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RELATIVE CONTROLLABILITY OF NONLINEAR NEUTRAL VOLTERRA INTEGRODIFFERENTIAL SYSTEMS

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Abstract

Sufficient conditions are derived for the relative controllability of nonlinear neutral Volterra integrodifferential systems with distributed delays in the control variables. The results are a generalization of previous results and are obtained by using Schauder's fixed-point theorem.

1. Introduction

The primary motivation for the study of neutral functional differential equations is the application to transmission-line theory. It is known that the mixed initial-boundary hyberbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed networks (see [15]). In particular, models for systems with delay in the control occur in population studies and in some complex economic systems. More specifically, models for systems with distributed delays in the control occur in the study of agricultural economics and population dynamics [2, 3]. Volterra integrodifferential equations occur often in applied mathematics [7]. In [12] a simplified model for compartmental systems with pipes is represented by nonlinear neutral Volterra integrodifferential equation.

The problem of controllability of linear neutral systems has been investigated by several authors [6, 13, 5]. Angell [1] and Chukwu [8] discussed the functional controllability of nonlinear neutral systems and Underwood and Chukwu [17] studied the null controllability for such systems. Further Chukwu [9] considered the Euclidean

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controllability of a neutral system with nonlinear base. Onwuatu [16] discussed the problem for nonlinear systems of neutral functional differential equations with limited controls. Gahl [11] derived a set of sufficient conditions for the controllability of nonlinear neutral systems through the fixed-point method. In [4] Balachandran established sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems. However for systems with delays in the control variables the problem of relative controllability is still to be studied. A number of papers have appeared on linear systems with different types of delays in control variables. Klamka [14] and Balachandran and Dauer [5] investigated the relative controllability of nonlinear systems with distributed delays in control. In this paper we shall study the relative controllability of nonlinear neutral Volterra integrodifferential systems with distributed delays in control. Our approach, similar to one used by Do [10] for nonlinear neutral systems, is to define the appropriate control and its corresponding solution by an integral equation. This equation is then solved by applying the Schauder fixed-point theorem.

2. Preliminaries

Let Q denote the Banach space of continuous $R^n \times R^m$-valued functions defined on $[0, t_1]$ with the norm $\|(x, u)\| = \|x\| + \|u\|$, where $\|x\| = \sup\{\, |x(t)|, t \in [0, t_1]\,\}$ and $\|u\| = \sup\{\, |u(t)|\,, t \in [0, t_1]\,\}$.

That is, $Q = C_n[0, t_1] \times C_m[0, t_1]$, where $C_n[0, t_1]$ is the Banach space of continuous R^n-valued functions defined on $[0, t_1]$ with the supremum norm. Put $J = [0, t_1]$.

Consider the linear neutral Volterra integrodifferential systems with distributed delays of the form

$$\frac{d}{dt}[x(t) - \int_0^t C(t, s)x(s)ds - g(t)]$$
$$= A(t)x(t) + \int_0^t G(t, s)x(s)ds + \int_{-h}^0 d_\theta B(t, \theta)u(t + \theta) \tag{1}$$

and the nonlinear system

$$\frac{d}{dt}[x(t) - \int_0^t C(t, s)x(s)ds - g(t)]$$
$$= A(t)x(t) + \int_0^t G(t, s)x(s)ds + \int_{-h}^0 d_\theta B(t, \theta)u(t + \theta) + f(t, x(t), u(t)). \tag{2}$$

Here $x \in R^n$ and u is an m-dimensional vector function with $u \in C_m[-h, t_1]$ and $B(t, \theta)$ is an $n \times m$ matrix continuous in t and of bounded variation in θ on $[-h, 0]$

for each $t \in J$. The $n \times n$ matrices $A(t)$, $C(t,s)$ and $G(t,s)$ are continuous in their arguments. The n-vector functions f and g are respectively continuous and absolutely continuous. The integral is in the Lebesgue-Stieltjes sense which is denoted by the symbol d_θ.

Let $h > 0$ be given. For a function $u : [-h, t_1] \to R^m$ and $t \in [0, t_1]$ we use the symbol u_t to denote the function on $[-h, 0)$ defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$. The following definitions of complete state and relative controllability of system (1) or (2) are assumed [14].

DEFINITION 1. The set $z(t) = \{x(t), u_t\}$ is said to be the complete state of the system (1) at time t.

DEFINITION 2. The system (1) or (2) is said to be relatively controllable on J if, for every initial complete state $z(0)$ and $x_1 \in R^n$, there exists a control function $u(t)$ defined on $[0, t_1]$ such that the solution of the system (1) or (2) satisfies $x(t_1) = x_1$.

The solution of (1) can be written as in [18]:

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t (\partial/\partial t) Z(t,s) g(s) ds$$
$$+ \int_0^t Z(t,s) \left(\int_{-h}^0 d_\theta B(s,\theta) u(s+\theta) \right) ds, \tag{3}$$

where $Z(t,s)$ and $(\partial/\partial t)Z(t,s)$ are continuous matrices satisfying

$$(\partial/\partial t) Z(t,s) - \int_0^t (\partial/\partial t) Z(t,\tau) C(\tau,s) d\tau + C(t,s)$$
$$= -Z(t,s)A(s) - \int_0^t Z(t,\tau) G(\tau,s) d\tau,$$

and where $Z(t,t) = I$ and the solution of the nonlinear system (2) is given by

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t (\partial/\partial t) Z(t,s) g(s) ds$$
$$+ \int_0^t Z(t,s) [\int_{-h}^0 d_\theta B(s,\theta) u(s+\theta) + f(s, x(s), u(s))] ds. \tag{4}$$

Using the asymmetric Fubini theorem, as in [14], equation (3) can be written as

$$x(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t (\partial/\partial t) Z(t,s) g(s) ds$$
$$+ \int_{-h}^t d_{B_\theta} \int_\theta^0 Z(t, s-\theta) B(s-\theta, \theta) u_o(s) ds$$
$$+ \int_0^t [\int_{-h}^0 Z(t, s-\theta) d_\theta B_t(s-\theta, \theta)] u(s) ds, \tag{5}$$

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where d_{B_θ} denotes that the integration is in the Lebesgue-Stieltjes sense with respect to the variable θ in B and

$$B_t(s,\theta) = \begin{cases} B(s,\theta), & s \le t \\ 0 & s > t. \end{cases}$$

Define

$$p(t) = Z(t,0)[x(0) - g(0)] + g(t) - \int_0^t (\partial/\partial t) Z(t,s) g(s) ds,$$

$$q(t) = \int_{-h}^0 d_{B_\theta} \int_\theta^0 Z(t, s-\theta) B(s-\theta, \theta) u_\circ(s) ds,$$

$$S(t,s) = \int_{-h}^0 Z(t, s-\theta) d_\theta B_t(s-\theta, \theta)$$

and the controllability matrix

$$W(0,t) = \int_0^t S(t,s) S^*(t,s) ds,$$

where the star (*) denotes the matrix transpose.

Then equations (5) and (4) become

$$x(t) = p(t) + q(t) + \int_0^t S(t,s) u(s) ds \tag{6}$$

and

$$x(t) = p(t) + q(t) + \int_0^t S(t,s) u(s) ds + \int_0^t Z(t,s) f(s, x(s), u(s)) ds. \tag{7}$$

It is easy to prove that, as in [4], the system (1) is relatively controllable on J if and only if W is nonsingular.

It is clear that x_1 can be obtained if there exist continuous x and u such that

$$u(t) = S^*(t_1, t) W^{-1}(0, t_1)[x_1 - p(t_1) - q(t_1) - \int_0^{t_1} Z(t_1, s) f(s, x(s), u(s)) ds] \tag{8}$$

and

$$x(t) = p(t) + q(t) + \int_0^t S(t,s) u(s) ds + \int_0^t Z(t,s) f(s, x(s), u(s)) ds. \tag{9}$$

Now we must find the conditions for the existence of such x and u. If $\alpha_i \in L^1(J)$, $i = 1, 2, \dots q$, the $\|\alpha_i\|$ is the L^1 norm of $\alpha_i(s)$. That is, $\|\alpha_i\| = \int_0^t |\alpha_i| \, ds$. Let us introduce the notation

$$\begin{aligned}
K &= \max\{\|Z(t,s)\| : 0 \le s \le t \le t_1\}, \\
k &= \max_{0 \le s \le t_1} \{S(t_1, s)t_1, 1\}, \\
a_i &= 3k \max_{0 \le t \le t_1} \left\{\|S^*(t_1,t)\| \left\|W^{-1}(0,t_1\right\| \|Z(t_1,t)\| \|\alpha_i\|\right\}, \\
b_i &= 3K \|\alpha_i\|, \\
c_i &= \max\{a_i, b_i\}, \\
d_1 &= 3k \max_{0 \le t \le t_1} \|S^*(t_1,t)\| \left\|W^{-1}(0,t_1\right\| [|x_1| + |p(t_1| + |q_1|], \\
d_2 &= 3[|p(t_1)| + |q(t_1)|], \\
d &= \max\{d_1, d_2\}.
\end{aligned}$$

3. Main result

Now we shall prove the main theorem which is a generalization of Theorem 2 in [4].

THEOREM. *Let measurable functions $\phi_i : R^{n+m} \to R^+$ and L^1 functions $\alpha_i : J \to R^+$, $i = 1, 2, \dots q$ be such that*

$$|f(t,x,u)| \le \sum_{i=1}^{q} \alpha_i(t)\phi_i(x,u) \quad \text{for every} \quad (t,x,u) \in J \times R^{n+m}.$$

Then the relative controllability of (1) implies the relative controllability of (2) if

$$\lim_{r\to\infty} \sup\left(r - \sum_{i=1}^{q} c_i \sup\{\phi_i(x,u) : \|(x,u)\| \le r\}\right) = \infty. \tag{10}$$

PROOF.

$$\text{Define} \quad T : Q \to Q \quad \text{by}$$

$$T(x,u) = (y,v),$$

where

$$v(t) = S^*(t_1,t)W^{-1}(0,t_1)\left[x_1 - p(t_1) - q(t_1) - \int_0^{t_1} Z(t_1,s)f(s,x(s),u(s))ds\right] \tag{11}$$

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and

$$y(t) = p(t) + q(t) + \int_0^t S(t,s)v(s)ds + \int_0^t Z(t,s)f(s,x(s),u(s))ds. \tag{12}$$

By our assumptions, the operator T is continuous. Clearly the solution u and x to (8) and (9) are fixed points of T. We shall prove the existence of such fixed points by using the Schauder fixed-point theorem.

Let $\Psi_i(r) = \sup\{\phi_i(x,u) : \|(x,u)\| \le r\}$. Since (10) holds, there exists $r_\circ > 0$ such that

$$\sum_{i=1}^{q} c_i \Psi_i(r_\circ) + d \le r_\circ.$$

Now let

$$Q_{r_\circ} = \{(x,u) \in Q : \|(x,u)\| \le r_\circ\}.$$

If $(x,u) \in Q_{r_\circ}$ then from (11) and (12) we have

$$\begin{aligned}
\|v\| &\le \|S^*(t_1,t)\| \left\|W^{-1}(0,t_1)\right\| \Big[|x_1| + |p(t_1)| + |q(t_1)| \\
&\qquad + \int_0^{t_1} \|Z(t_1,s)\| \sum_{i=1}^{q} \alpha_i(s)\phi_i(x(s),u(s))ds\Big] \\
&\le (d_1/3k) + (1/3k)\sum_{i=1}^{q} \alpha_i \Psi_i(r_\circ) \\
&\le (1/3k)(d + \sum_{i=1}^{q} c_i \Psi_i(r_\circ)) \\
&\le (r_\circ/3k) \le (r_\circ/3)
\end{aligned}$$

and

$$\begin{aligned}
\|y\| &\le |p(t)| + |q(t)| + \int_0^t \|S(t,s)\|\,\|v\|\,ds + \int_0^t \|Z(t,s)\| \sum_{i=1}^{q} \alpha_i \phi_i(x(s),u(s))ds \\
&\le (d/3) + k\,\|v\| + K \sum_{i=1}^{q} \|\alpha_i\|\, \Psi_i(r_\circ) \\
&\le (d/3) + k\,\|v\| + 1/3 \sum_{i=1}^{q} c_i \Psi_i(r_\circ) \\
&\le (1/3)(d + \sum_{i=1}^{q} c_i \Psi_i(r_\circ)) + k\,\|v\| \\
&\le (r_\circ/3) + (r_\circ/3) = 2(r_\circ/3).
\end{aligned}$$

Hence T maps Q_{r_0} into itself. Further it is easy to see that $T(Q_r)$ is equicontinuous for all $r > 0$ [10]. By the Ascoli-Arzela theorem, $T(Q_{r_0})$ is compact in Q. Since Q_{r_0} is closed, bounded and convex, the Schauder fixed-point theorem guarantees that T has a fixed point $(x, u) \in Q_{r_0}$ such that $T(x, u) = (x, u)$. It follows that, for $(x, u) \equiv (y, v)$, we have

$$x(t) = p(t) + q(t) + \int_0^t S(t,s)u(s)ds + \int_0^t Z(t,s)f(s, x(s), u(s))ds. \quad (13)$$

Thus the solutions of (8) and (9) exist. Hence the system is relatively controllable on J.

REMARK. To apply the above theorem we must construct α_is and ϕ_is such that (10) is satisfied. These constructions are different for different situations. However an obvious construction of α_is and ϕ_is is easily achieved by taking $q = 1$, $\alpha_1 = \alpha = 1$ and

$$\phi_1(x, u) = \phi(x, u) = \sup\{|f(t, x, u)| : t \in J\}.$$

In this case (10) holds if

$$\lim_{r\to\infty} \inf(1/r) \sup\{\phi(x, u) : \|(x, u)\| \leq r\} < 1/c_1.$$

Now we state a corollary which is a particular case of the above theorem.

COROLLARY. *If the continuous function f satisfies the condition*

$$\lim_{\|(x,u)\|\to\infty} |f(t, x, u)| / \|(x, u)\|$$

uniformly in $t \in J$ and if the system (1) *is relatively controllable on J, then the system* (2) *is relatively controllable on J.*

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