J. Austral. Math. Soc. (Series A) 31 (1981), 146–151

A CLASS OF IDEALS OF THE CENTRE OF A GROUP RING

MICHAEL F. O'REILLY

(Received 24 September 1979; revised 3 November 1980)

Communicated by H. Lausch

Abstract

Reynolds (1972), using character-theory, showed that the p-section sums span an ideal of the centre $Z(kG)$ of the group algebra of a finite group G over a field k of characteristic dividing the order of G. In O'Reilly (1973) a character-free proof was given. Here we extend these techniques to show the existence of a wider class of ideals of $Z(kG)$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 16 A 26, 20 C 05, 20 C 20.

1. Introduction and notation

Let G be a finite group, JG the group ring over the integers J, with centre $Z(JG)$. For $X \subset G$ write $\overline{X} = \sum_{g \in X} g$; for $L < G$, $K < \mathfrak{N}(X) \cap L$ (the normalizer of X in L) let $\overline{X}_K^L = \sum_{g \in \Omega} \overline{X}^g$ where Ω is a transversal of K in L. In particular $\overline{X}_K^G \in Z(JG)$ and a conjugacy class sum is of the form b_C^G where $C = C(b)$ is the centralizer of b.

The main result is

THEOREM 1. *Let n be a fixed divisor of $|G|$, L a fixed subgroup of G. The subspace $\mathfrak{W}(L, n)$ of $Z(JG)$ spanned by the set $\{(\overline{Hy})_N^G / H < L,\ y \in \mathfrak{N}(H),\ H < N < \mathfrak{N}(Hy),\ N\colon H \text{ divides } n\}$ is an ideal of $Z(JG)$.*

The ideal $\mathfrak{W}(L, n)$ will thus include integer multiples $|C(b)| b_{C(b)}^G$ of conjugacy class sums (taking $N = H = \{1\}$) but will only include the class sum itself if $|C(b)|$ divides n (taking $H = \{1\}$, $N = C(b)$).

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press

By extending the ring of coefficients to the p-adic integers and mapping canonically to $Z(kG)$, k the residue class field of characteristic p, we obtain ideals $\mathcal{W}'(L, n)$ of $Z(kG)$. In the special case where $n = p^\alpha$ the generating set may be restricted [Theorem 2] to elements where N is a Sylow p-subgroup of $\mathcal{N}(Hy)$. When $|L| = p^\beta$ Theorem 3 shows that a further restriction to subgroups H which lie in the Sylow p-subgroup of the centralizer of the p-regular part of y is permissible. The ideal of p'-sections is then $\mathcal{W}'(P, 1)$ where P is a Sylow p-subgroup.

2. The main theorem

For $X, Y \subset G$ and $S < \mathcal{N}(X)$, $T < \mathcal{N}(Y)$ the elements $\bar{X}_S^G$ and $\bar{Y}_T^G$ multiply according to the Mackey decomposition

$$(1) \qquad \bar{X}_S^G \bar{Y}_T^G = \sum_{g \in \Omega} (\bar{X}\bar{Y}^g)_{S \cap T^g}^G$$

where Ω is a set of (S, T) double coset representatives in L. For $S < K < G$, we have trivially that

$$(2) \qquad (\bar{X}_S^K)_K^G = \bar{X}_S^G.$$

We first outline the proof of Theorem 1. It must be shown that if $(\bar{H}y)_N^G \in \mathcal{W}(L, n)$ and b_C^G is a conjugacy class sum then their product lies in $\mathcal{W}(L, n)$. By Eq. (1) this product is the sum of terms $(\bar{H}u)_S^G$ where $u = yb^g$ and $S = N \cap C^g$, which do not have the form required by the above spanning set of $\mathcal{W}(L, n)$. However we show [Lemma 3l] that Hu may be partitioned into conjugates of cosets $H_x x$, H_x being the maximum subgroup of H normalized by x. This gives [Lemma 4] $\bar{H}u$ as the sum of terms $(\bar{H}_x x)_{T(x,u)}^K$ where $K = \mathcal{N}(Hu) \cap N$ and $T(x, u) = K \cap \mathcal{N}(H_x x)$. From this and Eq. (2) we obtain $(\bar{H}u)_S^G$ as the sum of terms $(\bar{H}_x x)_{T(x,u)}^G$ which are shown to be in the given spanning set.

For $H < G$ and $u \in G$, H_u denotes the unique maximal subgroup of H which u normalizes.

LEMMA 1.

$$H \cap \mathcal{N}(H_u u) \overset{(a)}{=} H_u \overset{(b)}{<} H \cap H^u \overset{(c)}{=} H \cap \mathcal{N}(Hu) \overset{(d)}{\lhd} \mathcal{N}(Hu) \overset{(e)}{<} \mathcal{N}(H).$$

PROOF. We verify the chain from the right. $x \in N(Hu)$ implies $Hu = H^x u^x$ and so $Huu^{-1}H = H^x u^x (u^x)^{-1} H^x$, that is $H = H^x$ proving (e). Trivially then $\mathcal{N}(Hu)$ normalizes $H \cap \mathcal{N}(Hu)$ giving (d). Also trivially $H \cap H^u < H \cap \mathcal{N}(Hu)$. If $x \in H \cap \mathcal{N}(Hu)$ then as above $u^x \in Hu$ giving $x \in H^u$; so $H \cap \mathcal{N}(Hu) < H \cap H^u$ giving (c). (b) is immediate from the definition of H_u.

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press

Trivially $H_u \leqslant H \cap \mathfrak{N}(H_u u)$. If $h \in H \cap \mathfrak{N}(H_u u)$ then the inclusion $u^h \in H_u u$ may be rewritten $uhu^{-1} \in hH_u \subset H \cap N(H_u u)$. So u^{-1} and hence u normalize $H \cap \mathfrak{N}(H_u u)$. By definition of H_u, $H \cap \mathfrak{N}(H_u u) \leqslant H_u$ proving (a).

Corollary. *$x \in \mathfrak{N}(Hu)$ if and only if $x \in \mathfrak{N}(H)$ and $[x, u^{-1}] \in H$.*

Proof. Necessity is immediate from the proof of (e). If $[x, u^{-1}] \in H$ and $x \in \mathfrak{N}(H)$ then $u^x \in Hu$ and so $(Hu)^x = H^x u^x \subset HHu = Hu$.

Lemma 2. *If $H \cap H^u \leqslant K \leqslant \mathfrak{N}(Hu)$ then $K \cap \mathfrak{N}(H_u u)$: H_u divides KH: H.*

Proof. From Lemma 1, $H \cap H^u \triangleleft \mathfrak{N}(Hu)$, $H \cap K = H \cap H^u$ and $\{K \cap \mathfrak{N}(H_u u)\} \cap \{H \cap H^u\} = H_u$. So

$$\frac{K \cap \mathfrak{N}(H_u u)}{H_u} \cong \frac{\{K \cap \mathfrak{N}(H_u u)\}\{H \cap H^u\}}{H \cap H^u} < \frac{K}{H \cap H^u} \cong \frac{KH}{H}.$$

Next we obtain a partition of the coset Hu into cosets of the form $H_x x$. First note that if $y \in H_x x$ then $H_y = H_x$ and so $H_y y = H_x x$; for y normalizes H_x giving $H_x \leqslant H_y$ and then $x \in H_y y$ giving $H_y \leqslant H_x$. The cosets $H_x x$, and $H_y y$ are thus either equal or disjoint and so we get a partition of G into cosets of form $H_x x$.

Lemma 3. (a) *The set $\mathcal{P} = \{H_x x, x \in G\}$ is a partition of G, permuted by conjugation by $\mathfrak{N}(H)$.*

(b) *The set $\mathcal{P}' = \{H_x x, x \in Hu\}$ is a partition of Hu, permuted by conjugation by $\mathfrak{N}(Hu)$.*

Proof. For $g \in \mathfrak{N}(H)$, $H_x^g = H_z$ where $z = x^g$. So $(H_x x)^g = H_z z \in \mathcal{P}$, proving (a). If $g \in \mathfrak{N}(Hu)$ then $H_x x \in \mathcal{P}'$ implies $H_z z \in \mathcal{P}'$, proving (b).

We can immediately obtain a decomposition of an arbitrary coset sum $\overline{H}u$.

Lemma 4. *Let $K \leqslant \mathfrak{N}(Hu)$ and let $\{H_x x, x \in \Lambda(K)\}$ be a set of representatives of the distinct K-orbits of $\mathcal{P}'$. Then*

$$\overline{H}u = \sum_{x \in \Lambda(K)} \left(\overline{H}_x x\right)^K_{K \cap \mathfrak{N}(H_x x)}.$$

The proof is trivial when it is noted that each summand is the sum of all the distinct cosets within a K-orbit.

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press

Proof of Theorem. Let $(\overline{H}y)_N^G \in \mathcal{W}(L, n)$, let b_C^G be a conjugacy class sum and Ω a set of (N, C) double coset representatives. By (1)

$$\begin{aligned}(\overline{H}y)_N^G b_C^G &= \sum_{g \in \Omega} (\overline{H}yb^g)_{N \cap C^g}^G \\ &= \sum_u (K: N \cap C^g)(\overline{H}u)_K^G\end{aligned}$$

where $u = yb^g$ and $K = \mathcal{N}(Hu) \cap N$. By Lemma 4 and Eq. (2) $(\overline{H}u)_K^G$ is the sum of terms $(\overline{H}_x x)_{T(x,u)}^G$ where $T(x, u) = K \cap \mathcal{N}(H_x x)$. We show that these terms lie in the given spanning set of $\mathcal{W}(L, n)$. By definition we have that $H_x \leqslant H \leqslant L$ and $x \in \mathcal{N}(H_x)$. Also $H_x \leqslant H \cap H^x = H \cap H^u = H \cap \mathcal{N}(Hu) \leqslant N \cap \mathcal{N}(Hu) = K$; so $H_x \leqslant T(x, u) \leqslant \mathcal{N}(H_x x)$. Finally since $T(x, u) = K \cap \mathcal{N}(H_x x)$, by Lemma 2, $T(x, u)$: H_x divides KH: H which divides N: H which divides n.

It may be noted that a slight generalization of Theorem 1 may be obtained by replacing $\mathcal{N}(Hy)$ by $\mathcal{N}(Hy) \cap T$ where $L \leqslant T \leqslant G$.

3. The modular case

Extending the coefficient ring from J to R, the ring of p-adic integers, gives ideals $\mathcal{W}_R(L, n)$ of $Z(RG)$. If $|G|$ is a unit in R then $\mathcal{W}_R(L, n) = Z(RG)$ for each conjugacy class sum may be written $\{b\}_1^G/|C(b)|$. However on passing from R to k, the residue class field by the natural homomorphism, the ideals $\mathcal{W}'(L, n)$ of $Z(kG)$ so obtained are non-trivial when p divides $|G|$. In this case we may restrict n and N.

Theorem 2. *For $n = mp^\alpha$ and $(m, p) = 1$ the ideal $\mathcal{W}'(L, n)$ equals $\mathcal{W}'(L, p^\alpha)$ and is spanned by the set $\{(\overline{H}y)_P^G / H \leqslant L, y \in \mathcal{N}(H)$, P a Sylow p-subgroup of $\mathcal{N}(Hy)$, P: $H \cap P$ divides $p^\alpha\}$*

Proof. Let $\beta = (\overline{H}y)_N^G$ $(\in \mathcal{W}'(L, n))$ and P be a Sylow p-subgroup of N. Then N: HP is a unit and $\beta = (\overline{H}y)_{HP}^G / N: HP$. Here HP: H $(= P: H \cap P)$ is the maximum power of p dividing N: H and so divides p^α. So $\mathcal{W}'(L, n) \subset \mathcal{W}'(L, p^\alpha)$ and trivially $\mathcal{W}'(L, p^\alpha) \subset \mathcal{W}'(L, n)$. Since $(\overline{H}y)_N^G = (\overline{H}y)_P^G / N: P$, $\mathcal{W}'(L, n)$ is spanned by the elements $(\overline{H}y)_P^G$, which are non-zero only if P is a Sylow p-subgroup of $\mathcal{N}(Hy)$.

We now restrict further to the case where L is a p-subgroup and obtain a further restriction of the spanning set. Let $y = rs = sr$ with r p-regular, s a p-element, P a subgroup of L and $y \in \mathcal{N}(P)$.

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press

LEMMA 5. $\mathfrak{N}(Py) \leqslant \mathfrak{N}(Pr)$.

PROOF. By the corollary to Lemma 1, $x \in \mathfrak{N}(Py)$ if and only if $x \in \mathfrak{N}(P)$ and $x^{-1}yxy^{-1} \in P$, that is $y^x \in Py$. Since $r = y^m$ for some integer m, $r \in \mathfrak{N}(P)$ and $r^x = (y^m)^x = (y^x)^m \in Py^m = Pr$.

LEMMA 6. *Let y normalize both P and $Q = P_0 < P$. Define recursively $P_{i+1} = \mathfrak{N}(P_i) \cap P$, $i = 0, 1, 2, \ldots$. Then $y \in \mathfrak{N}(P_i)$ and $(\overline{Qy})_Q^P = 0$ if and only if for some i, $\mathfrak{N}(P_iy) \cap P_{i+1} > P_i$. Otherwise $(\overline{Qy})_Q^P = \overline{Py}$.*

PROOF. For some l, $P_l = P$. The proof is by induction on the minimal such l. Since $y \in \mathfrak{N}(Q)$ and y normalizes P, y normalizes $P \cap \mathfrak{N}(Q) = P_1$. If $\mathfrak{N}(Qy) \cap P_1 > Q$ then $(\overline{Qy})_Q^{P_1} = 0$ whence $(\overline{Qy})_Q^P = 0$. Otherwise let T be a transversal of Q in P_1 and so

$$(\overline{Qy})_Q^{P_1} = \sum_{u \in T} (\overline{Qy})^u = \sum_{u \in T} \overline{Qy^u}.$$

Here $y^u = (u^{-1}yuy^{-1})y = q_uy$ where $q_u = u^{-1}(yuy^{-1}) \in P_1$. $q_u \in Qq_v$ implies $y^u \in Qy^v$, that is $uv^{-1} \in \mathfrak{N}(Qy) \cap P_1 = Q$. So the cosets Qq_u, $u \in T$, are distinct and $(\overline{Qy})_Q^{P_1} = \sum_{u \in T} \overline{Qq_uy} = \overline{P_1y}$. Applying the hypothesis to the chain from P_1 to P, we have the result.

Let $N = \mathfrak{N}(Py) \leqslant \mathfrak{N}(P)$; then y, r, and $s \in N$. Let $C = C(r) \cap N$ and $Q = P \cap C$. Let D be a Sylow p-subgroup of C. Then C and hence D normalize Q and so $D \leqslant \mathfrak{N}(Qr)$. Further by the corollary to Lemma 1 since $D < \mathfrak{N}(Py)$ we have $y^dy^{-1} \in P$ for all $d \in D$; trivially also $y^dy^{-1} = d^{-1}(y\,dy^{-1}) \in C$ and so $y^dy^{-1} \in P \cap C = Q$. So by the same corollary, $D < \mathfrak{N}(Qy)$.

LEMMA 7. $(\overline{Qy})_D^N = (N\colon PD)\overline{Py} \neq 0$.

PROOF. $(\overline{Qr})_D^N$ is the sum of N-conjugacy classes, the only p-regular class term being $r_D^N = (C\colon D)r_C^N \neq 0$. So $(\overline{Qr})_D^N \neq 0$. In particular $(\overline{Qr})_D^{PD} \neq 0$. Since a transversal of D in PD is a transversal of Q in P, we have $(\overline{Qr})_D^{PD} = (\overline{Qr})_Q^P$; and so by Lemma 6 $\mathfrak{N}(P_ir) \cap P_{i+1} = P_i$ and $(\overline{Qr})_D^N = (\overline{Pr})_{PD}^N = (N\colon PD)\overline{Pr}$. Thus PD is a Sylow p-subgroup of N. Since $\mathfrak{N}(P_iy) \cap P_{i+1} < \mathfrak{N}(P_ir) \cap P_{i+1} = P_i$, again by Lemma 6, we have $(\overline{Qy})_Q^P = \overline{Py}$ and so $(\overline{Qy})_P^N = (N\colon PD)\overline{Py} \neq 0$, as required.

THEOREM 3. *Let L be a p-subgroup of G. Then $\mathfrak{W}'(L, p^\alpha)$ is spanned by the set $\{(\overline{Py})_D^G / P < D \leqslant C(r) \cap L$, r the p-regular part of $y \in \mathfrak{N}(P)$, D a Sylow p-subgroup of $\mathfrak{N}(Py)$, $D\colon P$ divides $p^\alpha\}$.*

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press

The proof is an immediate consequence of Lemma 7 since an arbitrary generator $(\bar{P}y)_N^G$ of $\mathcal{W}'(L, p^\alpha)$ is a non-zero multiple of $(\bar{Q}y)_D^G$ which lies in the above set.

We conclude by noting that when $\alpha = 0$ and L is a Sylow p-subgroup of G, the elements of the above spanning set are of the form $(\bar{D}y)_D^G = (\bar{D}r)_D^G$ since $s \in D$. But these elements e just the p'-section sums of Lemma 2 in O'Reilly (1973), giving the ideal of Reynolds (1972) Theorem 1. This ideal has also been studied in Broué (1978) and Iizuka (1973).

If L is a Sylow p-subgroup of G, $\mathcal{W}'(L, p^\alpha)$ will contain only those p-regular classes, and hence block idempotents, of defect $\leqslant p^\alpha$.

References

M. Broué (1978) 'Radicals, hauteurs, p-sections et blocs', *Ann. of Math.* **107**, 89–107.

K. Iizuka, Y. Ito, A. Watanabe (1973), 'A remark on the representations of finite groups IV', *Memoirs of the Faculty of General Education*, Kumamato Univ., Natural Science Series, 8, 1–5.

M. F. O'Reilly (1973) 'Ideals in the centre of a group ring', *Proc. Second Internat. Conf. Theory of Groups*, pp. 536–540 (Lecture Notes in Mathematics 372, Springer Verlag, Berlin).

W. F. Reynolds (1972) 'Sections and ideals of centers of group algebras', *Algebra* **20**, 176–181.

Department of Mathematics
Rhodes University
Grahamstown, 6140
South Africa

https://doi.org/10.1017/S1446788700033425 Published online by Cambridge University Press