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Continuity of derivations on topological algebras of power series

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Let A be an algebra of formal power series in one indeterminate over the complex field, D a derivation on A. It is shown that if A has a Fréchet space topology under which it is a topological algebra, then D is necessarily continuous provided the coordinate projections satisfy a certain equicontinuity condition. This condition is always satisfied if A is a Banach algebra and the projections are continuous. A second result is given, with weaker hypothesis on the projections and correspondingly weaker conclusion.

In this paper we show that derivations on a certain class of metrizable topological algebras of formal power series are necessarily continuous. This result is contained in one of Johnson [3], however the proof as outlined in that paper requires the algebras to satisfy a certain algebraic condition which we do not assume here. Algebras satisfying this condition are considered in [4], where it is shown that they are necessarily semisimple in the Banach algebra case. The class we consider here, however, contains non-semisimple Banach algebras. Indeed, we consider a particular case below of a radical Banach Algebra.

Throughout this paper A will denote an algebra of formal power series in an indeterminate t over the complex field, containing the element t, but not necessarily having an identity. Elements of A will be denoted by expressions of the form $\sum a_i t^i$. We suppose further that A is a

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topological algebra under a complete locally convex metrizable topology determined by a sequence $\{\|\cdot\|_n\}_{n\geq 1}$ of seminorms, and note that multiplication in A is necessarily jointly continuous ([1], Theorem 5).

THEOREM 1. *Suppose that the projections* $p_j : \sum a_i t^i \to a_j$ *are continuous and, moreover, that there is a sequence* $\{\gamma_n\}_{n\geq 0}$ *of positive real numbers such that the family* $\left\{\gamma_n^{-1} p_n\right\}_{n\geq 0}$ *is equicontinuous. Then any derivation on* A *is continuous.*

Proof. Suppose to the contrary that D is a discontinuous derivation on A. Since $\{p_n\}_{n\geq 0}$ is a separating family of continuous linear functionals on A it follows by the closed graph theorem that at least one of the functionals $\{p_n D\}_{n\geq 0}$, and hence one with least index, $p_k D$ say, is discontinuous. In order to give a proof valid for all values of k we make the convention that empty sums, of the form $\sum_{i=1}^{j} (\cdot)_i$ with $j < 1$, have the value zero.

Since the functionals $p_0 D, \ldots, p_{k-1} D$ are continuous there is a neighbourhood U of zero in A such that if $x \in U$ then $|p_j(Dx)| \leq 1$, $0 \leq j \leq k-1$. Also, by the equicontinuity of the family $\left\{\gamma_n^{-1} p_n\right\}_{n\geq 0}$ there is a neighbourhood V of zero in A such that if $x \in V$ then $|p_n(x)| \leq \gamma_n$ for all $n \geq 0$. Finally, by the joint continuity of multiplication in A there is a sequence $\{M_n\}_{n\geq 1}$ of positive integers, and a sequence $\{\delta_n\}_{n\geq 1}$ of positive numbers such that if $\|x\|_i < \alpha$, $\|y\|_i < \beta$ for $1 \leq i \leq M_n$, where $\alpha\beta < \delta_n$, then $\|xy\|_j < 1$, $1 \leq j \leq n$.

Let $\{\mu_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\gamma_n \mu_n^{-1} \to 0$ as $n \to \infty$. Define inductively a sequence $\{x_n\}_{n\geq 1} \subseteq A$ such that

(i) $x_n \in U$, $Dt.x_n \in V$.

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(ii) $\|x_n\|_i < 2^{-n}\,\delta_n \min\limits_{\substack{1\le j\le n\\ 1\le m\le n}} \|t^j\|_m^{-1}$ for $1 \le i \le M_n$.

(iii) $|p_k(Dx_n)| \ge \mu_{n+k} + k + \sum\limits_{i=1}^{n+k+1} i\gamma_{n+k+1-i} + \sum\limits_{i=1}^{n-1} \left|p_{i+k}(Dx_{n-i})\right|$.

It follows from (ii) that if $m > j$ then $\left\|t^{m-j}x_m\right\|_i < 2^{-m}$, $1 \le i \le m$, and so for each j the series $\sum\limits_{m>j} t^{m-j}x_m$ converges in A to some element y_j . Set $y = y_0 = \sum\limits_{m\ge1} t^m x_m$. Then

$$
\begin{aligned}
p_n(Dy) &= p_n D\left\{t^{n+2}y_{n+2} + \sum_{i=1}^{n+2} t^i x_i\right\} \\
&= p_n D\left\{\sum_{i=1}^{n+2} t^i x_i\right\} \\
&= p_n\left\{Dt.x_1 + \sum_{i=2}^{n+2} it^{i-1}Dt.x_i + \sum_{i=1}^{n+2} t^i Dx_i\right\} \\
&= p_n(Dt.x_1) + \sum_{j=0}^{n}\left\{\sum_{i=2}^{n+2} ip_j(t^{i-1})p_{n-j}(Dt.x_i) + \sum_{i=1}^{n+2} p_j(t^i)p_{n-j}(Dx_i)\right\} \\
&= \sum_{i=1}^{n+1} ip_{n+1-i}(Dt.x_i) + \sum_{i=0}^{n-1} p_i(Dx_{n-i}) .
\end{aligned}
$$

Thus if $n \ge k+2$

$$
|p_n(Dy)| \ge |p_k(Dx_{n-k})| - \sum_{i=1}^{n+1} i|p_{n+1-i}(Dt.x_i)| - \left[\sum_{i=0}^{k-1} + \sum_{i=k+1}^{n-1}\right] |p_i(Dx_{n-i})| .
$$

But by (i) $\sum\limits_{i=0}^{k-1} |p_i(Dx_{n-i})| \le k$ and $\sum\limits_{i=1}^{n+1} i\left|p_{n+1-i}(Dt.x_i)\right| \le \sum\limits_{i=1}^{n+1} i\gamma_{n+1-i}$, and so by (iii) $|p_n(Dy)| \ge \mu_n$.

Now choose a non-zero scalar λ so small that $\lambda Dy \in V$. Then for each $n \ge 0$, $|\lambda|\mu_n \le |p_n(\lambda Dy)| \le \gamma_n$, and so $\gamma_n\mu_n^{-1} \ge |\lambda| > 0$. But this is impossible for all n by the definition of the sequence $\{\mu_n\}$; and so the result follows.

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REMARK. The hypothesis of Theorem 1 is satisfied by any Banach algebra of power series in which the projections are continuous, for clearly the family $\left\{\|p_n\|^{-1}p_n\right\}_{n\geq 0}$ is equicontinuous. The algebra of entire functions on the complex plane, considered as a power series algebra, is another example. Cauchy's inequalities show easily that in this case the family $\{p_n\}_{n\geq 0}$ is equicontinuous.

In Newman [5] it is shown that there are no non-zero continuous derivations on a certain radical Banach algebra R, the elements of which are formal power series $\sum_{i\geq 1} a_i t^i$ with $\sum |a_i||\lambda_i{}^i| < \infty$ for a certain sequence $\{\lambda_i\}_{i\geq 1}$, this latter expression defining the norm. This algebra clearly satisfies the conditions of Theorem 1, and we conclude that R admits no non-zero derivations. This establishes the falsity of the conjecture of Singer and Wermer stated in [5], namely that a commutative Banach algebra which admits no non-zero derivations is necessary semisimple. Newman's result had shown it false if the derivations were restricted to being continuous. The conjecture is true for finite dimensional algebras by [2], Theorem 4.3.

The proof of Theorem 1 made use of the fact that for each $x \in A$ there is a positive number $K(x)$ such that $\left|\gamma_n^{-1}\, p_n(x)\right| \leq K(x)$ for all $n \geq 0$. If we use this condition as hypothesis we can obtain the following result, which requires no continuity restrictions on the projections.

THEOREM 2. *Suppose that* A *satisfies the following two conditions:*

(a) if $\sum a_i t^i \in A$ *then* $\sum_{i\geq n} a_i t^i \to 0$ *in* A *as* $n \to \infty$;

(b) there is a sequence $\{\gamma_n\}$ *of positive real numbers such that for each* $x \in A$ *there is* $K(x)$ *such that* $\left|\gamma_n^{-1}\, p_n(x)\right| \leq K(x)$ *for all* $n \geq 0$.

Let D *be a derivation on* A *with* $Dt = 0$. *Then* $D = 0$.

Proof. The proof follows the same line as that of Theorem 1 and we give an outline only.

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Suppose $p_k D$ is discontinuous but that $p_0 D, \ldots, p_{k-1} D$ are continuous. Let U, $\{\delta_n\}$, $\{M_n\}$ be as in the proof of Theorem 1, and let $\{\mu_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\gamma_n \mu_n^{-1} \to 0$ as $n \to \infty$. Define inductively a sequence $\{x_n\}_{n\geq 1} \subseteq A$ such that

(i)' $x_n \in U$.

(ii)' $\|x_n\|_i < 2^{-n}\delta_n \min_{\substack{1\leq j\leq n \\ 1\leq m\leq n}} \left\|t^j\right\|_m^{-1}$ for $1 \leq i \leq M_n$.

(iii)' $\left|p_k\left(Dx_n\right)\right| \geq \mu_{n+k} + k + \sum_{i=1}^{n-1} \left|p_{i+k}\left(Dx_{n-i}\right)\right|$.

Then $y = \sum_{m\geq 1} t^m x_m \in A$ and $\left|p_n(Dy)\right| \geq \mu_n$. But then $\left|\gamma_n^{-1} p_n(Dy)\right| \geq \gamma_n^{-1}\mu_n \to \infty$ as $n \to \infty$ by the definition of $\{\mu_n\}$ contradicting hypothesis *(b)*. It follows that $p_n D$ is continuous for all $n \geq 0$.

To see that $D = 0$, suppose $D\left(\sum a_i t^i\right) = \sum b_i t^i$. Then for each j, $p_j D\left(\sum_{i\geq n} a_i t^i\right) = b_j$ for all n, and so by *(a)* it follows that $b_j = 0$. Thus $D\left(\sum a_i t^i\right) = 0$, as required.

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