

EULER CLASSES OF COMBINATORIAL MANIFOLDS

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0. Introduction. Every finite simplicial complex has a tangent bundle in the category of simplicial bundles (see [9]). The goal of this paper is to classify simplicial bundles, and, as an application of this result, to construct Euler classes for a large class of combinatorial manifolds. This construction is closely related to [3] and [4].

In Section 1, after briefly reviewing the definition of simplicial bundles and the construction of tangent bundles of simplicial complexes, we classify simplicial bundles over a given simplicial complex as classes of maps from the original complex to a fixed classifying space. In Section 2 we use this result, and in particular the representation of tangent bundles, to construct Euler classes as follows:

Let M be a closed combinatorial m -manifold. We associate to M a closed cellular m -manifold M^* , the linear dual of M (M^* is a “linearized” form of the dual cell complex of M). There is a natural chain complex $(C_*^{\text{co}}(M^*), \partial)$ of cooriented cells associated to M^* , and associating to each oriented p -simplex of M its cooriented $(m - p)$ -dimensional dual defines a map g_* from the oriented simplicial p -chains $C_p^{\circ}(M)$ on M (with Z coefficients) to $C_{m-p}^{\text{co}}(M^*)$ for which the following diagram commutes

$$\begin{array}{ccc} C_p^{\circ}(M) & \xrightarrow{g_*} & C_{m-p}^{\text{co}}(M^*) \\ \downarrow \partial & & \downarrow \partial \\ C_{p-1}^{\circ}(M) & \xrightarrow{g_*} & C_{m-(p-1)}^{\text{co}}(M^*) \end{array}$$

For each m there is a chain complex of groups $(\mathcal{C}_*(m), \partial)$, and $H^*(\mathcal{C}(m))$ denotes the homology of the associated complex $(\text{Hom}_Z(\mathcal{C}_*(m), \mathbf{R}), d)$. There is a class $e \in H^{2n}(\mathcal{C}(2n))$ which plays the role of a “universal Euler class”: If M is an oriented $2n$ -dimensional combinatorial manifold, the classifying map f_{M^*} of the tangent bundle of its linear dual M^* induces a map $(f_{M^*})_*$ from $C_{2n-p}^{\text{co}}(M^*)$ to $\mathcal{C}_p(2n)$ for

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which the following diagram commutes

$$\begin{array}{ccc}
 C_{2n-p}^{\text{co}}(M^*) & \xrightarrow{(f_{M^*})_*} & \mathcal{C}_p(2n) \\
 \downarrow \partial & & \downarrow \partial \\
 C_{2n-(p-1)}^{\text{co}}(M^*) & \xrightarrow{(f_{M^*})_*} & \mathcal{C}_{p-1}(2n)
 \end{array} ,$$

and $g^*(f_{M^*})_*e \in H^{2n}(M)$ is the Euler class of M .

The relationship between $e \in H^{2n}(\mathcal{C}(2n))$ and the usual universal Euler class $\tilde{e} \in H^{2n}(BSO(2n))$ is explained in Section 3: For each m there is a natural chain map h_* from $\mathcal{C}_*(m)$ to the singular chains $C_*(BSO(m))$ on $BSO(m)$, and if $m = 2n$, then $h^*\tilde{e} = e$. It should be noted that the composite

$$h_*(f_{M^*})_*g_* : C_*^o(M) \rightarrow C_*(BSO(m))$$

is closely related to the homological Gauss map arising in the combinatorial formula for the first Pontrjagin class as described in [3], [4], and [6].

A construction similar to our construction of Euler classes could be made without using the classification theorem for simplicial bundles if the linear dual M^* of M could be rectilinearly embedded (or immersed) in some Euclidean space. However, in contrast to the behavior of simplicial complexes (i.e. simplicial polyhedra) it is not true that all cellular polyhedra (such as M^*) rectilinearly embed (or immerse) in any Euclidean space even locally (see Example 7). Thus the use of the classification theorem in this construction avoids a non-trivial topological problem.

The construction we present is valid for any closed combinatorial manifold for which a linear dual exists. Although it is not clear whether or not all combinatorial manifolds have linear duals, linear duals do exist for all combinatorial 2-manifolds, for combinatorial spheres, and for combinatorial projective spaces.

The classifying map f_{M^*} of the tangent bundle of the linear dual M^* can clearly be replaced by the classifying map of any of a class of simplicial bundles over M^* . The results obtained by doing this will be considered in a future paper.

1. The classification theorem. We first clarify the following terms as they will be used in the sequel: A *simplicial complex* K is a set of simplices which can be described in terms of a fixed set of vertices and various face relations. A *polyhedron* $|K|$ is the geometric realization of a simplicial complex K (i.e. a topological space) together with a fixed triangulation of the space given by the structure of K . Thus, for example, if K' is the first barycentric subdivision of K then $K \neq K'$ (since K and K'

are different sets), and moreover $|K| \neq |K'|$ (since the triangulations of the two spaces are different).

This terminology is certainly non-standard since one normally considers a simplicial complex and its first barycentric subdivision to have the same polyhedra. Geometrically, however, they are different: there are, for example, more piecewise smooth differential forms on the first barycentric subdivision of a polyhedron than there are on the original polyhedron. Since the origins of this paper are geometric (see [8], [9], and [10]) it is thus natural to adopt this terminology.

Henceforth all simplicial complexes and polyhedra are finite.

We now define simplicial bundles. In [9] it was shown that every polyhedron has a tangent bundle in the category of simplicial bundles in much the same way that every smooth manifold has a tangent bundle in the category of smooth vector bundles.

Henceforth “vector space” means \mathbf{R}^N , $N = 0, 1, 2, \dots$ with the usual topology.

1. *Definition.* A *simplicial bundle* is a continuous surjection $\xi : E \rightarrow P$ from a topological space E to a polyhedron P such that $\xi^{-1}(x)$ is a vector space for each $x \in P$ and

a) for each simplex $\sigma_\alpha \in P$ there is a vector space F_α and a fiber preserving embedding

$$\phi_\alpha : \sigma_\alpha \times F_\alpha \rightarrow \xi^{-1}(\sigma_\alpha)$$

which is a linear injection on each fiber and which is a linear isomorphism on each fiber which lies over the interior of σ_α ,

b) if $\sigma_\alpha \subseteq \sigma_\beta$ for simplices σ_α and σ_β in P then

$$\phi_\beta(x, -) = \phi_\alpha(x, -) \circ l_{\alpha\beta}$$

for each $x \in \sigma_\alpha$ and some fixed linear injection $l_{\alpha\beta} : F_\beta \rightarrow F_\alpha$, and

c) $l_{\alpha\gamma} = l_{\alpha\beta} \circ l_{\beta\gamma}$ whenever $\sigma_\alpha \subseteq \sigma_\beta \subseteq \sigma_\gamma$.

Simplicial bundles and the appropriate morphisms form a category. This category contains a subcategory of vector bundles. Simplicial bundles differ from vector bundles, however, since fiber dimensions of simplicial bundles are allowed to vary. The standard example of a simplicial bundle is illustrated in Diagram 1; observe that $\xi^{-1}(x) = \mathbf{R}^1$ for $x \in P - \{x_0\}$ and $\xi^{-1}(x_0) = \mathbf{R}^2$.

Remark. Definition 1 is not precisely the same definition of simplicial bundle given in [9]; first axiom 5 of the definition given in [9] has been dropped, and second the point of view of the resulting definition has been changed. Axiom 5 has been dropped since it makes the definition of simplicial bundles too restrictive in the sense that pullbacks of simplicial bundles are not always simplicial bundles. The point of view of the resulting definition has been changed since that definition would permit

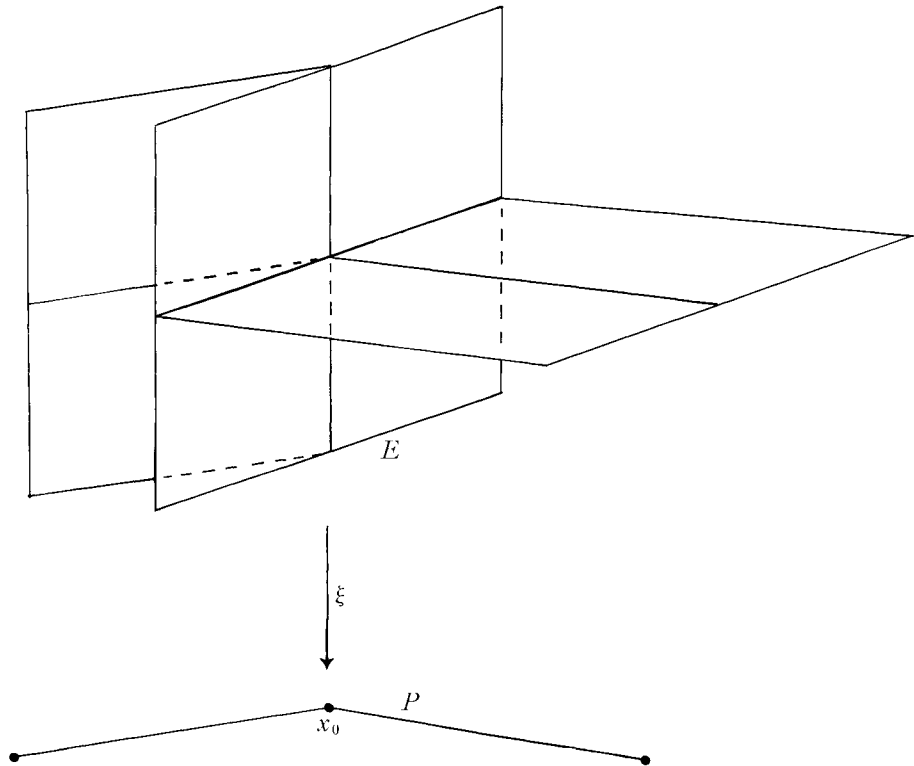


Diagram 1

the existence of two simplicial bundles $\xi' : E' \rightarrow P$ and $\xi'' : E'' \rightarrow P$ for which $E' = E''$, $\xi' = \xi''$ (as maps), and yet $\xi' \neq \xi''$ (as simplicial bundles) since the analytic data describing these bundles are different. Definition 1 alleviates these problems, yet it still describes tangent bundles of polyhedra as we now show. (This calculation will be referred to later on.)

Let P be a polyhedron. A *small open neighborhood* U of $x_0 \in P$ is the intersection of the open star of x_0 with any other open neighborhood of x_0 . The intersection U_α of a small open neighborhood U of x_0 with any simplex σ_α containing x_0 is a *wedge* of U . A *small open subset* of P is a subset which is a small open neighborhood of some point in P .

If U is a small open neighborhood of x_0 , a *coordinate system* on U with origin x_0 is a collection $u = \{u_i\}$ of real-valued piecewise affine functions u_i defined on U such that for each wedge U_α of U , $u_i|_{U_\alpha} = 0$ for all but a subset of $\dim U_\alpha$ functions $u_i \in u$, the set of $u_i|_{U_\alpha}$ for u_i in this subset forms a coordinate system on U_α with origin x_0 , and if $u_i|_{U_\alpha}$ and $u_j|_{U_\beta}$ are coordinate functions on U_α and U_β , respectively, then $u_i|_{U_\alpha \cap U_\beta}$ is a coordinate function on $U_\alpha \cap U_\beta$. Such coordinate systems on small open subsets generalize barycentric coordinate systems on vertex stars.

The rules for change of coordinates are:

1) If u is a coordinate system on U with origin x_0 , and $W \subseteq U$ is a small open neighborhood of y_0 , then there is an induced coordinate system on W with origin y_0 given by

$$w = \{u_i - u_i(y_0) : W \rightarrow R \mid u_i \in u \text{ and } u_{i|W} \neq 0\}$$

where $u_i(y_0)$ is a constant function.

2) If u and w are coordinate systems on U with origin x_0 then there are real constants c_{ij} for which $u_i = \sum_j c_{ij}w_j$ for each $u_i \in u$ where $w_j \in w$. In fact if the first p functions of both u and w coordinatize the p -simplex in whose interior x_0 is contained, then, after suitably permuting u_{p+1}, \dots, u_N , the matrix (c_{ij}) is of the form

$$(c_{ij}) = \left(\begin{array}{c|c} \text{invertible} & * \\ \hline p \times p \text{ matrix} & \text{invertible} \\ \hline 0 & \text{diagonal} \\ & \text{matrix} \end{array} \right).$$

The tangent bundle $t(P) : TP \rightarrow P$ of the polyhedron P is constructed as follows:

Locally: Let U be a small open neighborhood of x_0 and let u be a coordinate system on U with origin x_0 of cardinality N . Let $\pi^{-1}(U, u)$ be the set of all $(x, v = \sum_i c_i(\partial/\partial u_i))$ in $U \times \mathbf{R}^N$ such that $c_i = 0$ if $x \notin \text{support } u_i$, and let $\pi : \pi^{-1}(U, u) \rightarrow U$ denote projection onto the first factor.

Globally: Let $TP = \coprod \pi^{-1}(U, u)/\sim$, the free union of $\pi^{-1}(U, u)$ for all possible combinations of U and u modulo the equivalence relation \sim : for (x_0, v_0) in $\pi^{-1}(U_0, u_0)$ and (x_1, v_1) in $\pi^{-1}(U_1, u_1)$, $(x_0, v_0) \sim (x_1, v_1)$ if and only if $x_0 = x_1$ and after inducing coordinate systems w_0 and w_1 on a small open neighborhood $W \subseteq U_0 \cap U_1$ of $x_0 = x_1$,

$$\begin{aligned} v_1 &= \sum_j c_j(\partial/\partial w_j) \text{ for } w_j \in w_1 \Rightarrow \\ v_0 &= \sum_{ij} c_j c_{ij}(\partial/\partial w_i) \text{ for } w_i \in w_0 \end{aligned}$$

where (c_{ij}) is the transition matrix from w_0 to w_1 . Let $t(P) : TP \rightarrow P$ be the induced projection. Then $t(P)$ is a (simplicial bundle) called the tangent bundle of P .

For example, the simplicial bundle in Diagram 1 is the tangent bundle of the polyhedron P . In contrast, the tangent bundle of the polyhedron P consisting of just one 1-simplex is the product bundle $t(P) : TP = P \times \mathbf{R}^1 \rightarrow P$. Note that tangent bundles of polyhedra are not, in general, vector bundles, and that the structure of the tangent bundle of a polyhedron depends on the triangulation of the polyhedron.

2. PROPOSITION. *The tangent bundle $t(P)$ of a polyhedron P is a simplicial bundle.*

Proof. For a simplex σ_α in P let $F_\alpha = t(P)^{-1}x_\alpha$ for some fixed x_α in the interior of σ_α . To define the embeddings ϕ_α , fix σ_α and let $x \in \sigma_\alpha$. Then x_α is in the open star $\text{St } x$ of x . If u is a coordinate system on $\text{St } x$ with origin x , then $t(P)^{-1}x$ may be identified as the vector space generated by $\{(\partial/\partial u_i)_x \mid u_i \in u\}$. Since a subset of u generates a coordinate system on $\text{St } x_\alpha$ with origin x_α by translation, $t(P)^{-1}x_\alpha$ may be identified as the vector space generated by the vectors $(\partial/\partial u_i)_{x_\alpha}$ for u_i in the translated coordinate system. Define ϕ_α on generators by

$$\phi_\alpha(x, (\partial/\partial u_i)_{x_\alpha}) = (x, (\partial/\partial u_i)_x).$$

If σ_α and σ_β are simplices of P for which $\sigma_\alpha \subseteq \sigma_\beta$, then $x_\beta \in \text{St } x_\alpha$, and translation defines a linear injection $l_{\alpha\beta} : F_\beta \rightarrow F_\alpha$.

Morphisms of simplicial bundles are defined in the obvious manner. Simplicial bundles $\xi' : E' \rightarrow P$ and $\xi'' : E'' \rightarrow P$ are *equivalent* if and only if there are morphisms $f : \xi' \rightarrow \xi''$ and $g : \xi'' \rightarrow \xi'$ for which $f \circ g$ and $g \circ f$ are the appropriate identities.

3. PROPOSITION. *Every simplicial bundle $\xi : E \rightarrow P$ is equivalent to a simplicial bundle of the form $\pi : \coprod(\sigma_\alpha \times F_\alpha)/\sim \rightarrow P$ where $(x_\alpha, v_\alpha) \sim (x_\beta, v_\beta)$ if and only if $\phi_\alpha(x_\alpha, v_\alpha) = \phi_\beta(x_\beta, v_\beta)$ and π is induced by first factor projections.*

Proof. This is simply the fact that $\{\phi_\sigma(\sigma_\alpha \times F_\alpha) \mid \sigma_\alpha \subseteq P\}$ is a neighborhood finite closed cover of P .

Associating to each polyhedron P the set $\text{Sxl } P$ of simplicial bundles over P defines a contravariant functor on the category of polyhedra with values in the category of sets.

For $N = 0, 1, 2, \dots$, a *decomposition* $D = \{V_\alpha\}_{\alpha \in A}$ in \mathbf{R}^N is a finite set of subspaces $V_\alpha \subseteq \mathbf{R}^N$ such that the smallest subspace $V_D \subseteq \mathbf{R}^N$ which contains $\cup_{\alpha \in A} V_\alpha$ is in D ; V_D is the *maximal subspace* of D . (At times we identify a decomposition D in \mathbf{R}^N with a decomposed vector space $V \subseteq \mathbf{R}^N$: V is the maximal subspace of D , and the decomposition of V is determined by the elements of D .) Decompositions $D' = \{V'_\alpha\}_{\alpha \in A}$ and $D'' = \{V''_\alpha\}_{\alpha \in A}$ in \mathbf{R}^N (both with index set A) are isomorphic, written $D' \sim D''$, if and only if there is a $T \in \text{Gl}(\mathbf{R}, N)$ for which $D' = TD''$ (i.e. $\{V'_\alpha\}_{\alpha \in A} = \{TV''_\alpha\}_{\alpha \in A}$). Isomorphism of decompositions is an equivalence relation, and the *decomposition space* $[D]$ is the equivalence class of all decompositions in \mathbf{R}^N isomorphic to D with the topology which has for a basis all sets of the form $\{TD \mid T \in U\}$ for U a basic open neighborhood of $1 \in \text{Gl}(\mathbf{R}, N)$.

For an abstract simplicial complex K let $C(K, \coprod_p G_p(\mathbf{R}^\infty))$ denote the set functions $f : K \rightarrow \coprod_p G_p(\mathbf{R}^\infty)$, $G_p(\mathbf{R}^\infty)$ the Grassman manifold of p -planes in \mathbf{R}^∞ , for which

(*) if $\sigma \subseteq \tau$ for simplices σ and τ of K then $f(\tau) \subseteq f(\sigma)$

(this condition is an analog of continuity). Let $(K, \coprod_p G_p(\mathbf{R}^\infty))$ denote the set of equivalence classes of $C(K, \coprod_p G_p(\mathbf{R}^\infty))$ under the relation \sim where $f_0 \sim f_1$ if and only if for each simplex σ of K , $D_0 = \{f_0(\tau) \mid \sigma \subseteq \tau\}$ and $D_1 = \{f_1(\tau) \mid \sigma \subseteq \tau\}$ are isomorphic decompositions, and there is a function $f : K \times [0, 1] \rightarrow \coprod_p G_p(\mathbf{R}^\infty)$ such that

- a) $f(-, 0) = f_0$ and $f(-, 1) = f_1$,
- b) $f(-, t) \in C(K, \coprod_p G_p(\mathbf{R}^\infty))$ for each $t \in [0, 1]$, and
- c) for each simplex σ of K , $f(\sigma, -)$ is a path in the decomposition space determined by σ from D_0 to D_1 .

The following result is a special version of a complete classification theorem for simplicial bundles which we will soon state. The proof of the special version is presented since this special version is what will be used in the application; the proof of the complete classification is sketched.

4. THEOREM (Special Classification). *There is an injective natural transformation of contravariant functors defined on the category of simplicial complexes*

$$F : (-, \coprod_p G_p(\mathbf{R}^\infty)) \rightarrow \text{Sxl } | - |$$

where, for a fixed simplicial complex K , $\text{Sxl } P$ is the set of isomorphism classes of simplicial bundles over the polyhedron $P = |K|$ of K .

Note: What distinguishes a result as a ‘‘classification theorem’’ is that the result replaces something uncomputable by something computable. The fact that Theorem 4 is in this sense indeed a ‘‘classification theorem’’ will be illustrated by the application in Section 2.

Proof. For $p = 0, 1, 2, \dots$, let $\gamma^p : E_p(\mathbf{R}^\infty) \rightarrow G_p(\mathbf{R}^\infty)$ denote the universal p -plane bundle over $G_p(\mathbf{R}^\infty)$.

Given $[f] \in (K, \coprod_p G_p(\mathbf{R}^\infty))$ we let $I[f] = [\xi_f]$ in $\text{Sxl } P$ where E_f is the subspace of $\coprod_p E_p(\mathbf{R}^\infty) \times P$

$$E_f = \{((f(\sigma), v), x) \mid x \in \sigma \text{ and } v \in f(\sigma)\}$$

and $\xi_f : E_f \rightarrow P$ is given by

$$\xi_f((f(\sigma), v), x) = x;$$

ξ_f is a simplicial bundle: For every simplex σ_α in P we let $F_\alpha = f(\sigma_\alpha)$, and we define

$$\phi_\alpha : \sigma_\alpha \times F_\alpha \rightarrow \xi_f^{-1}(\sigma_\alpha)$$

by

$$\phi_\alpha(x, v) = ((f(\sigma_\alpha), v), x).$$

If σ_α and σ_β are simplices of P for which $\sigma_\alpha \subseteq \sigma_\beta$ then $f(\sigma_\beta) \subseteq f(\sigma_\alpha)$ so

$F_\beta \subseteq F'_\alpha$. If $l_{\alpha\beta} : F_\beta \rightarrow F'_\alpha$ is inclusion, then

$$\phi_\beta(x, -) = \phi_\alpha(x, -) \circ l_{\alpha\beta}$$

for each $x \in \sigma_\alpha$ and if $\sigma_\alpha \subseteq \sigma_\beta \subseteq \sigma_\gamma$, then $l_{\alpha\gamma} = l_{\alpha\beta} \circ l_{\beta\gamma}$.

To show that F is well defined, one proceeds along the same lines as in the proof of the corresponding fact in the classification theorem for vector bundles (see [5]). There are essentially only two new facts which must be used:

For a specific decomposition $D = \{V_\alpha\}_{\alpha \in A}$ in \mathbf{R}^N , the decomposition space $[D]$ is homeomorphic to $Gl(\mathbf{R}, N)/H$ where

$$H = \{T \in Gl(\mathbf{R}, N) \mid TD = \{TV_\alpha\}_{\alpha \in A} = D\};$$

H is a closed subgroup of $Gl(\mathbf{R}, N)$, hence a Lie subgroup of $Gl(\mathbf{R}, N)$, and consequently the canonical surjection from $Gl(\mathbf{R}, N)$ to $[D]$ is a fibration. The first fact needed is that if

$$f : K \times [0, 1] \rightarrow \coprod_p G_p(\mathbf{R}^\infty)$$

is a homotopy between $f_0, f_1 \in C(K, \coprod_p G_p(\mathbf{R}^\infty))$, as described earlier, then for every simplex σ of P (and in particular for every vertex v of P) there is a map $\gamma_\sigma : [0, 1] \rightarrow Gl(\mathbf{R}, N)$ for which

$$f(\sigma, t) = \gamma_\sigma(t) \cdot f(\sigma, 1) \quad \text{and} \quad \gamma_\sigma(1) = 1.$$

The second fact is that the construction of γ_σ is natural in the following sense: if v_0 and v_1 are vertices of the simplex σ , so that $f(\sigma, t) \subseteq f(v_0, t)$ and $f(\sigma, t) \subseteq f(v_1, t)$ for every t , then

$$\gamma_{v_0}(t) \cdot f(\sigma, 1) = \gamma_{v_1}(t) \cdot f(\sigma, 1)$$

as decomposed vector spaces for each t (i.e. $\gamma_{v_0}(t)l^* = \gamma_{v_1}(t)l^*$ for each subspace l^* of $f(\sigma, 1)$). To see this, let $v = v_0, v_1$. Since $v \in \sigma, f(\sigma, 1) \subseteq f(v, 1)$. Let

$$S = \{T \in Gl(\mathbf{R}, N) \mid Tf(\sigma, 1) \subseteq f(v, 1)\};$$

clearly $1 \in S$. Introduce the equivalence relation \sim on S : $T' \sim T''$ if and only if $T'V = T''V$ for every subspace V of $f(\sigma, 1)$. It is easy to show that there are only a finite number of equivalence classes of S with respect to this relation, and that these classes are the connected components of S . Since $f(\sigma, t) \subseteq f(v, t)$ for every t ,

$$\gamma_v^{-1}(t) \cdot \gamma_\sigma(t) \in S \quad \text{for each } t;$$

since $\gamma_v^{-1}(1) \cdot \gamma_\sigma(1) = 1$, this implies that $\gamma_v^{-1} \cdot \gamma_\sigma$ maps $[0, 1]$ to the component of S containing 1. Thus

$$\gamma_v^{-1}(t) \cdot \gamma(t)l^* = l^*$$

for every $t \in [0, 1]$ and subspace V of $f(\sigma, 1)$. Since this is true for both $v = v_0$ and $v = v_1$, it follows that for every $t \in [0, 1]$ and subspace $V \in f(\sigma, 1)$,

$$\gamma_{v_0}(t)V = \gamma_\sigma(t)V = \gamma_{v_1}(t)V.$$

To show that F is injective, assume that $f_0, f_1 \in C(K, \coprod_p G_p(\mathbf{R}^\infty))$ and that $k : E_{f_0} \rightarrow E_{f_1}$ gives an equivalence between ξ_{f_0} and ξ_{f_1} . Altering f_0 and f_1 by homotopies, if necessary, we can assume, without loss of generality, that

$$f_0(K) = \cup_\sigma f_0(\sigma) \subseteq \mathbf{R}^{\text{even}} = \{v \in \mathbf{R}^\infty \mid v = (0, v_0, 0, v_1, 0, \dots)\}$$

and that

$$f_1(K) = \cup_\sigma f_1(\sigma) \subseteq \mathbf{R}^{\text{odd}} = \{v \in \mathbf{R}^\infty \mid v = (v_0, 0, v_1, 0, v_2, \dots)\}.$$

If

$$\begin{aligned} h : E_{f_0} \times [0, 1] &\rightarrow \mathbf{R}^\infty \\ ((f_0(\sigma), v), x, t) &\rightarrow (1 - t)v + tk(v) \end{aligned}$$

then we can define the desired homotopy by

$$\begin{aligned} f : K \times [0, 1] &\rightarrow \coprod_p G_p(\mathbf{R}^\infty) \\ (\sigma, t) &\rightarrow V = \{h((f_0(\sigma), v), x, t) \mid x \in \sigma \text{ and } v \in f(\sigma)\}. \end{aligned}$$

The proof that F is natural is straightforward.

As the simplicial bundle over the triangle $P = \Delta^1 = S^1$ in Diagram 2 illustrates, F need not be surjective: there is no $f \in C(K, \coprod_p G_p(\mathbf{R}^\infty))$ for which $\xi_f = \xi$.

5. THEOREM. *For every polyhedron $P = |K|$ there is an $[f] \in (K, \coprod_p G_p(\mathbf{R}^\infty))$ for which $F[f] = t(P)$.*

Proof. Given a representative $t : TP \rightarrow P$ of $t(P) \in \text{Sxl } P$, we construct a Gauss map $g : TP \rightarrow \mathbf{R}^\infty$ such that if $f \in C(K, \coprod_p G_p(\mathbf{R}^\infty))$ is given by $f(\sigma) = g(t^{-1}(x))$ for x in the interior of σ , then $F[f] = t(P)$.

For each simplex σ_α in K pick a specific structure map

$$\phi_\alpha : \sigma_\alpha \times F_\alpha \rightarrow TP$$

of $t(P)$, and let $l_{\alpha\beta} : F_\beta \rightarrow F_\alpha$ be the associated transition maps. Let v_0, \dots, v_N be the vertices of K . For each v_j , the dimension of the associated fiber F_j is the number of vertices v_k in the link of v_j (= the number of 1-simplices (v_j, v_k) for v_k in the link of v_j); in fact for each such v_k there is a canonical vector $e_k^j \in F_j$ which lies along (v_j, v_k) . For $i = 0, \dots, N$, K_i denotes the subcomplex of K consisting of all simplices which have vertices v_0, \dots, v_i , and $P_i = |K_i|$; thus

$$\{v_0\} = P_0 \subseteq P_1 \subseteq \dots \subseteq P_N = P.$$

The construction of g proceeds inductively on i :

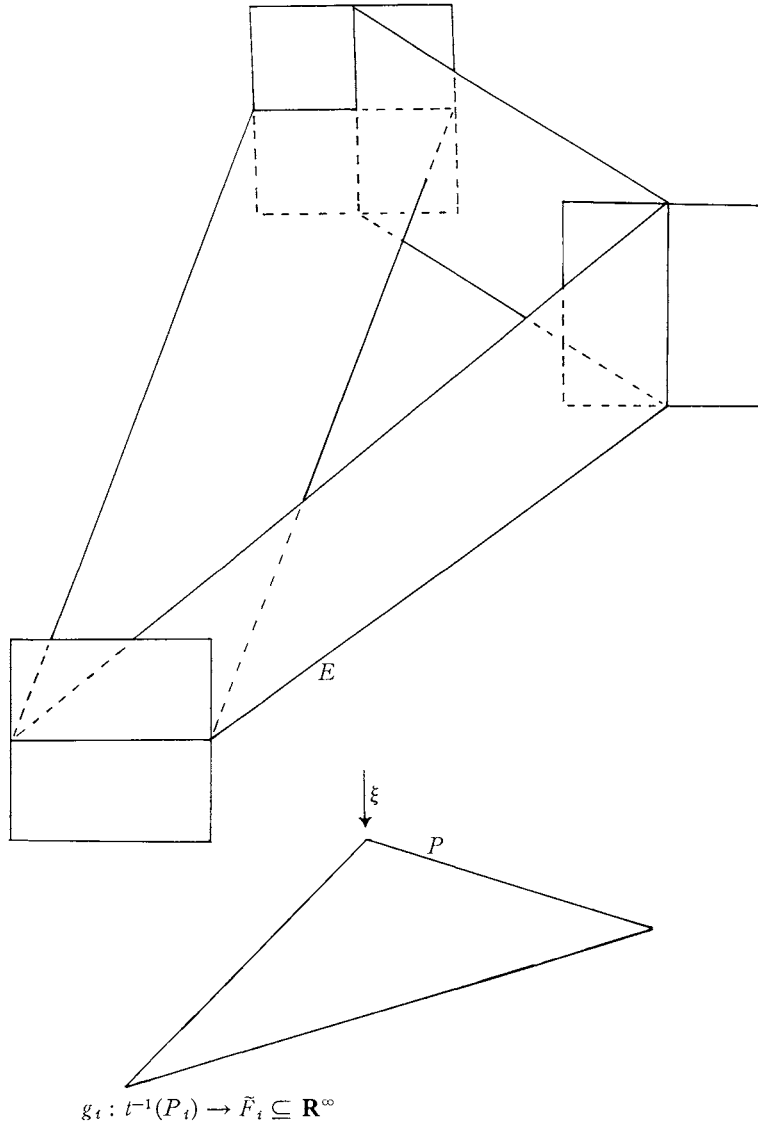


Diagram 2

First there is an obvious isomorphism

$$g_0 = \phi_0^{-1}(v_0, -) : t^{-1}(v_0) \rightarrow \tilde{F}_0 = F_0 \subseteq \mathbf{R}^\infty.$$

Next assume (inductively) that

$$g_i : t^{-1}(P_i) \rightarrow \tilde{F}_i \subseteq \mathbf{R}^\infty$$

is given and that for any fixed $k > i$ the vectors $\{g_i(\phi_j(v_j, e_k^j)) \mid j \leq i\}$ are linearly independent. We define g_{i+1} as follows:

Let F_{i+1}^0 denote the subspace of F_{i+1} generated by $\{e_j^{i+1} \mid j \leq i\}$, and let $l : F_{i+1}^0 \rightarrow \tilde{F}_i$ be the linear map defined by

$$l(e_j^{i+1}) = g_i(\phi_j(v_j, -e_{i+1}^j));$$

l is injective by the induction assumption. We define \tilde{F}_{i+1} to be the push-out of $F_{i+1} \leftarrow F_{i+1}^0 \xrightarrow{l} \tilde{F}_i$ in the category of vector spaces (this makes sense since l is injective). If

$$W = \cup_{x \in \text{link } v_{i+1}} \{\phi_\alpha(x, -)F_\alpha \mid x \in \sigma_\alpha \subseteq \text{St } v_{i+1}\} \subseteq t^{-1}(P_i),$$

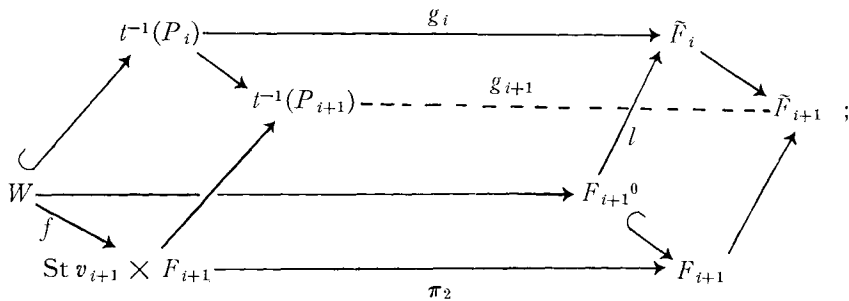
$\text{St } v_{i+1}$ denoting the closed star of v_{i+1} in K_{i+1} , then there is a map

$$f : W \rightarrow \text{St } v_{i+1} \times F_{i+1}$$

given by

$$f(\phi_\alpha(x, v)) = (x, l_{i+1, \alpha} v),$$

and $t^{-1}(P_{i+1})$ is $t^{-1}(P_i)$ attached to $\text{St } v_{i+1} \times F_{i+1}$ by f . There is an obvious map from W to F_{i+1}^0 , and a commutative diagram



g_{i+1} exists by universality of the attaching construction. The induction assumption at this stage (i.e. that for any fixed $k > i + 1$ the vectors

$$\{g_{i+1}(\phi_j(v_j, e_k^j)) \mid j \leq i + 1\}$$

are linearly independent) follows from the fact that for a fixed $k > i + 1$ the vectors e_j^{i+1} , for $j \leq i$, and e_k^{i+1} in F_{i+1} are linearly independent.

The complete classification theorem for simplicial bundles follows by replacing $C(K, \prod_p G_p(\mathbf{R}^\infty))$ by the set $C(P, \prod_p G_p(\mathbf{R}^\infty))$ of set functions $f : P = |K| \rightarrow \prod_p G_p(\mathbf{R}^\infty)$ such that for each simplex σ of K ,

- a) the restriction of f to the interior of σ extends to a (necessarily unique) continuous function $f_\sigma : \sigma \rightarrow G_p(\mathbf{R}^\infty)$, for some p ,
- b) if $x_0 \in \sigma$, then

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \sigma}} f(x) \subseteq f(x_0),$$

c) associating to each $x \in \sigma$ the decomposition

$$D = \left\{ \lim_{\substack{y \rightarrow x \\ y \in \tau}} f_\tau(y) \mid \sigma \subseteq \tau \right\}$$

defines a continuous map from σ to $[D]$.

If $(P, \coprod_p G_p(\mathbf{R}^\infty))$ denotes the appropriate set of equivalence classes of such functions, then it is not hard to show that there is a natural equivalence of contravariant functors defined on the category of polyhedra

$$F : (-, \coprod_p G_p(\mathbf{R}^\infty)) \rightarrow \text{Sxl } -.$$

The proof of this result is similar to the proof of Theorem 4; the only difference is that now F is surjective since we can use Proposition 3 and a partition of unity argument to construct a Gauss map for any simplicial bundle.

2. An application. A *combinatorial m -manifold* (or a Brouwer m -manifold, see [2]) M is a triangulated m -manifold such that the closed star of each vertex is an m -cell which can be mapped homeomorphically into \mathbf{R}^m in such a manner that the image of each m -cell of the star is an m -simplex.

Henceforth all manifolds are closed.

Every combinatorial m -manifold M has a dual cell decomposition (see [7]). An *affine structure* $r_j : \tau_j^p \rightarrow \mathbf{R}^p$ on a p -dimensional cell τ_j^p of the dual cell decomposition of M is a piecewise affine homeomorphism into \mathbf{R}^p . A *linear dual* M^* of M is the standard dual cell decomposition of M together with a family $r_j : \tau_j^p \rightarrow \mathbf{R}^p$ of affine structures on the cells of the dual cell decomposition of M such that:

- if $\tau_j^p \subseteq \tau_k^q$ then $r_k r_j^{-1} : r_j(\tau_j^p) \rightarrow r_k(\tau_k^q)$ is affine, and
- each $r_j(\tau_j^p)$ is strictly convex (each $(p-1)$ -dimensional face of $r_j(\tau_j^p)$ determines a $(p-1)$ -dimensional hyperplane in \mathbf{R}^p ; any other $(p-1)$ -dimensional face of $r_j(\tau_j^p)$ intersects this hyperplane in at most a $(p-2)$ -dimensional face).

It follows that there is a bijection between the vertices (0-cells) w_i of M^* and the m -simplices σ_i^m of M ; and the vertices w_{i_1}, \dots, w_{i_N} of M^* are the vertices of a p -cell in M^* if and only if the intersection $\cap \sigma_{i_j}^m$ of the corresponding m -simplices of M is an $(m-p)$ -simplex σ_i^{m-p} in M and the closed star of σ_i^{m-p} is the union $\cup \sigma_{i_j}^m$ of these simplices.

6. LEMMA. *Every combinatorial 2-manifold M has a linear dual M^* .*

Proof. For each vertex v_j of M choose a piecewise linear realization of the closed star of v_j in \mathbf{R}^2 . Draw a small circle in \mathbf{R}^2 with center v_j , and for every 2-simplex σ_i^2 of M choose a point w_i on the intersection of this circle and the interior of σ_i^2 . Let τ_j^2 denote the convex hull of the w_i . We

denote the 1-dimensional face of τ_j^2 determined by the vertices w_{i_1} and w_{i_2} by (w_{i_1}, w_{i_2}) .

Let $M^* = \coprod_j \tau_j^2 / \sim$ where \sim is the equivalence relation: $x_j \sim x_k$ for $x_j \in \tau_j^2$ and $x_k \in \tau_k^2$ if and only if either

- a) x_j and x_k are both vertices which correspond to the same 2-simplex of M , or
- b) $x_j \in (w_{j_1}, w_{j_2})$ and $x_k \in (w_{k_1}, w_{k_2})$, where w_{j_1} and w_{k_1} (respectively w_{j_2} and w_{k_2}) both correspond to the same 2-simplex of M , and $Tx_j = x_k$ where
 $T : (w_{j_1}, w_{j_2}) \rightarrow (w_{k_1}, w_{k_2})$
 is the unique affine map taking w_{j_1} to w_{k_1} and w_{j_2} to w_{k_2} .

Thus we get an idea of what a linear dual M^* of a combinatorial 2-manifold M looks like locally as follows: Piecewise linearly realize part of M in \mathbf{R}^2 . The vertices of M^* are the barycenters w_i of the 2-simplices σ_i^2 of M . Vertices w_{i_1} and w_{i_2} are joined by a 1-cell if and only if the intersection of the corresponding 2-simplices $\sigma_{i_1}^2$ and $\sigma_{i_2}^2$ of M is a 1-simplex of M ; in general, every vertex of a linear dual M^* of a combinatorial m -manifold M is contained in precisely $m + 1$ 1-cells of M^* . Finally, w_{i_1}, \dots, w_{i_N} are the vertices of a 2-cell of M^* if and only if the intersection of the corresponding 2-simplices $\sigma_{i_1}^2, \dots, \sigma_{i_N}^2$ of M is a vertex v_i of M and the closed star of v_i in M is precisely the union of $\sigma_{i_1}^2, \dots, \sigma_{i_N}^2$.

One could try to generalize Lemma 6 to show that combinatorial manifolds of arbitrary dimension m have linear duals: The first step would be to construct the m -dimensional duals τ_i^m of the vertices of M . The next step would be to find compatible affine maps between corresponding faces of different m -cells τ_i^m ; however this is not necessarily possible; there is, for example, no affine transformation taking a square to an equilateral trapezoid.

Although it is unclear whether or not all combinatorial manifolds of arbitrary dimension have linear duals, there are examples of combinatorial manifolds which do have them: Spheres (triangulated as tetrahedra) have linear duals; real projective spaces have triangulations which have linear duals (see Example 7 for the case $m = 2$); and by using brute force I have found a triangulation of CP^2 which has a linear dual.

One can get an idea of what a linear dual looks like locally. However, in contrast to the fact that every polyhedron can be embedded in some high dimensional Euclidean space, it is not true that every linear dual M^* can be rectilinearly embedded in any Euclidean space (i.e. in such a manner that the realization of each cell is flat) even locally. To illustrate this, I present the following example for which I would like to thank Ben Halpern.

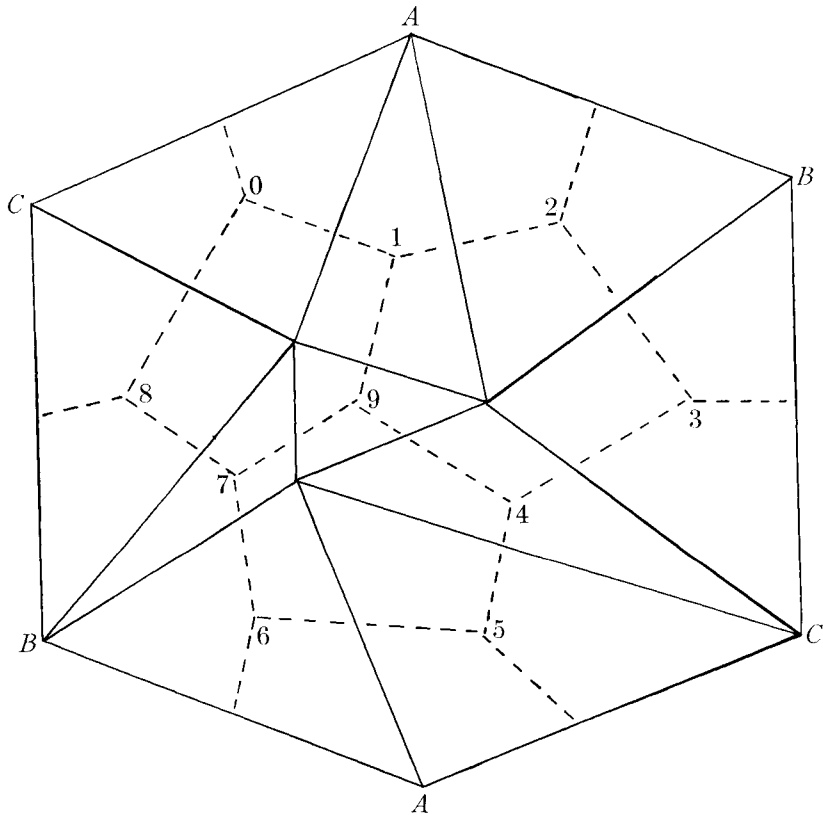


Diagram 3

7. *Example.* Let M denote the simplicial subdivision of the projective plane given in Diagram 3; M consists of nine triangles and M^* consists of six pentagons. If M^* rectilinearly embedded in some Euclidean space, the 1-cells (1, 9), (4, 9), and (7, 9) would all have to lie in a three dimensional subspace \mathbf{R}^3 . Since the realization of each cell would be flat, the pentagons (1, 2, 3, 4, 9), (4, 5, 6, 7, 9), and (7, 8, 0, 1, 9) would be in \mathbf{R}^3 , so in particular the vertices 0, 2, 3, 5, 6, and 8 would also be in \mathbf{R}^3 . Consequently the other three pentagons would also be in \mathbf{R}^3 which is impossible since this would imply that we had an embedding of the projective plane in \mathbf{R}^3 .

Since the construction of tangent bundles of simplicial complexes is local, it follows that every linear dual has a tangent bundle in the appropriately defined category of simplicial bundles. There is also a classification theorem for simplicial bundles over linear duals (since the proof of the corresponding result for polyhedra is local). In particular, Theorems 4 and 5 are valid for linear duals except that "simplex" must be

replaced by “cell” and that K must be replaced by the set of geometric cells of M^* together with the appropriate incidence relations.

We let $(C_*^\circ(M), \partial)$ denote the oriented simplicial chain complex of M (with integer coefficients). If we define a *cooriented cell* of M^* to be a cell of M^* together with an orientation of the associated dual simplex of M , and if we let $C_*^{\text{co}}(M^*)$ denote the graded group of cooriented cells of M^* (with integer coefficients and oppositely cooriented cells differing by a minus sign), then the bijection g between the p -simplices of M and the $(m - p)$ -cells of M^* induces a homomorphism $g_* : C_p^\circ(M) \rightarrow C_{m-p}^{\text{co}}(M^*)$. If $\rho \subseteq \sigma$ for ρ and σ in $C_*^\circ(M)$, then $g_*(\sigma) \subseteq g_*(\rho)$ in $C_{m-*}^{\text{co}}(M^*)$. Furthermore there is a naturally defined boundary operator ($\partial^2 = 0$)

$$\partial : C_p^{\text{co}}(M^*) \rightarrow C_{p+1}^{\text{co}}(M^*)$$

for which the following diagram commutes

$$\begin{CD} C_p^\circ(M) @>g_*>> C_{m-p}^{\text{co}}(M^*) \\ @V\partial VV @VV\partial V \\ C_{p-1}^\circ(M) @>g_*>> C_{m-(p-1)}^{\text{co}}(M^*) \end{CD}$$

if we write a cooriented cell in $C_{m-p}^{\text{co}}(M^*)$ in the form

$$\tau^{m-p} = \underbrace{\langle w_{i_1}, \dots, w_{i_N} \rangle}_{\text{vertices of cell of } M^*} \underbrace{\langle v_{j_0}, \dots, v_{j_p} \rangle}_{\text{orientation of dual cell in } M}$$

then

$$\partial \tau^{m-p} = \sum_i (-1)^i \langle g(v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_p}); v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_p} \rangle.$$

Now let $G_m^\circ(\mathbf{R}^\infty) = BSO(m)$ denote the space of oriented m -planes in \mathbf{R}^∞ . For $p = 0, 1, \dots, m$, we consider the set of all $(p + 1)$ -tuples (V_0, \dots, V_p) of distinct $V_i \in BSO(m)$ for which

- (**) for each k -tuple $J \subseteq \{0, \dots, m\}$ the dimension of $\bigcap_{j \in J} V_j$ is $m - k + 1$

modulo the natural action of the even permutations on $p + 1$ letters; an equivalence class $\Gamma = \langle V_i \rangle = \langle V_0, \dots, V_p \rangle$ is a $((p + 1)$ -fold) *oriented configuration*.

We let $\mathcal{C}(m)$ denote the graded group $\bigoplus_p \mathcal{C}_p(m)$ whose p th grading $\mathcal{C}_p(m)$ consists of all formal finite sums $\sum_j n_j \Gamma_j$ for $n_j \in \mathbf{Z}$ and Γ_j a $(p + 1)$ -fold oriented configuration (configurations with opposite $\langle - \rangle$ orientations or opposite orientations in a single component V_i differing by

a minus sign) for $p = 0, \dots, m$, and zero otherwise. There is a natural boundary operator ($\partial^2 = 0$) given by

$$\partial : \mathcal{C}_p(m) \rightarrow \mathcal{C}_{p-1}(m)$$

$$\partial \langle V_0, \dots, V_p \rangle = \sum_{i=0}^p (-1)^i \langle V_0, \dots, \hat{V}_i, \dots, V_p \rangle.$$

We let $H^*(\mathcal{C}(m))$ denote the homology of the associated cochain complex $(\text{Hom}_{\mathbf{Z}}(\mathcal{C}(m), \mathbf{R}), d)$ where d is the differential induced by ∂ .

The $2n$ -dimensional Euler class $e \in H^{2n}(\mathcal{C}(2n))$ is described as follows (this construction is motivated by [1]):

Suppose $\langle V_i \rangle \in \mathcal{C}_{2n}(2n)$. Since $\langle V_i \rangle$ satisfies condition (**), the smallest subspace of \mathbf{R}^∞ containing V_0, \dots, V_{2n} is $(2n + 1)$ -dimensional (one can, in fact, obtain a basis $\{e_i\}$ for this space by letting e_i denote any non-zero vector along the line $V_0 \cap \dots \cap \hat{V}_i \cap \dots \cap V_{2n}$ for $i = 0, \dots, 2n$). Without loss of generality we assume that this subspace is \mathbf{R}^{2n+1} . The orientations of the V_i describe a unique simplicial cone $C \subseteq \mathbf{R}^{2n+1}$: The orientation of each V_i determines a unique positive normal direction to V_i , and hence a particular positive half space of \mathbf{R}^{2n+1} ; C is the intersection of all such half spaces.

Following (1), a linear height function $\xi : \mathbf{R}^\infty \rightarrow \mathbf{R}$ is *general* for C if and only if for every 1-dimensional edge l_i of C ,

$$x \neq y \text{ for } x, y \in l_i \Rightarrow \xi(x) \neq \xi(y).$$

If ξ is general for C and f is any face of C we define

$$i(f, c, \xi) = \begin{cases} 1 & \text{if } 0 = \xi(0) \geq \xi(x) \text{ for all } x \in f \\ 0 & \text{otherwise} \end{cases}$$

and

$$i(C, \xi) = \sum_{r=0}^{\infty} (-1)^r \sum_{f^r \in C} i(f^r, C, \xi)$$

where the second summation is taken over all r -dimensional faces f^r of C .

The $2n$ -dimensional Euler class $e \in H^{2n}(\mathcal{C}(2n))$ is the class represented by the homomorphism $e \in \text{Hom}_{\mathbf{Z}}(\mathcal{C}_{2n}(2n), \mathbf{R})$ which is defined by associating to each $\langle V_i \rangle \in \mathcal{C}_{2n}(2n)$

$$e \langle V_i \rangle = \epsilon \cdot \frac{1}{\text{vol } S^{2n}} \int_{S^{2n}} i(C, \xi) d\omega^{2n} \in \mathbf{R}$$

where $\text{vol } S^{2n}$ is the volume of the unit $2n$ -sphere $S^{2n} \subseteq \mathbf{R}^{2n+1}$, $C \subseteq \mathbf{R}^{2n+1}$ is the simplicial cone associated to $\langle V_i \rangle$, the integral is taken over all $\xi \in S^{2n}$ which give general height functions $\xi : \mathbf{R}^{2n+1} \rightarrow \mathbf{R}$ for C (by projection to ξ), and where $\epsilon = \pm 1$ is chosen as follows: There are two (independent) orientations of C determined by $\langle V_i \rangle$; one is determined by

the orientations of the individual V_i 's and the other is determined by the order of the V_i in $\langle V_i \rangle$ (i.e. the " $\langle - \rangle$ "); $\epsilon = +1$ if these orientations agree, and $\epsilon = -1$ otherwise.

8. LEMMA. *If $k > 2n$ then*

$$\frac{1}{\text{vol } S^{2n}} \int_{S^{2n}} i(C, \xi) d\omega^{2n} = \frac{1}{\text{vol } S^k} \int_{S^k} i(C, \xi) d\omega^k$$

the integral on the left taken over $\xi \in S^{2n}$, and the integral on the right taken over $\xi \in S^k$.

Proof. This follows from the fact that the planes V_0, \dots, V_{2n} determine a triangulation of S^{2n} into spherical simplices, the fact that $i(C, \xi)$ is constant over the interior of each such spherical $2n$ -simplex, and the fact that if

$$\pi : S^{k+1} - \{\text{north pole, south pole}\} \rightarrow S^k$$

is given by orthogonal projection to \mathbf{R}^{k+1} followed by radial projection to S^k , and $\sigma \subseteq S^k$ is a spherical simplex then

$$\text{vol } \sigma / \text{vol } S^k = \text{vol } \pi^{-1}(\sigma) / \text{vol } S^{k+1}.$$

The following result indicates the sense in which $e \in H^{2n}(\mathcal{C}(2n))$ is an Euler class.

9. THEOREM. *Let M be a (closed) oriented combinatorial $2n$ -manifold which has a linear dual. If $f : M^* \rightarrow \coprod_p G_p(\mathbf{R}^\infty)$ is any map which classifies the tangent bundle of M^* , then for each p there is an induced map*

$$f_* : C_{2n-p}^{\text{co}}(M^*) \rightarrow \mathcal{C}_p(2n)$$

for which the following diagram commutes

$$\begin{array}{ccc} C_{2n-p}^{\text{co}}(M^*) & \xrightarrow{f_*} & \mathcal{C}_p(2n) \\ \partial \downarrow & & \downarrow \partial \\ C_{2n-(p-1)}^{\text{co}}(M^*) & \xrightarrow{f_*} & \mathcal{C}_{p-1}(2n) \end{array}$$

*and $g^*f_*e \in H^{2n}(M)$ is the Euler class of M .*

Proof. The induced map f_* is defined by associating to $\tau \in C_{2n-p}^{\text{co}}(M^*)$ the configuration in $\mathcal{C}_p(2n)$ described by

$$\{f(\tau_i^{2n}) \in BSO(2n) \mid \tau \subseteq \tau_i^{2n}\}$$

(each $f(\tau_i^{2n})$ receiving an orientation from the orientation of M^* induced by the one on M) and the order (" $\langle - \rangle$ orientation") determined by the

coorientation of τ . To show that g^*f^*e is the Euler class of M we integrate:

$$\begin{aligned} \langle g^*f^*e, [M] \rangle &= \sum_{\sigma_i^{2n} \subseteq M} \langle g^*f^*e, \sigma_i^{2n} \rangle \\ &= \sum_{w_j \in M^*} \langle f^*e, w_j \rangle = \sum_{w_j \in M^*} \langle e, f^*w_j \rangle \\ &= \sum_{w_j \in M^*} \frac{1}{\text{vol } S^N} \int_{S^N} i(C_j, \xi) d\omega^N \end{aligned}$$

for some large N , by Lemma 8, where C_j is the cone associated to $f(w_j)$. Thus

$$\langle g^*f^*e, [M] \rangle = \frac{1}{\text{vol } S^N} \int_{S^N} \sum_{w_j \in M^*} i(C_j, \xi) d\omega^N$$

the integral taken over all $\xi \in S^N$ which are general for all C_j . (The integral and summation can be interchanged since the set of linear maps $\xi \in S^N$ which are general for all C_j is open and dense: In fact the set of $\xi \in S^N$ for which ξ is not general for all C_j is the (finite) union of the hyperplanes $\{\xi \in S^N \mid \xi(f^i) = 0\}$, the union taken over all C_j and all 1-dimensional edges f^i of each C_j .) But

$$\sum_{w_j \in M^*} i(C_j, \xi) = \sum_{\tau=0}^{\infty} (-1)^\tau \sum_{\tau' \subseteq M^*} \sum_{w_j \in \tau'} i(f^{\tau'}, C_j, \xi),$$

where $f^{\tau'}$ is the face of C_j determined by τ' , and

10. *Claim.* For a fixed $\tau' \subseteq M^*$ and ξ which is general for all C_j ,

$$\sum_{w_j \in \tau'} i(f^{\tau'}, C_j, \xi) = 1.$$

Proof. Pick a vertex w_j of τ' . There is a unique affine realization $h : \tau' \rightarrow f(\tau')$ of $\tau' \subseteq \mathbf{R}^{2n}$ taking w_j to the origin and such that for each cell $\tau^s \subseteq \tau'$, the image $h(\tau^s)$ of τ^s in $f(\tau')$ is parallel to

$$\cap \{f(\tau_i^{2n}) \mid \tau^s \subseteq \tau_i^{2n}\}.$$

The result follows from the fact that $\xi \circ h$ is a general linear height function on τ' (in the sense of [1]), and since τ' is strictly convex, $\xi \circ h$ achieves its maximum at precisely one vertex of τ' .

Thus

$$\sum_{w_j \in M^*} i(C_j, \xi) = \sum_{\tau=0}^{\infty} (-1)^\tau \sum_{\tau' \subseteq M^*} 1 = \chi(M^*) = \chi(M),$$

so

$$\langle g^*f^*e, [M] \rangle = \frac{1}{\text{vol } S^N} \int_{S^N} \chi(M) d\omega^N = \chi(M).$$

If the linear dual M^* of the combinatorial $2n$ -manifold M can be

rectilinearly embedded (or immersed) in some Euclidean space, then, having chosen a specific rectilinear realization, the map

$$f_* : C_{2n-p}^{co}(M^*) \rightarrow \mathcal{C}_p(2n)$$

could be replaced by the map which associates to each cooriented $(2n - p)$ -cell τ^{2n-p} of M^* the oriented p -dimensional tangent “cone” to $\overrightarrow{\tau^{2n-p}}$ in \mathbf{R}^∞ (i.e. the set of all non-negative multiples of vectors of the form \overrightarrow{PQ} where $P \in \tau^{2n-p}$ and Q is in the star of τ^{2n-p}). From this point of view, we use the universal Euler class $e \in H^{2n}(\mathcal{C}(2n))$ to measure the curvature of M by redistributing the curvature of M to the vertices of M^* , i.e. the interior of the $2n$ -simplices of M . (A linear dual M^* of M is a linearized form of the dual cell complex of M ; it is used since it is important that in the redistribution process no curvature winds up on the $(2n - 1)$ -skeleton of M .)

As Example 7 illustrates, however, linear duals cannot always be rectilinearly realized in Euclidean space; it is this problem which the use of the classification theorem avoids.

3. A final remark. For each m there is a natural chain map h_* from $\mathcal{C}_*(m)$ to the singular chains $C_*(BSO(m))$ on $BSO(m)$ (with \mathbf{Z} coefficients), and if $m = 2n$ then the image of the usual universal Euler class $\bar{e} \in H^{2n}(BSO(2n))$ (cohomology with real coefficients) under

$$h_* : H^{2n}(BSO(2n)) \rightarrow H^{2n}(\mathcal{C}(2n))$$

is e .

In order to define h_* we first need the following two lemmas.

LEMMA. *If $p = 1, 2, \dots, m$ and V_0, \dots, V_p is a collection of m -dimensional subspaces of \mathbf{R}^∞ satisfying (**), then the span of V_0, \dots, V_p is of dimension $m + 1$.*

Proof. This follows by induction on p .

LEMMA. *Let $p = 1, 2, \dots, m$ and V_0, \dots, V_p be a collection of oriented m -dimensional subspaces of \mathbf{R}^∞ satisfying (**). If \mathbf{R}^{m+1} is the span of V_0, \dots, V_p , and for each $i = 0, \dots, p, n_i \in \mathbf{R}^{m+1}$ is the unit vector in the positive normal direction to V_i , then $\{n_i \mid i = 0, \dots, p\}$ is a linearly independent set of $p + 1$ vectors.*

Proof. Again use induction on p . Here is the inductive step: If, for example, $n_0 = c_1 n_1 + \dots + c_p n_p$ for some constants c_i , then n_0 is perpendicular to $V_1 \cap \dots \cap V_p$, a subspace of dimension $m - p + 1$. But this implies that

$$\bigcap_{i=1}^p V_i \subseteq \bigcap_{i=0}^p V_i,$$

and consequently that the dimension of $V_0 \cap \dots \cap V_p$ is at least $m - p + 1$; this contradicts (**).

Given $\langle V_i \rangle$ in $\mathcal{C}_p(m)$, we define $h_*(\langle V_i \rangle)$ as follows: Let \mathbf{R}^{m+1} be the span of V_0, \dots, V_p , and $n_i \in S^m \subseteq \mathbf{R}^{m+1}$ the corresponding unit vectors in the positive normal direction to the $V_i, i = 0, \dots, p$. The convex hull of the n_i 's forms a p -simplex in \mathbf{R}^{m+1} , and there is a well defined simplicial map from the standard p -simplex Δ^p to the convex hull of the n_i 's given by mapping each vertex $v_i \in \Delta^p$ to n_i . Projecting the convex hull of the n_i 's to S^m determines a spherical simplex $\bar{\sigma} : \Delta^p \rightarrow S^m$. Following this map by the map \perp which associates to each unit vector $u \in S^m$ the m -plane in \mathbf{R}^{m+1} normal to u , we obtain a singular p -simplex

$$\sigma = \perp \circ \bar{\sigma} : \Delta^p \rightarrow BSO(m), \text{ and}$$

$$h_*(\langle V_i \rangle) = \sigma \in C_p(BSO(m)).$$

Having defined h_* on generators of $\mathcal{C}_*(m)$, we define h_* on all of $\mathcal{C}_*(m)$ by extending additively.

11. THEOREM. *If $\bar{e} \in H^{2n}(BSO(2n))$ is the universal Euler class, then*

$$h^*\bar{e} = e \in H^{2n}(\mathcal{C}(2n)).$$

Proof. If $BSO(2n)$ is regarded as a limit of finite Grassmanians, there is a canonical representative for \bar{e} obtained by integration over singular smooth simplices of an invariant differential form (see [11]). Using this representative, for a given $\langle V_i \rangle \in \mathcal{C}_{2n}(2n)$,

$$\begin{aligned} \langle h^*\bar{e}, \langle V_i \rangle \rangle &= \langle \bar{e}, h_*\langle V_i \rangle \rangle = \langle \bar{e}, \sigma \rangle = \langle \bar{e}, \perp \circ \bar{\sigma} \rangle \\ &= \langle \bar{e}, \perp_*(\bar{\sigma}) \rangle = \langle \perp^*\bar{e}, \bar{\sigma} \rangle. \end{aligned}$$

But

$$\langle \perp^*\bar{e}, \bar{\sigma} \rangle = \frac{1}{\text{vol } S^{2n}} \int_{S^{2n}} i(C, \xi) d\omega^{2n}$$

so $\langle h^*\bar{e}, \langle V_i \rangle \rangle = \langle e, \langle V_i \rangle \rangle$.

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