+1)\theta - \cos(n-1)\theta| \le 1$, $x = \cos\theta$. Thus

$$\sum_{j\in K_x} |A_j(x)| = O\left(\frac{1}{n^2}\right) \sum_{j=1}^{n-1} \frac{n^2}{j^2} = O(1) ,$$

where the constant bounding $\sum_{j\in K_x} |A_j(x)|$ is independent of x because of the monotonicity of δ_x. Thus, given any $\gamma > 0$, and $x \in [-1+\gamma,\, 1-\gamma]$, there exists a constant M such that $\sum_{j=0}^{n} |A_j(x)| \le M$, for all n. Now consider any $\varepsilon > 0$ and choose, by the Weierstrass

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Theorem, a polynomial $p_N(x) \in P_N$ such that

$$\|f-p_N\|_\infty \equiv \max_{-1\le x\le 1} |f(x)-p_N(x)| < \varepsilon .$$

By the remark following formula (5), $\left\|H_n^1(T_m)-T_m\right\|_\infty$ can be made arbitrarily small for m fixed and n sufficiently large, and that $\left\|H_n^1(p_N)-p_N\right\|_\infty < \varepsilon$ for n sufficiently large. Therefore,

$$(15) \quad \max_{-1+\gamma\le x\le 1-\gamma} \left|H_n^1(f;\, x)-f(x)\right| \le \max_{-1+\gamma\le x\le 1-\gamma} \left|H_n^1(f-p_N;\, x)\right| + \left\|H_n^1(p_N)-p_N\right\|_\infty + \|p_N-f\|_\infty$$

$$\le \|f-p_N\|_\infty\left(\sum_{j=0}^{n} |A_j(x)|+1\right) + \left\|H_n^1(p_N)-p_N\right\|_\infty$$

$$< (M+1)\varepsilon + \varepsilon , \text{ for } n \text{ sufficiently large.}$$

Thus we have proved,

THEOREM 1. *$\left\{H_n^1(f;\, x)\right\}_{n=1}^{\infty}$ is uniformly convergent to any $f \in C[-1, 1]$ on any closed subinterval $[-1+\gamma, 1-\gamma]$.*

Since the interpolating polynomials are constructed such that $H_n^1(f;\, \pm 1) = f(\pm 1)$, for all f , we have the immediate result,

COROLLARY 1. *Given any $x \in [-1, 1]$ and any $f \in C[-1, 1]$, $\lim_{n\to\infty} H_n^1(f;\, x) = f(x)$.*

Because of these results and because of the striking similarity between the properties of Lagrangian interpolation on S_1 versus S_3 (Salzer [7], see especially comparison of Lebesgue constants and remainder norms), one might expect Fejér's result to be true for $H_n^1(f)$ as well as $L_n(f)$, in spite of Berman's negative results in [1] and [2]. We shall now show that this is indeed the case.

Define the continuous linear operator

(16) $$R_n(f;\, x) = f(x) - H_n^1(f;\, x)\ , \quad f \in C[-1,\, 1]\ .$$

Now,

$$\|R_n\| \equiv \sup_{\|f\|_\infty=1} \|R_n(f)\|_\infty \le \left\| 1 + \sum_{j=0}^{n} |A_j(x)| \right\|_\infty$$

$$= 1 + \sum_{j=0}^{n} \left|A_j(\alpha_n)\right|\ , \text{ for some } \alpha_n \in [-1,\, 1]\ .$$

Since $A_j(t_k) = \delta_{jk}$, it is obvious that $\alpha_n \ne \cos\frac{j\pi}{n}$, $0 \le j \le n$, for each n . By choosing $\hat{f} \in C[-1,\, 1]$ such that $\hat{f}(t_j) = -\operatorname{sgn}\left(A_j(\alpha_n)\right)$, $\hat{f}(\alpha_n) = 1$, $\|\hat{f}\|_\infty = 1$, we see that this upper bound is attained and

(17) $$\|R_n\| = 1 + \sum_{j=0}^{n} \left|A_j(\alpha_n)\right|\ .$$

Furthermore, we see from Theorem 1, that if the sequence $\left\{\|R_n\|\right\}_{n=1}^{\infty}$ is unbounded, then the sequence $\{\alpha_n\}_{n=1}^{\infty}$ must cluster either at 1 or -1 . We shall show that the norms are uniformly bounded by showing that $\sum\limits_{j=0}^{n} \left|A_j(\alpha_n)\right|$ is uniformly bounded for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ converging to ± 1 .

Let $\{\alpha_n\}_{n=1}^{\infty}$ be any sequence converging to 1 from the left, $\alpha_n \ne 1$, and n sufficiently large such that $\frac{2}{3} < \alpha_n < 1$. Then there exists a corresponding sequence $\{\theta_n\}$, such that $\alpha_n = \cos\theta_n$, $\theta_n \to \theta^+$. Again let $r(x,\, t) = 1 + xt - 2t^2$, $-1 < x$, $t < 1$. Then $r(\alpha_n,\, \alpha_n) = 1 - \alpha_n^2 > 0$, $r(\alpha_n,\, 1) < 0$, and $r(\alpha_n,\, -1) < 0$. Then $\tau_n = \frac{1}{4}\left(\alpha_n + \sqrt{\alpha_n^2+8}\right)$ and $\sigma_n = \frac{1}{4}\left(\alpha_n - \sqrt{\alpha_n^2+8}\right)$ are the only two zeros of $r(\alpha_n,\, t)$ and $-1 < \sigma_n < 0 < \alpha_n < \tau_n < 1$. (Note that $A_0(x) \ge 0$ and $A_n(x) \ge 0$, $x \in [-1,\, 1]$, and thus we need only consider $1 \le j \le n-1$.)

Let $J_n = \left\{j \neq 0 : \cos \frac{j\pi}{n} \geq \tau_n\right\}$. Then for all $j \in J_n$, $r(\alpha_n, t_j) \leq 0$, and so $|r(\alpha_n, t_j)| < |r(\alpha_n, 1)| = 1 - \alpha_n$. Also, for all $j \in J_n$,

$$\left(\cos \frac{j\pi}{n} - \alpha_n\right)^{-1} \leq (\tau_n - \alpha_n)^{-1} \leq \frac{4}{\sqrt{\alpha_n^2 + 8 - 3\alpha_n}} ,$$

and since $\frac{2}{3} < \alpha_n < 1$, it is easily seen that there exists a constant K_1 , independent of n , such that

$$\text{(18)} \qquad \frac{1}{\left(\cos \frac{j\pi}{n} - \alpha_n\right)^2} \leq \frac{16}{\left(\sqrt{\alpha_n^2 + 8 - 3\alpha_n}\right)^2} \leq \frac{K_1}{(1-\alpha_n)^2} = \frac{K_1}{(1-\cos\theta_n)^2} .$$

Similarly, for $0 \leq \theta \leq \frac{\pi}{2}$, there exists a constant K_2 , independent of θ , such that $(\sin\theta)^{-1} \leq K_2/\theta$, and thus for $j \in J_n$,

$$\text{(19)} \qquad \frac{1}{1-t_j^2} = \frac{1}{\sin^2 \frac{j\pi}{n}} \leq \frac{K_2^2 n^2}{j^2 \pi^2} .$$

Now, $W(\alpha_n)^2 \leq \sin^2\theta_n$, and so for $j \in J_n$,

$$\text{(20)} \qquad |A_j(\alpha_n)| = \frac{W(\alpha_n)^2}{n^2} \frac{|r(\alpha_n, t_j)|}{\left(1-t_j^2\right)(t_j - \alpha_n)^2}$$

$$\leq \frac{K_2^2}{j^2\pi^2} \frac{K_1 \sin^2\theta_n}{1-\cos\theta_n} \leq \frac{M_1}{j^2} ,$$

since $0 \leq \theta_n \leq \frac{\pi}{2}$, and M_1 is independent of n . Thus,

$$\text{(21)} \qquad \sum_{j \in J_n} |A_j(\alpha_n)| \leq \sum_{j=1}^{n} \frac{M_1}{j^2} < \frac{M_1 \pi^2}{6} .$$

Now, let $I_n = \left\{j \neq n : \cos \frac{j\pi}{n} \leq \sigma_n\right\}$. For $j \in I_n$,

$\left(\alpha_n - \cos \frac{j\pi}{n}\right)^{-2} \le 1$, $|r(\alpha_n, t_j)| \le 4$, and

$$\sum_{j\in I_n} |A_j(x)| \le \sum_{j=1}^{n} \frac{M_2 \sin^2\theta_n}{j^2} < \frac{\pi^2}{6} M_2 \sin^2\theta_n \; . \tag{22}$$

For all $j \notin (J_n \cup I_n)$, $r(\alpha_n, t_j) > 0$, and thus $A_j(\alpha_n) > 0$. Since $1 = \sum_{j=0}^{n} A_j(\alpha_n)$, then again

$$\sum_{j=0}^{n} |A_j(\alpha_n)| \le 1 + 2 \sum_{j\in J_n \cup I_n} |A_j(\alpha_n)| \le M \text{ , for all } n \; . \tag{23}$$

These steps can obviously be mimicked for any sequence $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \to -1$. Thus, since the sequence $\{\|R_n\|\}_{n=1}$ is uniformly bounded and $\|R_n(p)-p\|_\infty \to 0$, for any polynomial p , we have by the Banach-Steinhaus Theorem (Goldstein [6], p. 108):

THEOREM 2. *$H_n^1(f)$ is uniformly convergent to f , for all $f \in C[-1, 1]$.*

References

[1] D.L. Berman, "A study of the Hermite-Fejér interpolation process", *Soviet Math. Dokl.* 10 (1969), 813-816.

[2] D.L. Berman, "Extended Hermite-Fejér interpolation processes diverging everywhere", *Soviet Math. Dokl.* 11 (1970), 830-833.

[3] E.W. Cheney, *Introduction to approximation theory* (McGraw-Hill, New York, 1966).

[4] Philip J. Davis, *Interpolation and approximation* (Blaisdell, New York, Toronto, London, 1963; reprinted, 1965).

[5] Leopold Fejér, "Ueber Interpolation", *Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. Kl.* 1916, 66-91.

[6] Allen A. Goldstein, *Constructive real analysis* (Harper and Row, New York, Evanston, London; John Weatherhill, Tokyo; 1967).

[7] H.E. Salzer, "Lagrangian interpolation at the Chebyshev points $x_{n,\nu} \equiv \cos(\nu\pi/n)$, $\nu = 0(1)n$; some unnoted advantages", *Computer J.* 15 (1972), 156-159.

[8] Martin Avery Snyder, *Chebyshev methods in numerical approximation* (Prentice-Hall, Englewood Cliffs, New Jersey, 1966).

[9] Gabor Szegö, *Orthogonal polynomials* (Colloquium Publ. 23. Amer. Math. Soc., New York, 1939; revised edition 1959, reprinted with corrections 1967).

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