THE CLASSICAL MODULAR GROUP AS A SUBGROUP OF GL(2, $\mathbb{Z}$)†

by MORRIS NEWMAN

For R. A. Rankin, on the occasion of his 70th birthday

The title is somewhat misleading, since the classical modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ is certainly not a subgroup of $\mathrm{GL}(2, \mathbb{Z})$. What is meant of course are the faithful representations of Γ as a subgroup of $\mathrm{GL}(2, \mathbb{Z})$, where Γ is to be thought of as the free product of a cyclic group of order 2 and a cyclic group of order 3. No such representation is possible as a subgroup of $\mathrm{SL}(2, \mathbb{Z})$; it is necessary to have matrices of determinant -1 as well.

Thus we seek all matrices A, B of $\mathrm{GL}(2, \mathbb{Z})$ such that A is of period 2, B is of period 3, and the group generated by A and B satisfies

$$\{A, B\} = \{A\} * \{B\}, \tag{1}$$

the free product of the cyclic group $\{A\}$ of order 2 and the cyclic group $\{B\}$ of order 3.

It is clear that if (1) holds, then neither A nor B can be scalar. Furthermore, we are at liberty to replace A by SAS^{-1} and B by SB^eS^{-1}, where S is any element of $\mathrm{GL}(2, \mathbb{Z})$ and $e = \pm 1$. This allows us to choose a specially simple form for A, which facilitates the analysis.

For notational simplicity, the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will be written as

$$[a, b; c, d].$$

We require some preliminary lemmas.

LEMMA 1. *Let A be any non-scalar element of $\mathrm{GL}(2, \mathbb{Z})$ of period 2. Then A is similar over $\mathrm{GL}(2, \mathbb{Z})$ to $[1, f; 0, -1]$, where f may be chosen modulo 2.*

Proof. This lemma is well-known in a much more general form. For a convenient reference, see [**2**, p. 54].

LEMMA 2. *Suppose that*

$$T = [1, 0; 0, -1], \qquad R = [a, b; -c, -1-a],$$

where a, b, c are real numbers such that

$$bc = a^2 + a + 1, \qquad a \geq 0, \quad b > 0, \quad c > 0.$$

Then $\{T, R\}$, the group generated by T and R, is just $\{T\} * \{R\}$, the free product of the cyclic group $\{T\}$ of order 2 and the cyclic group $\{R\}$ of order 3.

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Proof. Certainly $T^2=R^3=I$. We have $TR=[a,b;c,1+a]$, $TR^2=TR^{-1}=-[1+a,b;c,a]$. The diagonal elements of TR are nonnegative, and at most one diagonal element may be 0. The off-diagonal elements of TR are positive. Both observations also hold for $-TR^2$. It is then easy to show that the semigroup generated by TR and TR^2 is free, which in turn implies the truth of the lemma. A detailed proof along these lines of a related result may be found in [**1**].

LEMMA 3. *Let* $A=[a,b;c,-a]$, $a^2+bc=1$, *be any non-scalar element of* $\mathrm{GL}(2,\mathbb{Z})$ *of period* 2. *Suppose that* A *is similar over* $\mathrm{GL}(2,\mathbb{Z})$ *to* $[1,f;0,-1]$. *Then* f *is even if and only if* b *and* c *are even.*

Proof. Let $S=[x,y;u,v]$ be an element of $\mathrm{GL}(2,\mathbb{Z})$ such that

$$SAS^{-1}=[1,f;0,-1]. \tag{2}$$

There are four cases.

(i) $A\equiv I \bmod 2$. Then (2) implies that $[1,f;0,-1]\equiv I \bmod 2$, so that f is even.

(ii) $A\equiv[1,0;1,1] \bmod 2$. Then (2) implies that $f\equiv y \bmod 2$, $v\equiv 0 \bmod 2$. But $v\equiv 0 \bmod 2$ implies that $y\equiv 1 \bmod 2$, so that f is odd.

(iii) $A\equiv[1,1;0,1] \bmod 2$. Then (2) implies that $f\equiv x \bmod 2$, $u\equiv 0 \bmod 2$. But $u\equiv 0 \bmod 2$ implies that $x\equiv 1 \bmod 2$, so that f is odd.

(iv) $A\equiv[0,1;1,0] \bmod 2$. Then (2) implies that $f\equiv x+y \bmod 2$, $u\equiv v \bmod 2$. But $u\equiv v \bmod 2$ implies that $1\equiv xu+yv\equiv u(x+y) \bmod 2$, which in turn implies that $x+y\equiv 1 \bmod 2$, so that f is odd.

This completes the proof.

The first theorem we wish to prove is the following.

THEOREM 1. *Let* A, B *be non-scalar elements of* $\mathrm{GL}(2,\mathbb{Z})$ *of periods* 2, 3 *respectively. Suppose that* $A\equiv I \bmod 2$. *Then* $\{A,B\}$, *the group generated by* A *and* B, *is just* $\{A\}*\{B\}$, *the free product of the cyclic group* $\{A\}$ *of order* 2 *and the cyclic group* $\{B\}$ *of order* 3.

Proof. By Lemmas 1 and 3, we may assume that an element S of $\mathrm{GL}(2,\mathbb{Z})$ exists such that

$$T=SAS^{-1}=[1,0;0,-1], \tag{3}$$

$$R=SBS^{-1}=[a,b;-c,-a-1], \tag{4}$$

where a, b, c are integers such that $bc=a^2+a+1$. Since in addition R may be replaced by R^{-1}, and $R^{-1}=[-a-1,-b;c,a]$, we may assume that in (4) $a\geqslant 0$. Now the similarity $TTT^{-1}=T$, $TRT^{-1}=[a,-b;c,-a-1]$ leaves T unchanged, leaves the diagonal elements of R unchanged, but changes the signs of the off-diagonal elements of R. Hence we may also assume that in (4) $b>0$, $c>0$, since b and c are different from 0 and of the same sign. But now Lemma 2 implies the truth of the theorem, and the proof is concluded.

We must now consider the case when the element of period 2 is not similar over $\mathrm{GL}(2,\mathbb{Z})$ to $[1,0;0,-1]$. Once again, Lemmas 1 and 3 allow us to assume that the

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element T of period 2 may be taken as

$$T=[1,-1;0,-1], \tag{5}$$

and the element R of period 3 may be taken as

$$R=[a,b;-c,-a-1], \tag{6}$$

where $bc=a^2+a+1$, and $b>0, c>0$ (it may be necessary to replace R by its inverse to achieve the positivity of b and c). We then get the next theorem.

THEOREM 2. *Suppose that T and R are as given in* (5) *and* (6), *respectively. Suppose in addition that in* (6)

$$a\geqslant 0. \tag{7}$$

Then $\{T,R\}=\{T\}*\{R\}$, the free product of the cyclic group $\{T\}$ of order 2 and the cyclic group $\{R\}$ of order 3.

Proof. We have $TR=[a+c,a+b+1;c,a+1]$, $TR^2=TR^{-1}=-[a+c+1,a+b;c,a]$. Thus the semigroup generated by TR and TR^2 is free, and the conclusion follows.

The case when $a<0$ cannot be treated completely (indeed, the desired theorem no longer holds) but a partial answer is possible, as in the next theorem.

THEOREM 3. *Suppose that T and R are as given in* (5) *and* (6), *respectively. Suppose in addition that in* (6)

$$a<0, 2a+c\geqslant 0. \tag{8}$$

Then $\{T,R\}=\{T\}*\{R\}$, the free product of the cyclic group $\{T\}$ of order 2 and the cyclic group $\{R\}$ of order 3.

Proof. Put $U=[1,-\frac{1}{2};0,1]$ (so that U does not belong to $\mathrm{GL}(2,\mathbb{Z})$, but does belong to $\mathrm{SL}(2,\mathbb{Q})$). Then

$$UTU^{-1}=[1,0;0,-1],$$
$$URU^{-1}=[a+\tfrac{1}{2}c,b';-c,-1-a-\tfrac{1}{2}c],$$

where $4b'=4a+4b+2+c$ must be positive, since c is positive and $4b'c=(2a+c+1)^2+3$. But then Lemma 2 implies the result, and the proof is concluded.

Finally, we point out that the desired result is not universally true; for example, if

$$T=[1,-1;0,-1], \qquad R=[-1,1;-1,0],$$

then

$$T^2=R^3=(TR)^2=I.$$

In this case $\{T,R\}$ is finite, and is in fact just the dihedral group of order 6.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA
CALIFORNIA 93106
U.S.A.