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ON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

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Abstract

Some results on fixed points of asymptotically regular mappings are obtained in complete metric spaces and normed linear spaces.

The structure of the set of common fixed points is also discussed in Banach spaces. Our work generalizes essentially known results of Das and Naik, Fisher, Jaggi, Jungck, Rhoades, Singh and Tiwari and several others.

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1. Introduction

Many authors have extended the well-known result of Jungck [35]. In addition to the authors specifically cited in this paper, Conserva [9], Cheh-Chih Yeh [5], Fisher [17], [20], Khan [27], Khan and Imdad [30], Park [46], Park and Rhoades [48], Singh [61] have proved their results in complete metric spaces, Khan [28] in uniform spaces and Cheh-Chih Yeh [6] in L-spaces.

Sessa [60] has generalized the result of [10], considering two selfmaps A, S of a complete metric space (X, d) which are weakly commuting, that is,

(1.1) $$d(ASx, SAx) \leqslant d(Sx, Ax)$$

for any $x \in X$.

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EXAMPLE 1. Let $X = [0, 1]$ equipped with the euclidean metric and $Sx = x/(x+16)$, $Ax = x/8$ for any $x \in X$. We have for any $x \in X$:

$$d(ASx, SAx) = \frac{x}{x+128} - \frac{x}{8x+128} = \frac{7}{x+128} \cdot \frac{x^2}{8x+128}$$
$$\leqslant \frac{x^2+8x}{8(x+16)} = \frac{x}{8} - \frac{x}{x+16} = d(Sx, Ax).$$

Thus S and A satisfy (1.1) but do not commute for any $x \neq 0$.

Using an idea developed in [53], the results of this paper are established in complete metric spaces without considering the usual sequence of successive approximations in order to show the existence of common fixed points. Further, in convex Banach spaces the structure of the set of common fixed points is investigated without assuming any hypothesis of commutativity of the mappings under discussion.

Two survey papers of the first author [50], [51] compare many contractive conditions. It is easily seen that most of the contractive conditions used imply the asymptotic regularity of the mappings under consideration, so the study of such mappings plays an important role in fixed point theory.

2. Results in complete metric spaces

The following definition appears in [53]:

DEFINITION 1. Let A and S be two selfmaps of X and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be *asymptotically S-regular* with respect to A if $d(Ax_n, Sx_n) \to 0$ as $n \to \infty$.

If A is the identity map of X, Definition 1 becomes that of Engl [15].

Drawing inspiration from the contractive conditions of Hardy and Rogers [24] and Jungck [35], we present our main theorem.

THEOREM 1. *Let A, S, T be three selfmaps of a complete metric space (X, d) satisfying*

(2.1) $$d(Sx, Ty) \leqslant a_1 d(Sx, Ax) + a_2 d(Tx, Ax) + a_3 d(Sy, Ay) + a_4 d(Ty, Ay) + a_5 d(Sx, Ay) + a_6 d(Tx, Ay) + a_7 d(Sy, Ax) + a_8 d(Ty, Ax) + a_9 d(Ax, Ay)$$

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for all x, y in X, where the $a_i = a_i(x, y)$, $i = 1, 2, \ldots, 9$, are nonnegative functions such that

$$(2.2)\quad \max\Big\{\sup_{x,y\in X}(a_1 + a_2 + a_5 + a_6),$$
$$\sup_{x,y\in X}(a_3 + a_4 + a_7 + a_8),\ \sup_{x,y\in X}(a_5 + a_6 + a_7 + a_8 + a_9)\Big\} < 1.$$

(2.3) *If A is continuous,*

(2.4) *A weakly commutes with S and T, and*

(2.5) *there exists a sequence which is asymptotically S-regular and T-regular with respect to A,*

then A, S and T have a unique common fixed point.

PROOF. Let $\{x_n\}$ be a sequence satisfying (2.5). Using (2.1),

$$\begin{aligned} d(Ax_n, Ax_m) &\leqslant d(Ax_n, Sx_n) + d(Sx_n, Tx_m) + d(Tx_m, Ax_m) \\ &\leqslant d(Ax_n, Sx_n) + a_1 d(Sx_n, Ax_n) + a_2 d(Tx_n, Ax_n) \\ &\quad + a_3 d(Sx_m, Ax_m) + a_4 d(Tx_m, Ax_m) + a_5 d(Sx_n, Ax_m) \\ &\quad + a_6 d(Tx_n, Ax_m) + a_7 d(Sx_m, Ax_n) + a_8 d(Tx_m, Ax_n) \\ &\quad + a_9 d(Ax_n, Ax_m) + d(Tx_m, Ax_m) \end{aligned}$$

where $a_i = a_i(x_n, x_m)$. Therefore

$$\begin{aligned} (1 - a_5 - a_6 - a_7 - a_8 - a_9) \cdot d(Ax_n, Ax_m) &\leqslant (1 + a_1 + a_5) \\ &\quad \cdot d(Ax_n, Sx_n) + (a_2 + a_6) \cdot d(Tx_n, Ax_n) + (a_3 + a_7) \\ &\quad \cdot d(Sx_m, Ax_m) + (a_4 + a_8 + 1) \cdot d(Tx_m, Ax_m) \end{aligned}$$

which, from (2.2) and (2.5), implies that $\{Ax_n\}$ is Cauchy.

Since X is complete, let $z = \lim Ax_n$.

Being $d(Sx_n, z) \leqslant d(Sx_n, Ax_n) + d(Ax_n, z)$, $\{Sx_n\} \to z$. Similarly, $\{Tx_n\} \to z$. Also, using (2.3), $\{A^2x_n\} \to Az$, $\{ASx_n\} \to Az$ and $\{ATx_n\} \to Az$.

From (2.4),

$$\begin{aligned} d(SAx_n, Az) &\leqslant d(SAx_n, ASx_n) + d(ASx_n, Az) \\ &\leqslant d(Ax_n, Sx_n) + d(ASx_n, Az), \end{aligned}$$

whence $\{SAx_n\} \to Az$. Similarly, $\{TAx_n\} \to Az$.

Further, from (2.1) with $a_i = a_i(Ax_n, z)$,

$$\begin{aligned}
d(Az,Tz) &\leqslant d(Az,SAx_n) + d(SAx_n,Tz)\\
&\leqslant d(Az,SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n)\\
&\quad + a_3 d(Sz,Az) + a_4 d(Tz,Az) + a_5 d(SAx_n, Az)\\
&\quad + a_6 d(TAx_n, Az) + a_7 d(Sz, A^2x_n) + a_8 d(Tz, A^2x_n)\\
&\quad + a_9 d(A^2x_n, Az)\\
&\leqslant d(Az,SAx_n) + a_1 d(SAx_n, A^2x_n) + a_2 d(TAx_n, A^2x_n)\\
&\quad + (a_3 + a_4 + a_7 + a_8)\cdot \max\{d(Az,Sz),\, d(Az,Tz)\}\\
&\quad + (a_5 + a_6 + a_7 + a_8 + a_9)\\
&\quad \cdot \max\{d(SAx_n, Az),\, d(TAx_n, Az),\, d(A^2x_n, Az)\}.
\end{aligned}$$

Taking the limsup, we have

$$d(Az,Tz) \leqslant \sup_{x,y\in X}(a_3 + a_4 + a_7 + a_8)\cdot \max\{d(Az,Sz),\, d(Az,Tz)\}.$$

Similarly,

$$d(Az,Sz) \leqslant \sup_{x,y\in X}(a_1 + a_2 + a_5 + a_6)\cdot \max\{d(Az,Sz),\, d(Az,Tz)\}.$$

Then, from (2.2) it follows $Az = Sz = Tz$.

From (2.1), with $a_i = a_i(x_n, Ax_n)$,

$$\begin{aligned}
d(Sx_n, TAx_n) &\leqslant a_1 d(Sx_n, Ax_n) + a_2 d(Tx_n, Ax_n) + a_3 d(SAx_n, A^2x_n)\\
&\quad + a_4 d(TAx_n, A^2x_n) + a_5 d(Sx_n, A^2x_n) + a_6 d(Tx_n A^2x_n)\\
&\quad + a_7 d(SAx_n, Ax_n) + a_8 d(TAx_n, Ax_n) + a_9 d(Ax_n, A^2x_n)\\
&\leqslant a_1 d(Sx_n, Ax_n) + a_2 d(Tx_n, Ax_n) + a_3 d(SAx_n, A^2x_n)\\
&\quad + a_4 d(TAx_n, A^2x_n) + (a_5 + a_6 + a_7 + a_8 + a_9)\\
&\quad \cdot \max\{d(Sx_n, A^2x_n),\, d(Tx_n, A^2x_n),\\
&\qquad d(SAx_n, Ax_n),\, d(TAx_n, Ax_n),\, d(Ax_n, A^2x_n)\}.
\end{aligned}$$

Taking limsup of both sides, yields

$$d(z,Az) \leqslant \sup_{x,y\in X}(a_5 + a_6 + a_7 + a_8 + a_9)\cdot d(z,Az),$$

which, from (2.2), implies $z = Az$, and hence z is a common fixed point of A, S and T.

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To prove the uniqueness of z, suppose z and w are common fixed points of A, S and T. From (2.1), with $a_i = a_i(z,w)$,

$$\begin{aligned} d(z,w) = d(Sz,Tw) &\leqslant a_1 d(Sz,Az) + a_2 d(Tz,Az) + a_3 d(Sw,Aw) \\ &\quad + a_4 d(Tw,Aw) + a_5 d(Sz,Aw) + a_6 d(Tz,Aw) \\ &\quad + a_7 d(Sw,Az) + a_8 d(Tw,Az) + a_9 d(Az,Aw) \\ &= (a_5 + a_6 + a_7 + a_8 + a_9) \cdot d(z,w) \end{aligned}$$

which, from (2.2), implies $z = w$.

This completes the proof.

REMARK 1. Theorem 2.1 may be regarded as an extension of the well known result of Hardy and Rogers [24], which considered the following condition:

$$\begin{aligned} d(Tx,Ty) &\leqslant b_1 d(Tx,x) + b_2 d(Ty,y) + b_3 d(Tx,y) \\ &\quad + b_4 d(Ty,x) + b_5 d(x,y) \end{aligned} \tag{2.6}$$

for all x, y in X, where the control constants $b_i \geqslant 0$, $i = 1, \ldots, 5$, satisfy $b_1 + b_2 + b_3 + b_4 + b_5 < 1$. No such restriction is required in Theorem 2.1.

REMARK 2. Condition (2.6) has been also used by Guay and Singh [23] assuming $b_i \geqslant 0$, $i = 1, \ldots, 5$, $b_1 = b_2$, $b_3 = b_4$ (such an assumption is not restrictive) and

$$\max\{b_3 + b_4 + b_5, b_1 + b_3\} < 1. \tag{2.7}$$

Our condition (2.1) written for $S = T$ and A the identity map of X becomes (2.6) with $b_1 = a_1 + a_2$, $b_2 = a_3 + a_4$, $b_3 = a_5 + a_6$, $b_4 = a_7 + a_8$, $b_5 = a_9$ and clearly (2.2) becomes (2.7).

We also cite the papers of Emmanuele [14] and Taskovic [64], where asymptotically regular mappings are investigated under different contractive conditions.

REMARK 3. In Jungck [35], the continuity of the mapping $S = T$ is a consequence of his contractive condition and it is used in his proof. But in Theorem 2.1 the continuity of the mappings S and T is neither assumed nor is implied by the contractive condition (2.1).

REMARK 4. Das and Naik [10] generalize Jungck's theorem by considering the following condition

$$\begin{aligned} d(Sx,Sy) \leqslant c \max\{ & d(Sx,Ax), d(Sy,Ay), \\ & d(Sx,Ay), d(Sy,Ax), d(Ax,Ay)\} \end{aligned} \tag{2.8}$$

for all x, y in X, where $0 \leqslant c < 1$.

As indicated in Massa [38], (2.8) is equivalent to the following condition

$$d(Sx, Sy) \leqslant a_1 d(Sx, Ax) + a_2 d(Sy, Ay) + a_3 d(Sx, Ay) + a_4 d(Sy, Ax) + a_5 d(Ax, Ay) \tag{2.9}$$

for all x, y in X, where $a_i = a_i(x, y)$, $i = 1, \ldots, 5$, and

$$\sup_{x, y \in X} (a_1 + a_2 + a_3 + a_4 + a_5) < 1. \tag{2.10}$$

Clearly (2.9) is obtained from (2.1) for $S = T$ and (2.10) implies (2.2). Das and Naik assume S and A commute, A continuous, $S(X)$ contained in $A(X)$. So, choosing x_0, x_1 in X such that $Sx_0 = Ax_1$, they define inductively a sequence $\{y_n\}$ as follows

$$Sx_n = Ax_{n+1} = y_n, \qquad n = 0, 1, 2, \ldots.$$

In their paper, they prove that the sequence $\{y_n\}$ converges to a point y and X and

$$\lim_{n \to \infty} d(Sy_n, Ay_n) = d(Ay, Ay) = 0.$$

Thus $\{y_n\}$ is asymptotically S-regular with respect to A. Since the remaining conditions of Theorem 2.1 are satisfied, Theorem 2.1 is a generalization of the result of Das and Naik, which has been extended also by Chang [3], Chang [4], Fisher [18], [19], Khan and Imdad [29] and Rhoades [51] under different contractive conditions. However, it is not hard to check that these last results are also valid using the weak commutativity concept. We refer the reader to the paper of Rhoades [51] for further details.

3. Some examples

EXAMPLE 2. Let $X = [0, 1]$ with the Euclidean metric and $S = T$, A: $X \to X$ given as in Example 1. S and A weakly commute and let $\{x_n\}$ be a sequence in X converging to 0. Since

$$d(Sx_n, Ax_n) = \frac{x_n(x_n + 8)}{8(x_n + 16)},$$

$\{x_n\}$ is asymptotically S-regular with respect to A. For every x, y in X,

$$d(Sx, Sy) = \left| \frac{x}{x + 16} - \frac{y}{y + 16} \right| = \frac{16|x - y|}{(x + 16)(y + 16)} \leqslant \frac{16|x - y|}{256}$$
$$= \frac{|x - y|}{16} = \frac{1}{2} \cdot \frac{|x - y|}{8} = \frac{1}{2} d(Ax, Ay).$$

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A is continuous in X and it suffices to assume $a_9 = 1/2$, $a_i = 0$ for $i = 1, \ldots, 8$ in order to satisfy Theorem 2.1. Of course, all the results of the preceding authors are not applicable to this example since S and A do not commute.

By slightly modifying some examples of Fisher [20], we show that some of the assumptions of Theorem 2.1 cannot be dropped.

EXAMPLE 3. Let $X = \{x_1, x_2\}$ with any metric d and $S = T$, A: $X \to X$ defined by

$$Ax_1 = Ax_2 = Sx_2 = x_1, \quad Sx_1 = x_2.$$

Considering the constant sequence $\{x_2\}$, it is easily seen that all the hypothesis of Theorem 2.1 are valid except (2.4). Indeed, we have

$$d(SAx_2, ASx_2) = d(Sx_1, Ax_1) = d(x_2, x_1) > 0 = d(x_1, x_1) = d(Ax_2, Sx_2)$$

and A and S do not have a common fixed point.

EXAMPLE 4. Let $X = [1, \infty)$ with the Euclidean metric and $Sx = 2x$, $Tx = 4x$, $Ax = 22x$ for any x in X. Since we have $-4y \leqslant 8x$ for all x, y in X, then $2x - 4y \leqslant 10x$. This implies that if $x \geqslant 2y$, $d(Sx, Ty) = 2(x - 2y) \leqslant 10x = (22x - 2x)/2 = d(Sx, Ax)/2$. If $x < 2y \leqslant 6x$, then $4y \leqslant 12x$ which implies $d(Sx, Ty) = 2(2y - x) \leqslant 10x = (22x - 2x)/2 = d(Sx, Ax)/2$. If $x < 6x < 2y$, then obviously $x < 2x < 4y < 11y$ which implies

$$d(Sx, Ty) = 2(2y - x) < 11y - x = (22y - 2x)/2 = d(Sx, Ay)/2.$$

Thus (2.1) is satisfied with

$$\begin{aligned} &a_1 = \tfrac{1}{2} \quad \text{and} \quad a_5 = 0, \quad \text{if } x \geqslant 2y, \\ &a_1 = \tfrac{1}{2} \quad \text{and} \quad a_5 = 0, \quad \text{if } x < 2y \leqslant 6x, \\ &a_1 = 0 \quad \text{and} \quad a_5 = \tfrac{1}{2}, \quad \text{if } x < 6x < 2y \end{aligned}$$

and $a_i = 0$ for $i = 2, 3, 4, 6, 7, 8, 9$. The other assumptions of Theorem 2.1 are satisfied except condition (2.5) being for any sequence x_n of X,

$$\begin{aligned} d(Sx_n, Ax_n) &= 20x_n \to 0 \quad \text{iff } x_n \to 0, \\ d(Tx_n, Ax_n) &= 18x_n \to 0 \quad \text{iff } x_n \to 0, \end{aligned}$$

but 0 does not belong to X.

Condition (2.3) is also necessary in Theorem 2.1. To see this, consider the following

EXAMPLE 5. Let $X = [0, 1]$ with the Euclidean metric and $S = T$, A: $X \to X$ given by

$$Sx = \begin{cases} 1/2 & \text{if } x = 0, \\ x/4 & \text{if } x \neq 0, \end{cases} \qquad Ax = \begin{cases} 1 & \text{if } x = 0, \\ x/2 & \text{if } x \neq 0. \end{cases}$$

A commutes with S and one can readily verify, considering a sequence $\{x_n\}$, $x_n \neq 0$, converging to 0, that all the assumptions of Theorem 2.1 are satisfied with $a_9 = 1/2$ and $a_i = 0$ for $i = 1, \ldots, 8$ except (2.3). On the other hand, A and S have no fixed points.

4. Further results

Replacing the continuity of A with the continuity of S or T, we have the following theorem:

THEOREM 4.1. *Let A, S, T be three selfmaps of a complete metric space (X, d) satisfying conditions* (2.1), (2.5) *and*

(2.2′) a_1, a_2, a_3, a_4 *bounded on X and* $\sup_{x,y \in X} (a_5 + a_6 + a_7 + a_8 + a_9) < 1.$

If T is continuous and weakly commuting with A and S, then T has a fixed point.

PROOF. Let $\{x_n\}$ be a sequence as defined in (2.5). As in the proof of Theorem 2.1, the sequences $\{Ax_n\}$, $\{Sx_n\}$, $\{Tx_n\}$ converge to a point z in X. Since T is continuous, $\{TAx_n\} \to Tz$ and $\{T^2x_n\} \to Tz$. Using the weak commutativity of T and A,

$$d(ATx_n, Tz) \leqslant d(ATx_n, TAx_n) + d(TAx_n, Tz) \leqslant d(Ax_n, Tx_n) + d(TAx_n, Tz)$$

which implies that $\{ATx_n\} \to Tz$ as $n \to \infty$.

Since $\{TSx_n\} \to Tz$ and T weakly commutes with S, it is similarly proved that $\{STx_n\} \to Tz$ as $n \to \infty$.

From (2.1),

$$\begin{aligned} d(Sx_n, T^2x_n) \leqslant (a_1 + a_2 + a_3 + a_4) \\ \cdot \max\{ d(Sx_n, Ax_n),\ d(Tx_n, Ax_n), \\ d(STx_n, ATx_n),\ d(T^2x_n, ATx_n)\} \\ + (a_5 + a_6 + a_7 + a_8 + a_9) \\ \cdot \max\{ d(Sx_n, ATx_n),\ d(Tx_n, ATx_n),\ d(STx_n, Ax_n), \\ d(T^2x_n, Ax_n),\ d(Ax_n, ATx_n)\}, \end{aligned}$$

where $a_i = a_i(x_n, Tx_n)$. Taking the limsup,

$$d(z, Tz) \leqslant \sup_{x,y \in X} (a_5 + a_6 + a_7 + a_8 + a_9) \cdot d(z, Tz),$$

giving $Tz = z$ from (2.2′).

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An analogous theorem can be proved using the continuity of S instead of T. Note that z is not, in general, a comon fixed point of A, S and T as is shown in the following

EXAMPLE 6. Let $X = [0,1]$ with the Euclidean metric and A, S, T: $X \to X$ given by

$$Ax = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x \neq 0, \end{cases} \qquad Sx = \begin{cases} 1/4 & \text{if } x = 0, \\ x/2 & \text{if } x \neq 0, \end{cases}$$

$$Tx = x/2 \quad \text{for any } x \text{ in } X.$$

We have

$$d(AT0, TA0) = d(A0, T1) = 1/2 < 1 = d(T0, A0),$$

$$d(ST0, TS0) = d(S0, T1/4) = 1/8 < 1/4 = d(T0, S0)$$

and $STx = TSx = x/4$, $TAx = ATx = x/2$ for any $x \neq 0$. So A and S weakly commute with T which is continuous on X. Further, for $x = 0$ and y in X,

$$d(S0, Ty) = \tfrac{1}{2} \cdot \left|\tfrac{1}{2} - y\right| < \tfrac{1}{2} \cdot \tfrac{3}{4} = \tfrac{1}{2} d(A0, S0)$$

and for $x \neq 0$, y in X,

$$d(Sx, Ty) = \tfrac{1}{2}|x - y| = \tfrac{1}{2} d(Ax, Ay).$$

Then (2.1) is satisfied with $a_1 = 1/2$, $a_i = 0$ for $i = 2, \ldots, 8$, $a_9 = 1/2$.

Considering a sequence $\{x_n\}$, $x_n \neq 0$, converging to 0, one immediately verifies (2.5) and therefore all the assumptions of Theorem 4.1 hold but 0 is not fixed point of either S or A.

Using a proof similar to that of Theorem 2.1, one can easily verify the following

THEOREM 4.2. *Let $\{S_n\}$ be a sequence of selfmaps of a complete metric space (X, d) and A a continuous selfmap of X satisfying with $i \neq j$,*

$$\begin{aligned} d(S_i x, S_j x) \leqslant\ & a_1 d(S_i x, Ax) + a_2 d(S_j x, Ax) + a_3 d(S_i y, Ay) \\ & + a_4 d(S_j y, Ay) + a_5 d(S_i x, Ay) + a_6 d(S_j x, Ay) \\ & + a_7 d(S_i y, Ax) + a_8 d(S_j y, Ax) + a_9 d(Ax, Ay) \end{aligned}$$

for all x, y in X, where $a_k = a_k(x, y)$, $k = 1, \ldots, 9$, are nonnegative functions satisfying (2.2). *If A weakly commutes with each S_n and there exists an asymptotically S_n-regular sequence with respect to A for every $n = 1, 2, \ldots$, then the family $\{A, S_1, S_2, \ldots\}$ has a unique common fixed point.*

This theorem can be regarded as an improvement of Theorem 1 of Singh and Tiwari [62], where the authors assume that range of A contains the range of S_n for every n, but this is not required in our Theorem 4.2.

5. Results in Banach spaces

In this section, we present a result which deals with the structure of the set of common fixed points. The next theorem generalizes Theorem 29 of [50] without requiring the commutativity of the mappings under consideration. We first need the following

LEMMA 5.1. *Let (X, d) be a complete metric space, K a closed subset of X, A, S and T three selfmaps of K satisfying* (2.1) *for all x, y in K,*

$$(2.2'')\quad \max\left\{\sup_{x,y\in X}(a_1+a_2+a_5+a_6),\ \sup_{x,y\in X}(a_3+a_4+a_7+a_8)\right\}<1.$$

If A is continuous and a_9 is bounded on X, then the set F of common fixed points of A, S and T is closed.

PROOF. Let $\{x_n\}$ be a Cauchy sequence in F with limit x in K. Then $d(x, Ax) \leqslant d(x, x_n) + d(x_n, Ax) = d(x, x_n) + d(Ax_n, Ax) \to 0$ since A is continuous. Thus $Ax = x$.

From (2.1), with $a_i = a_i(x_n, x)$,

$$\begin{aligned} d(x,Tx) &\leqslant d(x,x_n)+d(x_n,Tx)=d(x,x_n)+d(Sx_n,Tx)\\ &\leqslant d(x,x_n)+a_1d(Sx_n,Ax_n)+a_2d(Tx_n,Ax_n)+a_3d(Sx,Ax)\\ &\quad +a_4d(Tx,Ax)+a_5d(Sx_n,Ax)+a_6d(Tx_n,Ax)\\ &\quad +a_7d(Sx,Ax_n)+a_8d(Tx,Ax_n)+a_9d(Ax_n,Ax)\\ &= d(x,x_n)+a_3d(Sx,x)+a_4d(Tx,x)+a_5d(x_n,x)+a_6d(x_n,x)\\ &\quad +a_7d(Sx,x_n)+a_8d(Tx,x_n)+a_9d(x_n,x)\\ &\leqslant (a_3+a_4+a_7+a_8)\cdot\max\{d(x,Tx),\,d(x,Sx)\}\\ &\quad +(1+a_5+a_6+a_7+a_8+a_9)\cdot d(x_n,x). \end{aligned}$$

By the assumptions,

$$\sup_{x,y\in X}(a_5+a_6+a_7+a_8+a_9)<\infty.$$

Taking the limsup of both sides in the above inequality, we have

$$d(x,Tx) \leqslant \sup_{x,y\in X}(a_3+a_4+a_7+a_8)\cdot\max\{d(x,Tx),\,d(x,Sx)\}.$$

Similarly, the inequality $d(x,Sx) \leqslant d(x,x_n) + d(Sx,Tx_n)$ yields

$$d(x,Sx) \leqslant \sup_{x,y\in X}(a_1+a_2+a_5+a_6)\cdot\max\{d(x,Tx),\,d(x,Sx)\}.$$

From (2.2″), it follows that $Tx = Sx = x$. Thus x is in F and F is closed.

THEOREM 5.2. *Let X be a strictly convex Banach space, K a convex closed subset of X, A, S and T three selfmaps of K satisfying* (2.1) *for all x, y in K*, (2.2″) *and*

$$\text{(2.2‴)}\qquad \max\Big\{\sup_{x,y\in X}(a_1+a_2+a_5+a_6+a_7+a_8+a_9),\ \sup_{x,y\in X}(a_3+a_4+a_5+a_6+a_7+a_8+a_9)\Big\} \leqslant 1.$$

If A is continuous and affine, then the set F of common fixed points of A, S and T is closed and convex.

PROOF. Since (2.2‴) implies a_9 is bounded on X, that F is closed follows from Lemma 5.1. To show convexity, let $x_1, x_2 \in F$, $x = (x_1+x_2)/2$. Since K is convex, x is in K and $Ax = x$ since A is affine.

Case I. Suppose $\|x - Sx\| \leqslant \|x - Tx\|$. Then

$$\|x - Tx\| \leqslant \tfrac{1}{2}\big(\|x_1 - Tx\| + \|x_2 - Tx\|\big).$$

Without loss of generality, we may assume $\|x_2 - Tx\| \leqslant \|x_1 - Tx\|$. Then, from (2.1),

$$\begin{aligned}\|x - Tx\| &\leqslant \|x_1 - Tx\| = \|Sx_1 - Tx\|\\ &\leqslant a_1\|Sx_1 - Ax_1\| + a_2\|Tx_1 - Ax_1\| + a_3\|Sx - Ax\|\\ &\quad + a_4\|Tx - Ax\| + a_5\|Sx_1 - Ax\| + a_6\|Tx_1 - Ax\|\\ &\quad + a_7\|Sx - Ax_1\| + a_8\|Tx - Ax_1\| + a_9\|Ax_1 - Ax\|,\end{aligned}$$

where $a_i = a_i(x_1, x)$. Thus

$$\begin{aligned}\|x - Tx\| \leqslant (a_3+a_4)\cdot\|x - Tx\| + (a_5+a_6+a_9)\cdot\|x_1 - x\|\\ + a_7\|x_1 - Sx\| + a_8\|x_1 - Tx\|.\end{aligned}$$

Case Ia. Assume $\|x_1 - Sx\| \leqslant \|x_1 - Tx\|$. Then

$$(5.1)\qquad (1 - a_3 - a_4)\cdot\|x - Tx\| \leqslant (a_5 + a_6 + a_9)\cdot\|x_1 - x\| + (a_7 + a_8)\cdot\|x_1 - Tx\|,$$

$$(5.2)\qquad \|x_1 - Tx\| = \|Sx_1 - Tx\| \leqslant (a_3 + a_4)\cdot\|x - Tx\| + (a_5 + a_6 + a_9)\cdot\|x_1 - x\| + (a_7 + a_8)\cdot\|x_1 - Tx\|.$$

Substituting (5.2) into (5.1) yields, $(1 - a)\cdot\|x - Tx\| \leqslant b\cdot\|x_1 - x\|$, where

$$a = a_3 + a_4 + \frac{(a_3 + a_4)\cdot(a_7 + a_8)}{1 - a_7 - a_8},$$

$$b = a_5 + a_6 + a_9 + \frac{(a_7 + a_8)\cdot(a_5 + a_6 + a_9)}{1 - a_7 - a_8}.$$

From (2.2‴), $a + b \leqslant 1$. Thus $\|x - Tx\| \leqslant \|x_1 - x_2\|/2$. Substituting in (5.2), $\|x_1 - Tx\| \leqslant \|x_1 - x\| = \|x_1 - x_2\|/2$. Then

$$\|x_1 - x_2\| \leqslant \|x_1 - Tx\| + \|x_2 - Tx\| \leqslant 2\|x_1 - Tx\| \leqslant \|x_1 - x_2\|$$

and, since X is strictly convex, $Tx = x$. Since $\|Sx - x\| \leqslant \|Tx - x\|$, we have $Sx = x$ too. Thus F is convex.

Case Ib. Assume $\|x_1 - Tx\| \leqslant \|x_1 - Sx\|$. Then

$$(5.3)\qquad (1 - a_3 - a_4)\cdot\|x - Tx\| \leqslant (a_5 + a_6 + a_9)\cdot\|x_1 - x\| + (a_7 + a_8)\cdot\|x_1 - Sx\|.$$

From (2.1),

$$\|Sx - x_1\| = \|Sx - Tx_1\| \leqslant a_1'\|Sx - Ax\| + a_2'\|Tx - Ax\| + a_3'\|Sx_1 - Ax_1\| + a_4'\|Tx_1 - Ax_1\| + a_5'\|Sx - Ax_1\| + a_6'\|Tx - Ax_1\| + a_7'\|Sx_1 - Ax\| + a_8'\|Tx_1 - Ax\| + a_9'\|Ax - Ax_1\|,$$

where $a_i' = a_i(x, x_1)$. Thus

$$(5.4)\qquad \|Sx - x_1\| \leqslant (a_1' + a_2')\cdot\|x - Tx\| + (a_5' + a_6')\cdot\|x_1 - Sx\| + (a_7' + a_8' + a_9')\cdot\|x - x_1\|.$$

Substituting in (5.3) yields

$$(1 - c)\cdot\|x - Tx\| \leqslant d\cdot\|x_1 - x\|$$

where

$$c = a_3 + a_4 + \frac{(a_7 + a_8)\cdot(a_1' + a_2')}{1 - a_5' - a_6'},$$

$$d = a_5 + a_6 + a_9 + \frac{(a_7 + a_8)\cdot(a_7' + a_8' + a_9')}{1 - a_5' - a_6'}.$$

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From (2.2‴), $c + d \leqslant 1$, so that $\|x - Tx\| \leqslant \|x_1 - x_2\|/2$. Substituting in (5.4) yields $\|x_1 - Sx\| \leqslant \|x_1 - x_2\|/2$. Thus

$$\|x_1 - x_2\| \leqslant \|x_1 - Tx\| + \|x_2 - Tx\| \leqslant 2\|x_1 - Tx\| \leqslant 2\|x_1 - Sx\| \leqslant \|x_1 - x_2\|$$

and, since X is strictly convex, $Tx = x$. As in case Ia, $Sx = x$ too and F is convex.

Case II. Assume $\|x - Tx\| \leqslant \|x - Sx\|$. The proof is similar to case I and will therefore be omitted.

This concludes the proof.

6. Fixed points of orbitally continuous mappings

Let T be a selfmap of a metric space (X, d). An orbit of T at x_0 is denoted by the set

$$\mathcal{O}(x_0, T) = \{x_0, Tx_0, T^2x_0 \cdots T^nx_0, \ldots\}.$$

Further, $\overline{\mathcal{O}(x_0, T)}$ stands for the closure of the orbit.

Definition 2 (Jaggi [33]). A selfmap T of X is x_0-orbitally continuous for some x_0 in X if its restriction to the set $\overline{\mathcal{O}(x_0, T)}$ is continuous.

If T is x_0-orbitally continuous for any x_0 in X, Then T is said to be orbitally continuous. Ciric [7] has shown that orbitally continuous mappings are not necessarily continuous and on other hand, Jaggi [33] gave an example of x_0-orbitally continuous mapping T, but not orbitally continuous on X.

Browder and Petryshyn [2] give the following

Definition 3. A selfmap T of X is asymptotically regular at a point x of X if $d(T^nx, T^{n+1}x) \to 0$ as $n \to \infty$.

The following theorem is a special case of Theorem 6 of Park [47] (the proof is enclosed for sake of completeness):

Theorem 6.1. *Let S be an x_0-orbitally continuous selfmap of a metric space (X, d) for some x_0 in X. If the sequence $\{S^nx_0\}$ has a cluster point z in X and S is asymptotically regular at x_0, then z is a fixed point of S.*

PROOF. Let $\{S^{k(n)}x_0\}$ be a subsequence of $\{S^n x_0\}$-converging to z. Since $d(z, Sz) \leqslant d(z, S^{k(n)}x_0) + d(S^{k(n)}x_0, S^{k(n)+1}x_0) + d(S(S^{k(n)}x_0), Sz)$, using the asymptotic regularity of S and its x_0-orbitally continuity, we have $z = Sz$.

It is not hard to check that Theorem 6.1 includes a multitude of results for mappings satisfying conditions (1)–(24) and (26)–(49) of Rhoades [50] and also those of Ciric [8], Jaggi [33], Meir and Keeler [44], Fisher [16].

The following theorem is motivated by the contractive condition invented by Yen [65]:

THEOREM 6.2. *Let K be a non-empty convex subset of a normed linear space X, T a selfmap of K satisfying*

$$\begin{aligned}(6.1)\qquad \|Tx - Ty\| \leqslant a_1 \max\{\|x - Tx\|, \|y - Ty\|\} \\ + a_2 \max\{\|x - Ty\|, \|y - Tx\|\} + a_3\|x - y\|\end{aligned}$$

for all x, y in K, where $a_i = a_i(x, y)$, $i = 1, 2, 3$, are nonnegative functions satisfying

$$(6.2)\qquad \sup_{x,y\in X}\{2a_1 + a_2 + a_3\} \leqslant 1, \qquad \inf_{x,y\in X} a_1(x, y) > 0.$$

For each λ, $0 \leqslant \lambda \leqslant 1$, let $T_\lambda = \lambda \cdot I + (1 - \lambda)$. T where I is the identity map of K. Let x_0 be a point of K such that the sequence $\{T_\lambda^n(x_0)\}$ clusters to a point z of K and assume that T_λ is x_0-orbitally continuous and asymptotically regular at x_0. Then z is the unique fixed point of T in K and $\{T_\lambda^n(x_0)\}$ converges to z.

PROOF. From Theorem 6.1, z is a fixed point of T_λ. Thus $Tz = z$ and suppose T has two distinct fixed points w, z. From (6.1) with $a_i = a_i(w, z)$,

$$\|w - z\| = \|Tw - Tz\| \leqslant (a_2 + a_3) \cdot \|w - z\|,$$

which implies $2a_1(w, z) = 0$, a contradiction to (6.2).

For any x in K,

$$(6.3)\qquad \|T_\lambda(x) - z\| \leqslant \lambda \cdot \|x - z\| + (1 - \lambda) \cdot \|Tx - Tz\|.$$

From (6.1) with $a_i = a_i(x, z)$,

$$\begin{aligned}\|Tx - Tz\| &\leqslant a_1 \max\{\|x - Tx\|, \|z - Tz\|\} \\ &\quad + a_2 \max\{\|x - Tz\|, \|z - Tx\|\} + a_3\|x - z\| \\ &= a_1\|x - Tx\| + a_2 \max\{\|x - z\|, \|z - Tx\|\} + a_3\|x - z\| \\ &\leqslant a_1(\|x - z\| + \|Tz - Tx\|) + a_2 \max\{\|x - z\|, \|Tz - Tx\|\} + a_3\|x - z\|.\end{aligned}$$

Suppose $\|x - z\| < \|Tx - Tz\|$. Then, it follows from (6.2) that $\|Tx - Tz\| < \|Tx - Tz\|$, a contradiction. So $\|Tx - Tz\| \leqslant \|x - z\|$ and from (6.3), $\|T_\lambda(x) - z\| \leqslant \|x - z\|$.

Since x is arbitrary in K, we note that for $x = T_\lambda^n(x_0)$,

$$\|T_\lambda^{n+1}(x_0) - z\| \leqslant \|T_\lambda^n(x_0) - z\|,$$

which guarantees the convergence of the sequence $\{T_\lambda^n(x_0)\}$ to z, since z is a cluster point of the same sequence.

This concludes the proof.

REMARK 5. For a non-expansive mapping T, the convergence of $\{T_\lambda^n(x_0)\}$ was investigated by Diaz and Metcalf [11], Edelstein [13], Kannan [25], Krasnoselskii [31] and Schaeffer [58] in either uniformly convex or strictly convex Banach spaces. Jaggi [34] discussed the convergence of $\{T_\lambda^n(x_0)\}$ in a normed linear space with no additional structure. Our Theorem 6.2 extends Theorem 1 of [34].

7. Approximating fixed points in Banach space

Let X be a Banach space and K be a convex subset of X. Dotson [12] gives the following

DEFINITION 4. A selfmap T of K is quasi-nonexpansive if T has a fixed point z in K and $\|Tx - z\| \leqslant \|x - z\|$ for any x in K.

An extensive literature exists about non-expansive and quasi-nonexpansive mappings. Here we cite the fine papers of Garegnani and Zanco [21], Goebel and Massa [22], Karlovitz [26], Kuhn [32], Maluta [36], Massa [39], [40], [41], Massa and Roux [43], Petryshyn and Williamson [49], Rhoades [52], Roux [54], [55], [56], Roux and Zanco [57], Soardi [63] and the Italian bibliography of Papini [45] for further information.

In the case of a Banach space, a slightly more general result than Theorem 6.2 concerning approximation of fixed points can be obtained by considering an iterative procedure of Mann [37]. Strictly speaking, let x_1 be a point of K and $M(x_1, t_n, t)$ stands for the sequence $\{x_n\}$ defined by $x_{n+1} = (1 - t_n) \cdot x_n + t_n \cdot Tx_n$, where $\{t_n\}$ is a sequence of $[a, b]$, $0 < a < b < 1$. $F(T)$ denotes the set of the fixed point of T. A selfmap T of K with $F(T) \neq \varnothing$ is said to satisfy

Condition I. If there is a non-decreasing function f: $[0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for any $r > 0$, such that $\|x - Tx\| > f(d(x, F(T)))$ for all x in K, where $d(x, F(T)) = \inf\{\|x - z\|: z \in F(T)\}$.

Condition II. If there is a real number $h > 0$ such that $\|x - Tx\| \geqslant h \cdot d(x, F(T))$ holds for all x in K.

It is easily verified that a mapping T satisfying condition II also satisfies condition I.

These conditions appear in Senter and Dotson [59], who obtained the following theorem.

THEOREM 7.1. *Let X be a uniformly convex Banach space, K a closed convex subset of X and T quasi-nonexpansive selfmap of K. If T satisfies condition* I, *then for arbitrary x_1 in K, $M(x_1, t_n, T)$ converges to an element of $F(T)$.*

We use this result to establish the following theorem:

THEOREM 7.2. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X. Suppose that T is a selfmap of K, with $F(T) \neq \varnothing$, satisfying* (6.1) *for all x, y in K and* (6.2). *Then for arbitrary x_1 chosen in K, $M(x_1, t_n, t)$ converges to the unique element of $F(T)$.*

PROOF. Since $F(T) \neq \varnothing$, let be z an element of $F(T)$. As in the proof of Theorem 6.2, it is immediately proved that z is the unique element of $F(T)$ and T is quasi-nonexpansive. From (6.1) with $a_i = a_i(x, z)$,

$$\begin{aligned}\|Tx - Tz\| = \|Tx - z\| &\leqslant a_1 \max\{\|x - Tx\|, \|z - Tz\|\}\\ &\quad + a_2 \max\{\|x - Tz\|, \|z - Tx\|\} + a_3\|x - z\|\\ &= a_1\|x - Tx\| + (a_2 + a_3)\cdot\|x - z\|.\end{aligned}$$

Therefore

$$a_1\|x - Tx\| + (a_2 + a_3)\cdot\|x - z\| \geqslant \|Tx - z\| \geqslant \|x - z\| - \|x - Tx\|$$

which implies $\|x - Tx\| \geqslant h \cdot \|x - z\|$ where $h = (1 - a_2 - a_3)/(1 + a_1)$. From (6.2), $h > 0$ and so T satisfies condition II. The thesis follows from Theorem 7.1.

Related results to Theorem 7.2 can be found in Bose and Mukherjee [1] and Massa [42], which improves the results of [1].

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References

[1] R. K. Bose and R. N. Mukherjee, 'Approximating fixed points of some mappings', *Proc. Amer. Math. Soc.* **82** (1981), 603–606.

[2] F. E. Browder and W. V. Petryshyn, 'The solution by iteration of nonlinear functional equations in Banach spaces', *Bull. Amer. Math. Soc.* **72** (1966), 571–575.

[3] C. C. Chang, 'On a fixed point theorem of contractive type', *Comment. Math. Univ. St. Paul* **32** (1983), 15–19.

[4] S. S. Chang, 'A common fixed point theorem for commuting mappings', *Proc. Amer. Math. Soc.* **83** (1981), 645–652.

[5] Cheh-Chih Yeh, 'Common fixed point of continuous mappings in metric spaces', *Publ. Inst. Math.* **27** (41) (1980), 21–25.

[6] Cheh-Chih Yeh, 'Remark on common fixed point of commuting mappings in L-spaces', *Matematiche (Catania)* **34** (1979), 81–85.

[7] L. Ciric, 'Fixed and periodic points of almost contractive operators', *Math. Balkanica* **3** (1973), 33–44.

[8] L. Ciric, 'On some maps with a non-unique fixed point', *Publ. Inst. Math.* **17**(31) (1974), 52–58.

[9] V. Conserva, 'Common fixed point theorems for commuting maps on a metric space', *Publ. Inst. Math.* **32**(46) (1982), 37–43.

[10] K. M. Das and K. V. Naik, 'Common fixed point theorems for commuting maps on a metric space', *Proc. Amer. Math. Soc.* **77** (1979), 369–373.

[11] J. B. Diaz and F. T. Metcalf, 'On the set of subsequential limits of successive approximations', *Trans. Amer. Math. Soc.* **135** (1969), 459–465.

[12] W. G. Dotson, Jr., 'Fixed points of quasi-nonexpansive mappings', *J. Austral. Math. Soc.* **13** (1972), 167–170.

[13] M. Edelstein, 'A remark on a theorem of M. A. Krasnoselskii', *Amer. Math. Monthly* **73** (1966), 509–510.

[14] G. Emmanuele, 'Fixed point theorems in complete metric spaces', *Nonlinear Anal.* **5**(3) (1981), 287–292.

[15] H. W. Engl, 'Weak convergence of asymptotically regular sequences for non-expansive mappings and connections with certain Chebyshef-centers', *Nonlinear Anal.* **1**(5) (1977), 495–501.

[16] B. Fisher, 'Results on common fixed points', *Math. Japon.* **22** (1977) 335–338.

[17] B. Fisher, 'Mappings with a common fixed point', *Mat. Sem. Notes Kobe Univ.* **7** (1979), 81–84; addendum **8** (1980), 513–514.

[18] B. Fisher, 'Common fixed points of commuting mappings', *Bull. Inst. Math. Acad. Sinica* **9** (1981), 399–406.

[19] B. Fisher, 'Three mappings with a common fixed point', *Math. Sem. Notes Kobe Univ.* **10** (1982), 293–302.

[20] B. Fisher, 'Common fixed points of four mappings', *Bull. Inst. Math. Acad. Sinica* **11** (1983), 103–113.

[21] G. Garegnani and C. Zanco, 'Fixed points of somehow contractive multivalued mappings', *Ist. Lombardo Accad. Sci. Lett. Rend.* A **114** (1980), 138–148.

[22] K. Goebel, and S. Massa, 'Some remarks on nonexpansive mappings in Hilbert spaces', *Boll. Un. Mat. Ital.* (6) **3**-A (1984), 139–145.

[23] M. D. Guay and K. L. Singh, 'Fixed points of asymptotically regular mappings', *Mat. Vesnik* **35** (1983), 101–106.

[24] G. E. Hardy and T. D. Rogers, 'A generalization of a fixed point theorem of Reich', *Canad. Math. Bull.* **16** (1973), 201–206.

[25] R. Kannan, 'Construction of fixed points of a class of nonlinear mappings', *J. Math. Anal. Appl.* **41** (1973), 430–438.

[26] L. A. Karlovitz, 'On non-expansive mappings', *Proc. Amer. Math. Soc.* **55** (1976), 321–325.
[27] M. S. Khan, 'Remarks on some fixed point theorems', *C. R. Bulg. Sci.* **33**(12) (1980), 1581–1583.
[28] M. S. Khan, 'Commuting mappings and fixed points in uniform spaces', *Bull. Acad. Pol. Sci. Ser. Sci. Mat.* XXIX, **9-10** (1981), 499–507.
[29] M. S. Khan and M. Imdad, 'Fixed point theorems for a class of mappings', *Indian J. Pure Appl. Math.* **14** (1983), 1220–1227.
[30] M. S. Khan and M. Imdad, 'Some common fixed point theorems', *Glasnik Mat. Sci.* III **18** (1983), 321–326.
[31] M. A. Krasnoselskii, 'Two remarks about the method of successive approximations', *Uspehi Math. Nauk.* **10** (1955), 123–127.
[32] G. Kuhn, 'Generalized non-expansive mappings: approximation of fixed points', *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **114** (1980), 93–101.
[32] D. S. Jaggi, 'Fixed point theorems for orbitally continuous functions', *Mat. Vesnik* **1**(14) (1977), 129–135.
[34] D. S. Jaggi, 'Fixed point theorems for orbitally continuous functions II', *Mat. Vesnik* **19**(2) (1977), 113–118.
[35] G. Jungck, 'Commuting maps and fixed points', *Amer. Math. Monthly* **83** (1976), 261–263.
[36] E. Maluta, 'Generalized non-expansive maps on the real axis: existence and approximation of fixed points', *Ist. Lombardo Accad. Sci. Lett. Rend.* A **112** (1978), 213–221.
[37] W. R. Mann, 'Mean value method in iteration', *Proc. Amer. Math. Soc.* **4** (1953), 506–510.
[38] S. Massa, 'Generalized contractions in metric spaces', *Boll. Un. Mat. Ital.* (4) **10** (1974), 689–694.
[39] S. Massa, 'Approximation of fixed points by Cesaro's means of iterates', *Rend. Ist. Mat. Univ. Trieste* **9** (1977), 127–133.
[40] S. Massa, 'On the approximation of fixed points for quasi nonexpansive mappings', *Ist. Lombardo Accad. Sci. Lett. Rend.* A **111** (1977), 188–193.
[41] S. Massa, 'Convergence of an iterative process for a class of quasi-nonexpansive mappings', *Boll. Un. Mat. Ital.* (5) **15**-A (1978), 154–158.
[42] S. Massa, 'Fixed point approximation for quasi non-expansive mappings', *Matematiche (Catania)* **37** (1982), 3–7.
[43] S. Massa and D. Roux, 'A fixed point theorem for generalized nonexpansive mappings, *Boll. Un. Mat. Ital.* (5) **15**-A (1978), 624–634.
[44] A. Meir and E. Keeler, 'A theorem on contraction mappings', *J. Math. Anal. Appl.* **28** (1969), 326–329.
[45] P. L. Papini, *Una bibliografia italiana sui punti fissi* 1970–1983 (Dipartimento di Matematica, Univ. di Bologna, 1983).
[46] S. Park, 'Fixed points of f-contractive maps', *Rocky Mountain J. Math.* **8** (1978), 743–750.
[47] S. Park, 'A unified approach to fixed points of contractive type', *J. Korean Math. Soc.* **16** (1980), 95–105.
[48] S. Park and B. E. Rhoades, 'Extension of some fixed point theorems of Hegedüs and Kasahara', *Math. Sem. Notes Kobe Univ.* **9** (1981), 113–118.
[49] W. V. Petryshyn and T. E. Williamson, Jr., 'Strong and weak convergence of the sequence of successive approximation for quasi-nonexpansive mappings', *J. Math. Anal. Appl.* **43** (1973), 459–497.
[50] B. E. Rhoades, 'A comparison of various definitions of contractive mappings', *Trans. Amer. Math. Soc.* **226** (1977), 257–290.
[51] B. E. Rhoades, 'Contractive definitions revisited, topological methods in nonlinear functional analysis', *Contemporary Math. AMS* **21** (1983), 189–205.
[52] B. E. Rhoades, 'Some fixed point theorems for generalized non-expansive mappings', *Nonlinear analysis and applications*, ed. Singh/Burry, pp. 223–228 (Lecture Notes Pure Appl. Math. 80 (1982), Marcel-Dekker).

[53] B. E. Rhoades, S. Sessa, M. S. Khan and M. D. Khan, 'Some fixed point theorems for Hardy-Rogers type mappings', *Internat. J. Math. Math. Sci.*, 7(1) (1984), 75–87.
[54] D. Roux, 'Applicazioni quasi non-espansive: approssimazione dei punti fissi', *Rend. Mat.* (6) **10** (1977), 597–605.
[55] D. Roux, 'Teoremi di punto fisso per applicazioni contrattive', *Atti del Convegno Linceo "Applicazioni del teorema di punto fisso all'Analisi Economica"*, pp. 89–110 (Roma 1977 (1978)).
[56] D. Roux, 'Applicazioni del teorema di Brouwer a spazi a infinite dimensioni', *Atti del Convegno Linceo "Applicazioni del teorema di punto fisso all'Analisi Economica"* (Roma 1977 (1978)).
[57] D. Roux and C. Zanco, 'Quasi non-expansive mappings: strong and weak convergence to a fixed point of the sequence of iterates', *Matematiche (Catania)* **32** (1977), 307–315.
[58] M. Schaeffer, 'Uber die method suksessiver approximation', *Jber. Deutsh Math.* **59** (1957), 131–140.
[59] M. F. Senter and W. G. Dotson, Jr., 'Approximating fixed points of non-expansive mappings', *Proc. Amer. Math. Soc.* **44** (1974), 375–379.
[60] S. Sessa, 'On a weak commutativity condition in fixed point considerations', *Publ. Inst. Math.* **32**(46) (1982), 149–153.
[61] S. L. Singh, 'On common fixed points of commuting mappings', *Math. Seminar Notes Kobe Univ.* **5** (1977), 131–134.
[62] S. L. Singh and B. M. L. Tiwari, 'Common fixed points of mappings in complete metric spaces', *Proc. Nat. Acad. Sci. India* **51** (A) I (1981), 41–44.
[63] P. Soardi, 'Existence of fixed points of non-expansive mappings in certain Banach lattices', *Proc. Amer. Math. Soc.* **73** (1979), 25–29.
[64] M. R. Taskovic, 'On common fixed points of mappings', *Math. Balk.* **8** (1978), 213–219.
[65] C. L. Yen, 'Some fixed point theorems', *Chung Juan J.* **4** (1975), 1–8.

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