CONVEX HULLS OF SIMPLE SPACE CURVES

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1. Introduction. The convex hull of an arbitrary set M in real Euclidean n-space is known to consist of all the points within the r-simplexes with $r + 1$ vertices from M, $r \leqslant n$. This note shows that if M is specialized to be a curve A_n of real order n, then its convex hull consists of all the points within the r-simplexes with $r + 1$ vertices on A_n, $n = 2r + 1$ or $n = 2r$. In the first case each interior point is within exactly one simplex. This result was given by Egerváry **(1)** for $n = 3$. If n is even each interior point of the convex hull of A_n is within a 1-parameter system of $\frac{1}{2}n$-simplexes. The class of curves A_n includes the twisted n-ics, the convex hulls of which have been studied by Karlin and Shapley **(2)**. Some of their results are consequences of the present results.

2. Some definitions. A curve A_n is *defined* to be a 1–1 continuous mapping in real Euclidean n-space of all the real numbers s computed modulo 1 or of the interval, $0 \leqslant s \leqslant 1$, which satisfies the *order condition* that no hyperplane contains more than n points of A_n.

The order condition implies that any linear k-space, $0 \leqslant k < n$, cannot contain more than $k + 1$ points of A_n. If a hyperplane H supports A_n at an inner point s' then s' is defined to have multiplicity two within H. By displacing the hyperplanes it is possible to show that the *sharpened order condition*[1] holds that no hyperplane contains more than n points of A_n if each point is counted with its proper multiplicity of one or two.

The symbol $[A, B, \ldots]$ denotes the intersection of all the linear spaces which include the point sets $A, B, \ldots$, while $\{A, B, \ldots\}$ denotes the convex hull of the union of the point sets $A, B, \ldots$. Two sets A and B are said to be separated by a hyperplane H provided A is in one of the closed half spaces bounded by H and B in the other.

3. The boundary of A_n. The following lemma is stated without proof.

LEMMA 1. *If a hyperplane H supports a compact set X, then $\{H \cap X\} = H \cap \{X\}$.*

THEOREM 1. *The boundary of $\{A_n\}$ consists of all the points within all the q-simplexes for which the vertices are $q + 1$ points of A_n including e endpoints, $2q \leqslant n - 2 + e$, $(e = 0, 1, 2)$.*

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Proof. If P be a boundary point of $\{A_n\}$, a hyperplane H exists which supports A_n and contains P. Let $s_0, s_1, \ldots, s_q$ be the distinct curve points in $H \cap \{A_n\}$. Because of the order condition, $\{s_0, s_1, \ldots, s_q\}$ is a q-simplex. By Lemma 1,

$$P \in H \cap \{A_n\} = \{H \cap A_n\} = \{s_0, s_1, \ldots, s_q\}.$$

As H supports A_n an interior point s_i of A_n must be included in H twice. Consequently if e denotes the number of endpoints of A_n in H, it follows from the order condition that

$$e + 2(q + 1 - e) \leqslant n \text{ or } 2q \leqslant n - 2 + e.$$

Thus each boundary point of $\{A_n\}$ is within a q-simplex, with the required properties.

Conversely let P be a point of a q-simplex $\{s_0, s_1, \ldots, s_q\}$ for which $2q \leqslant n - 2 + e$. Then a hyperplane exists which contains P and supports A_n. To construct such a hyperplane, for each point s_i interior to A_n, let N_i be an arc $s'_i < s < s_i$ and if A_n is not closed let N', N'' be neighbourhoods of the endpoints 0, 1 respectively. Let H be a hyperplane which contains n points of A_n including all s_i, s'_i and so that the remaining $n - 2(q + 1) + e$ curve points within H are distributed among the arcs N_i, N', N'' in such a way that no arc N_i contains an odd number of these points. This distribution is always possible because if A_n is closed n is even and $e = 0$. If $N_i \to s_i$, $N' \to 0$, $N'' \to 1$ then any limiting position of H contains P and supports A_n. As $P \in \{s_0, s_1, \ldots, s_q\} \subseteq \{A_n\}$, P is a boundary point of $\{A_n\}$. The proof is now complete.

4. The structure of $\{A_n\}$. If $2r = n$ or $2r + 1 = n$, S_r is *defined* to be an r-simplex with interior points of A_n as vertices except for even n when at most one of the vertices may be an endpoint of A_n.

THEOREM 2. *The interior points P of $\{A_n\}$ consist of all the interior points of the simplexes S_r.*

For odd n, S_r is uniquely determined by any one of its interior points P; for even n, S_r is uniquely determined by an interior point P and any one vertex which can be either endpoint of A_n or any arbitrary point of A_n if it is closed.

Proof. We show first that every interior point P of a simplex S_r is an interior point of $\{A_n\}$. As $S_r \subseteq \{A_n\}$ it will be sufficient to show P is not a boundary point of $\{A_n\}$. Let e be the number of vertices of S_r which are endpoints of A_n. If P were a boundary point of $\{A_n\}$ it would be within a hyperplane H which would support $\{A_n\}$. H would also support S_r and consequently, as P is an inner point of S_r, $S_r \subseteq H$. Therefore H would contain $2(r + 1 - e) + e$ points of A_n. This would contradict the order condition as, by the definition of S_r, $e = 0$ if $n = 2r + 1$ and $e \leqslant 1$ if $n = 2r$. Hence the inner points of the simplexes S_r are all inner points of $\{A_n\}$.

We next show that a given interior point P of $\{A_n\}$ is an interior point of a simplex S_r. Let a be any real number if A_n is closed and 0 if A_n is open. Denote by $A(a, s')$ the arc of points $s, a \leqslant s \leqslant s'$. Let s_P be the least upper bound of all s' for which $P \notin \{A(a, s')\}$.

We prove that $P \in \{A(a, s_P)\}$. If this were false, P and $\{A(a, s_P)\}$ would be separated by a hyperplane at a positive distance from $\{A(a, s_P)\}$. This hyperplane would also separate $A(a, s')$ and P for $s' > s_P$ provided s' were sufficiently close to s_P. Consequently $P \notin \{A(a, s')\}$ contrary to the choice of s_P.

P is on a supporting hyperplane of $\{A(a, s_P)\}$. To prove this let s_μ be an increasing sequence which converges to s_P. Because $P \notin \{A(a, s_\mu)\}$ a hyperplane H_μ exists which supports $\{A(a, s_\mu)\}$ and contains P. s_μ can be chosen so that H_μ converges. If H be its limit then $P \in H$ and H supports $\{A(a, s_\mu)\}$. But, as s_μ is arbitrary, H supports $\{A(a, s_P)\}$. From this result, together with the fact that $P \in \{A(a, s_P)\}$, it follows that P is a boundary point of $\{A(a, s_P)\}$.

Consequently, by Theorem 1, a simplex S_q exists which contains P, has vertices on $A(a, s_P)$ and for which $2q \leqslant n - 2 + e$, where e is the number of vertices of S_q which are endpoints of $A(a, s_P)$. The vertices of S_q are also on A_n. Let e' be the number of these vertices which are endpoints of A_n. As P is not a boundary point of $\{A_n\}$, $2q > n - 2 + e'$. Therefore $e' < e$ and so $0 < e$. If A_n is open, $e' = e - 1$ as 0 is a common endpoint of A_n and $A(a, s_P)$. The two inequalities yield the result $2q = n - 2 + e$. Hence, if $n = 2r$, then $e = 2$ and $q = r$ and, if $n = 2r + 1$, $e = 1$ and $q = r$. If A_n is closed n is even and $e' = 0$. In this case the inequalities show $e = 2$ and $r = q$. P cannot be a point of a face of S_r for such points, by Theorem 1, are boundary points of $\{A_r\}$. Therefore P is an interior point of the r-simplex S_r which satisfies the requirements of the theorem as $e' = 0$ for odd n and $e' \leqslant 1$ for even n. This completes the proof of the first part of the theorem.

For even n, $e = 2$ and consequently a is a vertex of S_r. If A_n is closed a is arbitrary and so in this case, for a given P, an S_r exists with an arbitrary vertex. If A_n is open $a = 0$. After a reversal of orientation of the points on the curve, the other endpoint of A_n can be represented by the number 0. Therefore S_r can be chosen so that either endpoint of A_n is a vertex provided n is even.

Suppose now P is a point within two distinct simplexes with vertices $s_0, s_1, \ldots, s_r; s'_0, s'_1, \ldots, s'_r$ and that P is not in a face of $\{s_0, s_1, \ldots, s_r\}$. Let k, $0 \leqslant k \leqslant r$, be the number of vertices common to both simplexes. It follows, with the use of the Steinitz replacement theorem, that the space

$$[s_0, s_1, \ldots, s_r, s'_0, s'_1, \ldots, s'_r]$$

has dimension at most $2r - k$. It contains $2(r + 1) - k$ points of A_n. This leads to a contradiction of the order condition unless $2r - k = n$ in which case $k = 0$ and $n = 2r$. This proves, for odd n, that P is within only one simplex S_r and, for even n, that P is never in more than one simplex S_r with a given vertex. The proof is now complete.

COROLLARY. *Every point P in the interior of $\{A_{2r}\}$ is an interior point of each of two suitably chosen simplexes S_r, S'_r which have no common vertex.*

Proof. If A_{2r} is open each interior point P of $\{A_{2r}\}$ is, by the Theorem, interior to a simplex S_r (S'_r) with the endpoint $s = 0$, ($s = 1$) as a vertex. If S_r, S'_r were to have a common vertex then, by the Theorem, they would be identical and both endpoints of A_{2r} would be vertices in contradiction to the definition of the simplexes. If A_{2r} is closed the result is clear.

LEMMA 2. *If the vertices of two r-simplexes S_r, S'_r which have no common vertex are all on A_{2r} and if an arc of A_{2r} exists which contains two vertices of S_r and no vertex of S'_r, then S_r, S'_r have no point in common.*

Proof. Let $s_0, s_1, \ldots, s_r$, $s_0 < s_1 < \ldots < s_r < s_0 + 1$ $(= s_{r+1})$ be the vertices of S_r. By the hypothesis an arc $s_k \leqslant s \leqslant s_{k+1}$ exists which contains no vertex of S'_r, $0 \leqslant k < r$, if A_{2r} is open and $0 \leqslant k \leqslant r$, if A_{2r} is closed. In the latter case the coordinates may be adjusted so that $0 \leqslant k < r$. As S_r, S'_r have no common vertex, *distinct* curve points $t'_1, t_1, \ldots, t'_r, t_r$ of A_{2r} exist so that

$$t'_1 \leqslant s_0 \leqslant t_1 < t'_2 \leqslant s_2 \leqslant t_2 < \ldots < t'_{k+1} \leqslant s_k < s_{k+1} \leqslant t_{k+1} < \ldots$$
$$< t'_r \leqslant s_r \leqslant t_r \leqslant t'_1 + 1$$

and so that none of the arcs $t'_1 \leqslant s \leqslant t_i$, $1 \leqslant i \leqslant r$, contains a vertex of S'_r. Let H be the hyperplane $[t'_1, t_1, \ldots, t'_r, t_r]$. As H intersects A_{2r} only in the $2r$ points t'_i, t_i, $1 \leqslant i \leqslant r$, all the points of the arcs $t'_i \leqslant s \leqslant t_i$, $1 \leqslant i \leqslant r$, are either on H or on the same side of H while all the points of A_{2r} not within the above arcs are on the opposite side of H. Thus H separates the vertices of S_r from those of S'_r. Furthermore all the vertices of S'_r are at a positive distance from H. Hence S_r and S'_r have no points in common. The Lemma is now proved.

Convex hulls are defined for affine space. The following result shows that the convex hull $\{A_{2r}\}$ can be defined in terms of projective concepts.

THEOREM 3. *If $s_0, s_1, \ldots, s_r$; $s'_0, s'_1, \ldots, s'_r$ are curve points of A_{2r} for which*

$$0 \leqslant s_0 < s'_0 < s_1 < \ldots < s_r < s'_r \leqslant 1,$$

for open A_{2r} and

$$s_0 < s'_0 < s_1 < \ldots < s_r < s'_r < s_0 + 1 \ (= s_{r+1})$$

for closed A_{2r} then the interior of $\{A_{2r}\}$ consists of all the intersections

$$[s_0, s_1, \ldots, s_r] \cap [s'_0, s'_1, \ldots, s'_r].$$

Proof. Let P be a given point in the interior of $\{A_{2r}\}$. By the Corollary to Theorem 2, simplexes S_r, S'_r exist, without a common vertex, both of which

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contain P as an interior point. Let $s_0, s_1, \ldots, s_r$, $0 \leqslant s_0 < s_1 < \ldots < s_r < s_0 + 1$ be the vertices of S_r. As S_r, S'_r have the common interior point P it follows from Lemma 2 that each arc $s_i \leqslant s \leqslant s_{i+1}$, $0 \leqslant i < r$, contains exactly one vertex of S'_r. Therefore if $s'_0, s'_1, \ldots, s'_r$ be the vertices of S'_r the subscripts may be adjusted so that, for closed A_{2r},

$$s_0 < s'_0 < s_1 < \ldots < s'_{r-1} < s_r < s'_r < s_0 + 1$$

and, for open A_{2r}, either

$$0 \leqslant s_0 < s'_0 < s_1 < \ldots < s_r < s'_r \leqslant 1$$

or

$$0 \leqslant s'_0 < s_0 < \ldots < s'_r < s_r \leqslant 1.$$

As P is a common point of the simplexes

$$P \in [s_0, s_1, \ldots, s_r] \cap [s'_0, s'_1, \ldots, s'_r].$$

Now let Q be any point of $[s_0, s_1, \ldots, s_r] \cap [s'_0, s'_1, \ldots, s'_r]$ where

$$s_0, s_1, \ldots, s_r, s'_0, s'_1, \ldots, s'_r$$

are points of A_{2r} which satisfy the inequality system. The r-spaces $[s_0, s_1, \ldots, s_r]$, $[s'_0, s'_1, \ldots, s'_r]$ must have at least one point in common as $2r = n$. They cannot have more than one point in common for then

$$[s_0, s_1, \ldots, s_r, s'_0, \ldots, s'_r]$$

would have dimension at most $2r - 1$ and contain $2r + 2$ points of A_{2r}, in contradiction to the order condition.

Q cannot be a point on a proper face of either simplex $\{s_0, s_1, \ldots, s_r\}$, $\{s'_0, s'_1, \ldots, s'_r\}$. Suppose, for example, Q to be within the face $\{s_0, s_1, \ldots, s_{r-1}\}$. Then the space

$$[s_0, s_1, \ldots, s_{r-1}, s'_0, \ldots, s'_r]$$

would have dimension at most $2r - 1$ and contain $2r + 1$ points of A_{2r} in contradiction to the order condition.

If $s_0, s_1, \ldots, s_r, s'_0, \ldots, s'_r$ move continuously so that the inequalities are always satisfied, Q is uniquely defined and moves continuously. We know, if $Q = P$, that Q is interior to $\{A_{2r}\}$ as well as to both simplexes $\{s_0, s_1, \ldots, s_r\}$, $\{s'_0, s'_1, \ldots, s'_r\}$. As Q cannot enter a proper face of either of these simplexes it must remain in the interior of both of them. Q cannot enter the boundary of $\{A_{2r}\}$. For otherwise it would be in a hyperplane H supporting $\{A_{2r}\}$ and consequently supporting $\{s_0, s_1, \ldots, s_r\}$. As Q is an interior point of the simplex, $[s_0, s_1, \ldots, s_r] \subseteqq H$. It follows from the inequality system that at most one vertex of $\{s_0, s_1, \ldots, s_r\}$ is an endpoint of A_{2r}. Hence H would contain at least $2(r + 1) - 1 = 2r + 1$ points of A_{2r} in contradiction to the order condition. Therefore Q must always remain in the interior of $\{A_{2r}\}$. The proof is now complete.

References

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