ON PEIFFER CENTRAL SERIES

by GRAHAM ELLIS

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1. Introduction. Let G be a group. A *precrossed G-module* is a group homomorphism $\partial : M \to G$ together with a group action $(g, m) \mapsto {}^{g}m$ of G on M, such that $\partial({}^{g}m) = g(\partial m)g^{-1}$. The *Peiffer commutator* $< m, m' >$ of two elements $m, m' \in M$ is defined as

$$< m, m' >= mm'm^{-1}({}^{\partial m}m')^{-1}.$$

If all Peiffer commutators are trivial, the precrossed G-module is said to be a *crossed G-module*. The subgroup $< M, M >$ generated by all Peiffer commutators is called the *Peiffer subgroup* of M; it is the second term of a *lower Peiffer central series* (see below). The following table indicates how these concepts reduce to more standard concepts when restrictions are placed on ∂ and G.

Concepts: \ **Restrictions:**	$\partial(M) = 1$	$\partial(M) = G$	$\ker(\partial) = 1$	$G = 1$
precrossed G-module	group with G-action	•	normal subgroup of G	group
crossed G- module	ZG-module	central extension of G	normal subgroup of G	abelian group
Peiffer commutator	commutator	•	trivial element	commutator
Peiffer subgroup	•	•	trivial subgroup	derived subgroup
Peiffer central series	•	•	•	central series.

Furthermore, any ZG-module A gives rise to a precrossed G-module $\partial : A \rtimes G \to G$, $(a, g) \mapsto g$ in which the action of G on the direct product $M = A \rtimes G$ is given by ${}^{g}(a, g') = ({}^{g}a, gg'g^{-1})$. In this example the Peiffer subgroup of M lies in the module A. More precisely, $< M, M >= IG.A$ where $IG = \ker(ZG \to Z)$ is the augmentation ideal of G.

Interest in precrossed G-modules stems from algebraic topology: a precrossed G-module corresponds exactly to that low dimensional part of a CW-space which gives a presentation of the fundamental group. Thus (pre)crossed modules arise in combinatorial group theory (see [3] and [13] for references) and in low dimensional homotopy (see [1] and [3] for references); they are also central to a body of work on nonabelian cohomology (see [6] and [10] for references).

It would be of interest to know just how much of the extensive algebraic theory on group commutators extends to Peiffer commutators. For instance, it is shown in a substantial paper of H. J. Baues and D. Conduché [**2**] that the Magnus-Witt result on the quotients of the lower central series of a free group extends to a result on lower Peiffer central series. Furthermore, it is shown in [**7**] that results of C. Miller [**12**] and J. Stallings [**14**] on homology and central series of groups extend to Peiffer central series.

The aim of the present paper is to obtain a Peiffer commutator version of the result of P. Hall [**11**] which states that $\gamma_{c+1}G$ is finite whenever $G/Z_c(G)$ is finite (where $\gamma_{c+1}(G)$ and $Z_c(G)$ denote terms of the lower and upper central series of the group G). The appropriate lower Peiffer central series was defined in [**2**], and a corresponding upper central series is introduced below. We also obtain a Peiffer commutator version of J. Wiegold's bound [**15**] on the order of $\gamma_2(G)$ given that $G/Z_1(G)$ is of prime power order p^a. Our proofs of the Peiffer versions of these results rely on the finiteness of a nonabelian tensor product of groups [**8**], which in turn relies on the transfer homomorphism in group homology.

2. Statement of results. Let $\partial : M \to G$ be a precrossed G-module. Given two subgroups N and N' of M, we let $< N, N' >$ denote the subgroup of M generated by the Peiffer commutators $< n, n' >$ for $n \in N$, $n' \in N'$. We let $<< N, N' >>$ denote the subgroup of M generated by the Peiffer commutators $< n, n' >$ and $< n', n >$ for $n \in N$, $n' \in N'$. We say that a subgroup N of M is *G-invariant* if ${}^g n \in N$ for all $g \in G$, $n \in N$.

Recall from [**2**] that the *lower Peiffer central series* $P\gamma_n(M)(n \geq 1)$ is defined by inductively setting

$$P\gamma_1(M) = M,$$
$$P\gamma_n(M) = << M, P\gamma_{n-1}(M) >> \quad \text{for } n \geq 2.$$

Note that $P\gamma_2(M)$ is just the Peiffer subgroup $< M, M >$, and that $P\gamma_n(M)$ contains $P\gamma_{n+1}(M)$. We observe in Section 3 that each $P\gamma_n(M)$ is a G-invariant normal subgroup of M.

Let us define the *Peiffer centre* to be

$$PZ(M) = \{a \in M : < x, a >= 1 =< a, x > \text{ for all } x \in M\}.$$

More generally, given two subsets Z and Γ of M, define

$$V(Z, \Gamma) = \{a \in M : < x, a > \in Z \text{ and } < a, x > \in Z \text{ for all } x \in \Gamma\}$$

Note that $PZ(M) = V(1, M)$. We observe in Section 3 that, if Z and Γ are G-invariant normal subgroups of M, then $V(Z, \Gamma)$ is a G-invariant (but not necessarily normal) subgroup of M.

Let us define an *upper Peiffer central series* $PZ_n(M)(n \geq 1)$ by inductively setting

$$PZ_0(M) = 1,$$
$$PZ_1(M) = PZ(M),$$
$$PZ_n(M) = \bigcap_{\substack{i+j=n \\ i \geq 0,\, j \geq 1}} V(PZ_i(M), P\gamma_j(M)) \ (n \geq 1).$$

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In other words $PZ_n(M)$ is the intersection of those subsets $V(PZ_i(M), P\gamma_j(M))$ with $i+j=n,\ i \geq 0,\ j \geq 1$.

Observe (by induction) that $PZ_n(M)$ is contained in $PZ_{n+1}(M)$. In Section 3 we show that each $PZ_n(M)$ is a G-invariant normal subgroup of M, and that $PZ_n(M) = M$ if and only if $P\gamma_{n+1}(M) = 1$.

Following [2] we say that the precrossed module $\partial : M \to G$ is *Peiffer nilpotent of class n* if $P\gamma_{n+1}(M) = 1$. Thus precrossed modules of Peiffer nilpotency class 1 are just crossed modules, and as such were introduced by J. H. C. Whitehead (cf. [1] [3]) as an algebraic model of homotopy 2-types. Precrossed modules of Peiffer nilpotency class 2 are an essential ingredient in the algebraic model of homotopy 3-types introduced and developed by Baues in [1].

Our main results are:

THEOREM 1. *For* $n \geq 0$, *if the quotient group* $M/PZ_n(M)$ *is finite, then so too is the subgroup* $P\gamma_{n+1}(M)$.

THEOREM 2. *If* $|M/PZ(M)| = p^a$ *for some prime* p, *then* $| < M, M > | \leq p^{a^2}$.

The bound of Theorem 2 is not "best possible". For instance, if $G = 1$ then M is just a group and $< M, M > = [M, M]$, $PZ(M) = Z_1(M)$. In this case Wiegold's bound [15] states that $|[M, M]| \leq p^{a(a-1)/2}$ when $|M/Z_1(M)| = p^a$.

3. Proof of results. Recall from [2], [7] that Peiffer commutators satisfy the following easily verified identities for all $x, y, z \in M$, $g \in G$, and $k \in \ker(\partial)$:

$$< x, yz > = < x, y > \, {}^{\partial x}y \; < x, z > \, {}^{\partial x}y^{-1}, \tag{1}$$

$$< xy, z > = x < y, z > x^{-1} \; < x, {}^{\partial y}z >, \tag{2}$$

$$^{g}< x, y > = < {}^{g}x, {}^{g}y >, \tag{3}$$

$$< k, m > = kmk^{-1}m^{-1}, \tag{4}$$

$$< k, m > < m, k > = k \, {}^{\partial m}k^{-1}. \tag{5}$$

We shall write $N \leq_G M$ to indicate that N is a G-invariant normal subgroup of M. The following lemma is an easy consequence of the Peiffer identities (1)–(3).

LEMMA 3. (i) *If* $N \leq_G M$ *then* $< M, N > \leq_G M$ *and* $< N, M > \leq_G M$.
(ii) *If* $N \leq_G M$ *then* $<< M, N >> \leq_G M$.

Assertion (ii) of this lemma implies that each term of the lower Peiffer central series $P\gamma_n(M)$ is a G-invariant normal subgroup of M.

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Identities (1) and (2) imply that the Peiffer centre PZ(M) is a subgroup of M. Identity (3) implies that PZ(M) is G-invariant (and hence normal in M). More generally we have:

LEMMA 4. *If* $Z \leq_G M$ *and* $\Gamma \leq_G M$ *then* $V(Z, \Gamma)$ *is a* G-*invariant subgroup of* M.

LEMMA 5. $PZ_n(M) \leq_G M$ *for all* $n \geq 0$.

Proof. Certainly $PZ_0(M) \leq_G M$. Suppose, as an inductive hypothesis, that $PZ_j(M) \leq_G M$ for $j < n$. Lemma 4 implies that $PZ_n(M)$ is a G-invariant subgroup of M. To prove normality, choose $a \in PZ_n(M)$ and $m \in M$, and note that

$$mam^{-1} = < m, a >^{\partial m} a.$$

Since $< m, a > \in PZ_{n-1}(M) \subseteq PZ_n(M)$ and ${}^{\partial m}a \in PZ_n(M)$, it follows that mam^{-1} lies in $PZ_n(M)$. □

For an indeterminate x we set

$$< x > = x,$$

and call $< x >$ a *bracketing of weight 1 with variable x*. For $n \geq 2$ we define a *bracketing of weight n* to be an arrangement $< u, u' >$ with u and u′ bracketings of weights $i, j \geq 1$ where $i + j = n$ and where u and u′ have distinct variables. The variables involved in u and u′ will be the *variables* of $< u, u' >$. For example,

$$<< v, w >, << x, y >, z >>$$

is a bracketing of weight 5 with variables v, w, x, y, z. We shall let

$$<< x_1, ..., x_n >>$$

denote an arbitrary bracketing of weight n with variables $x_1, ..., x_n$. For instance $<< w, v, z, x, y >>$ could denote the above bracketing of weight 5.

Lemma 2.11 in [**2**] implies that

$$< x, y > \in P\gamma_{i+j}(M) \quad \text{whenever} \quad x \in P\gamma_i(M), \quad y \in P\gamma_j(M). \tag{6}$$

Thus, if each variable of a bracketing $<< x_1, ..., x_n >>$ is set equal to some element of M, the bracketing determines an element of $P\gamma_n(M)$.

LEMMA 6. *For* $n \geq 1$ *the following two conditions on an element* a *in* M *are equivalent*:
(i) $a \in PZ_n(M)$;
(ii) a *is such that* $<< a, x_1, ..., x_t >> = 1$ *for all bracketings of weight* $t + 1$ *with* $1 \leq t \leq n$, $x_j \in P\gamma_{i_j}(M)$, *and* $i_j + ... + i_t \geq n$.

Proof. Let us first show that (ii) implies (i). This is certainly true for $n = 1$. As an inductive hypothesis suppose that (ii) implies (i) when $n = k$. Let $a \in M$ satisfy (ii) for $n = k + 1$. We need to show that $a \in PZ_{k+1}(M)$. We set $n = k + 1$. For an arbitrary integer $1 \leq i \leq n$, and an arbitrary element $y \in P\gamma_i(M)$, let us set $\alpha = < a, y >$ and $\alpha' = < y, a >$. We need to show that

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$\alpha, \alpha' \in PZ_{n-i}(M)$. But $<< \alpha, x_1, ..., x_s >>= 1$ for $x_j \in P\gamma_{i_j}(M)$ with $i_1 + ... + i_s \geq n - i$. The inductive hypothesis implies that $\alpha \in PZ_{n-i}(M)$. Similarly $\alpha' \in PZ_{n-i}(M)$. It follows by induction that (ii) implies (i).

Let us now show that (i) implies (ii). This is true for $n = 1$. As an inductive hypothesis suppose that (i) implies (ii) when $n = k$. Let $a \in PZ_{k+1}(M)$. We need to show that a satisfies (ii) for $n = k + 1$. So set $n = k + 1$. Let $<< a, x_1, ..., x_t >>$ be some bracketing of weight $t + 1$ with $1 \leq t \leq n$. Let $x_j \in P\gamma_{i_j}(M)$ be such that $i_1 + ... + i_t \geq n$. Then, using (6), we have

$$<< a, x_1, ..., x_t >>=<< \alpha, y_1, ..., y_s >>$$

with $\alpha =< y_o, a >$ or $\alpha =< a, y_o >$ and $y_j \in P\gamma_{i_j}(M)$ with $i_1 + ... + i_s \geq n - i_o$. Note that $\alpha \in PZ_{n-i_o}(M)$. By the inductive hypothesis $<< \alpha, y_1, ..., y_s >>= 1$.

It follows by induction that (i) implies (ii). □

NOTATION. Given group elements x and y, we let ${}^xy$ denote the conjugate xyx^{-1}, and we let [x,y] denote the commutator $xyx^{-1}y^{-1}$.

PROPOSITION 7. $PZ_n(M) = M$ *if and only if* $P\gamma_{n+1}(M) = 1$.

Proof. Suppose that $P\gamma_{n+1}(M) = 1$. Then Lemma 6 in conjunction with (6) implies that $PZ_n(M) = M$.

Conversely, suppose that $PZ_n(M) = M$. Then, by Lemma 6, $<< x_1, ..., x_t >>= 1$ for all bracketings of weight $t \geq n + 1$ and all $x_i \in M$. We claim that $P\gamma_{n+1}(M)$ is normally generated by all such t-fold Peiffer commutators $<< x_1, ..., x_t >>$. This claim implies $P\gamma_{n+1}(M) = 1$. The claim is certainly true for $n = 1$. Suppose the claim is true for $n = k - 1$. Then any element $c \in P\gamma_k(M)$ has the form $c = x_1c_1x_1^{-1}...x_mc_mx_m^{-1}$ with c_i a t-fold Peiffer commutator $<< y_1, ..., y_k >>$ and $t \geq k$, $x_i, y_i \in M$. Now $P\gamma_{k+1}(M)$ is generated by Peiffer commutators of the form $< c, m >$ and $< m, c >$ with $m \in M$. Identities (2) and (4) imply that $< c, m >$ is a product of conjugates of elements of the form $< x_ic_ix_i^{-1}, m' >=$ $[x_ic_ix_i^{-1}, m'] = x_i[c_i, m'']x_i^{-1} = x_i < c_i, m'' > x_i^{-1}$ where $m', m'' \in M$. Identity (1) implies that $< m, c >$ is a product of conjugates of elements of the form $< m', x_ix_ic_i^{-1} >=$ $< m', x_i, c_i >^{(\partial x_i)}c_i >=< m', < x_i, c_i >> m'' < m', {}^{(\partial x_i)}c_i > m''^{-1}$. The claim follows by induction. □

Our proofs of Theorems 1 and 2 involve a nonabelian tensor product $V \otimes W$, where V and W are two groups equipped with an action $(v, w) \mapsto {}^vw$ of V on W and an action $(w, v) \mapsto {}^wv$ of W on V. When $x, y \in V$, or $x, y \in W$, the expression ${}^xy$ denotes the conjugate xyx^{-1}. The *tensor product* $V \otimes W$ is the group generated by symbols $v \otimes w$ for $v \in V$ and $w \in W$ subject to the relations

$$vv' \otimes w = ({}^vv' \otimes {}^vw)(v \otimes w) \tag{7}$$

$$v \otimes ww' = (v \otimes w)({}^wv \otimes {}^ww') \tag{8}$$

for $v, v' \in V$, $w, w' \in W$. An account of this tensor product is given in [**4**]. The tensor product is of most interest when the given actions are *compatible* in the following sense:

$${}^{({}^vw)}v' = {}^v({}^w({}^{v^{-1}}v')) \quad \text{and} \quad {}^{({}^wv)}w' = {}^w({}^v({}^{w^{-1}}w'))$$

for $v, v' \in V$, $w, w' \in W$. Compatible actions occur for instance when V and W belong to precrossed G-modules $\partial : V \to G$, $\partial' : W \to G$ and V (resp. W) acts on W (resp. V) via ∂ (resp. ∂') and the actions of G.

For convenience we compile several known properties of the tensor product into the following proposition.

PROPOSITION 8. (i) [5] *Let* $\partial : V \to G$, $\partial' : W \to G$ *be two precrossed* G-*modules. Then* G *acts on the resulting tensor product* $V \otimes W$ *by* ${}^g(v \otimes w) = {}^g v \otimes {}^g w$ *for* $g \in G$, $v \in V$, $w \in W$. *Also, there is a homomorphism* $\delta : V \otimes W \to G$ *which is defined on generators by* $\delta(v \otimes w) = [\delta v, \partial' w]$. *Moreover, this homomorphism and action form a crossed* G-*module.*

(ii) [8] *Let* V *and* W *be two finite groups which act compatibly on each other. Then the resulting tensor product* $V \otimes W$ *is a finite group.*

(iii) [9] *Let* E *be a group with two normal subgroups* V *and* W *of finite prime power orders* $|V| = p^n$ *and* $|W| = p^{n'}$. *Let* $V \otimes W$ *be the tensor product formed using the actions given by conjugation in* E. *Then* $|V \otimes W| \leq p^{nn'}$.

Given subgroups $A \leq V$, $B \leq W$ we let ${}^B AA^{-1}$ denote the subgroup of V generated by the elements ${}^b a a^{-1}$ for $a \in A$, $b \in B$.

LEMMA 9. *Let* V *and* W *act compatibly on each other, let* A *be a normal subgroup of* V, *and let* B *be a normal subgroup of* W. *Suppose that* ${}^W AA^{-1} \subseteq A$, ${}^B VV^{-1} \subseteq A$, ${}^V BB^{-1} \subseteq B$, ${}^A WW^{-1} \subseteq B$. *Then* V/A *and* W/B *act compatibly on each other, as do* A *and* W, *and* V *and* B; *the actions are induced from the actions of* V *and* W. *The tensor products constructed from these actions fit into a short exact sequence*

$$\iota(A \otimes W)\,\iota(V \otimes B) \rightarrowtail V \otimes W \twoheadrightarrow V/A \otimes W/B$$

where $\iota : A \otimes W \to V \otimes W$, $\iota : V \otimes B \to V \otimes W$ *denote the canonical homomorphisms.*

Proof. The canonical homomorphism $\phi : V \otimes W \to V/A \otimes W/B$ is clearly surjective. Moreover, identities (7) and (8) imply that the tensors $1 \otimes w$ and $v \otimes 1$ both represent the identity element in $V \otimes W$ for $v \in V$, $w \in W$. Let $\bar{v}$ denote the image of $v \in V$ in V/A, and $\bar{w}$ denote the image of $w \in W$ in W/B. Then $\bar{a} \otimes \bar{w}$ and $\bar{v} \otimes \bar{b}$ both represent the identity element in $V/A \otimes W/B$ for $a \in A$, $b \in B$. Hence $\iota(A \otimes W)$ and $\iota(V \otimes B)$ both lie in the kernel of ϕ. To prove that $\iota(A \otimes W)\iota(V \otimes B) = \ker(\phi)$ one readily verifies that $\iota(A \otimes W)$ and $\iota(V \otimes B)$ are normal in $V \otimes W$, that the function

$$V/A \times W/B \to V \otimes W/\iota(A \otimes W)\,\iota(V \otimes B), \quad (\bar{v}, \bar{w}) \mapsto v \otimes w$$

is well-defined, and that it induces a homomorphism $\psi : V/A \otimes W/B \to V \otimes W/\iota(A \otimes W)\iota(V \otimes B)$. Since ψ is mutually inverse to the induced homomorphism $\bar{\phi} : V \otimes W/\iota(A \otimes W)\iota(V \otimes B) \to V/A \otimes W/B$, it follows that $\bar{\phi}$ is injective. Hence $\iota(A \otimes W)\,\iota(V \otimes B) = \ker(\phi)$.

Let us consider the precrossed G-module $\partial : M \to G$. Using the action of G on M we can form the semi-direct product $S = M \rtimes G$, in which elements are multiplied by the rule

$$(m, g)(m', g') = (m^g m', gg').$$

Let

$$\overline{M} = \{(m, g) \in M \rtimes G : m \in M \quad \text{and} \quad g = \partial(m^{-1})\}$$

and note that $\overline{M}$ is a normal subgroup of S. Since the inclusion homomorphisms $M \hookrightarrow S$, $\overline{M} \hookrightarrow S$ are examples of crossed S-modules, we can use Proposition 8(i) to form crossed S-modules $\delta : M \otimes M \to S$ and $\delta : M \otimes \overline{M} \to S$.

Let $n \geq 1$, let $\beta = (\beta_1, \ldots, \beta_n)$ be an arbitrary sequence of 0s and 1s (i.e. $\beta_i = 0$ or 1), and let $\beta' = (\beta_1, \ldots, \beta_{n-1})$. Using Proposition 8(i) we define a crossed S-module $\delta : T^\beta \to S$ by inductively setting

$$T^\beta = \begin{cases} M \otimes M & \text{if } n = 1 \text{ and } \beta_1 = 0, \\ \overline{M} \otimes M & \text{if } n = 1 \text{ and } \beta_1 = 1, \\ M \otimes T^{\beta'} & \text{if } n \geq 2 \text{ and } \beta_n = 0, \\ \overline{M} \otimes T^{\beta'} & \text{if } n \geq 2 \text{ and } \beta_n = 1. \end{cases}$$

Using Lemma 3(i) we can define a G-invariant normal subgroup M^β in M by inductively setting

$$M^\beta = \begin{cases} [M, M] & \text{if } n = 1 \text{ and } \beta_1 = 0, \\ < M, M > & \text{if } n = 1 \text{ and } \beta_1 = 1, \\ < M^{\beta'}, M > & \text{if } n \geq 2 \text{ and } \beta_n = 0, \\ < M, M^{\beta'} > & \text{if } n \geq 2 \text{ and } \beta_n = 1. \end{cases}$$

LEMMA 10. (i) *For* $n \geq 1$ *and for each sequence* $\beta = (\beta_1, \ldots, \beta_n)$ *of 0s and 1s with* $\beta_1 = 1$, *the image of the crossed* S*-module* $\delta : T^\beta \to S$ *satisfies*

$$\mathrm{im}(\delta) = M^\beta.$$

(ii) *For a fixed* $n \geq 1$, *the family of* G*-invariant normal subgroups* $\{M^\beta : \beta = (\beta_1, \ldots, \beta_n),\ \beta_1 = 1\}$ *generates* $P\gamma_{n+1}(M)$.

Proof. One readily verifies that the identity

$$[(y, \partial y^{-1}),\ (x, 1)] = (< y, \partial y - 1x >, 1) \tag{9}$$

holds in $S = M \rtimes G$ for all $x, y \in M$. Hence the crossed module $\delta : \overline{M} \otimes M \to S$ has image $\mathrm{im}(\delta) = [\overline{M}, M] =< M, M >$. Therefore assertion (i) holds for $n = 1$. The assertion can be proved inductively for $n \geq 2$ (using the inductive hypothesis $\delta T^{\beta'} = M^{\beta'}$) : when $\beta_n = 1$ we have

$$\delta(T^\beta) = \delta(\overline{M} \otimes T^{\beta'}) = [\overline{M}, \delta T^{\beta'}] = [\overline{M}, M^{\beta'}] =< M, M^{\beta'} >= M^\beta;$$

when $\beta_n = 0$ we have

$$\delta(T^\beta) = \delta(M \otimes T^{\beta'}) = [M, \delta T^{\beta'}] = [M, M^{\beta'}] = [M^{\beta'}, M]$$

and (as we shall see)

$$[M^{\beta'}, M] = \langle M^{\beta'}, M \rangle .$$

To prove this last equality it suffices to note that there are inclusions

$$M^{\beta'} \subseteq \langle M, M \rangle \subseteq \ker(\partial : M \to G)$$

for any sequence $\beta' = (\beta_1, \ldots, \beta_{n-1})$ with $\beta_1 = 1$.

Assertion (ii) clearly holds.

Suppose that A is a G-invariant normal subgroup of M such that $\langle\langle A, M \rangle\rangle \subseteq A$. Let us set

$$\overline{A} = \{(a, \partial a^{-1}) \in S : a \in A\}.$$

Note that conjugation in S yields an action of G on $\overline{A}$. Moreover, $\overline{A}$ is a G-invariant normal subgroup of $\overline{M}$ and, for $N = A\overline{A}$, we have $A = N \cap M$ and $\overline{A} = N \cap \overline{M}$. Note also that if M/A is finite then so too is $\overline{M}/\overline{A}$ since one can readily verify that $|M/A| = |\overline{M}/\overline{A}|$.

Taking $A = PZ_1M$, we have a commutative diagram of group homomorphisms

$$\begin{array}{ccccccc} 1 \to \iota(\overline{PZ_1M} \otimes M)\iota(\overline{M} \otimes PZ_1M) & \to & \overline{M} \otimes M & \to & (\overline{M}/\overline{PZ_1M}) \otimes (M/PZ_1M) \to 1 \\ & & \downarrow \delta & \swarrow \bar{\delta} & \\ & & \langle M, M \rangle & & \\ & & \downarrow & & \\ & & 1 & & \end{array}$$

in which the row and column are exact. The exact row follows from Lemma 9. The surjectivity of δ follows from Lemma 10(i). The homomorphism δ induces a homomorphism $\bar{\delta}$ thanks to the exactness of the row and Lemma 6.

Suppose that M/PZ_1M is finite. Then so too is $\overline{M}/\overline{PZ_1M}$, and so Proposition 8(ii) implies the finiteness of $(\overline{M}/\overline{PZ_1M}) \otimes (M/PZ_1M)$. The surjectivity of $\bar{\delta}$ then implies that $\langle M, M \rangle$ is finite, thus proving Theorem 1 for $n = 1$.

Suppose that $|M/PZ_1M| = p^a$ for some prime p. The $|\overline{M}/\overline{PZ_1M}| = p^a$. Consider the normal subgroup $N = (PZ_1M)(\overline{PZ_1M})$ in S. Since $N \cap M = PZ_1M$ and $N \cap \overline{M} = \overline{PZ_1M}$, both M/PZ_1M and $\overline{M}/\overline{PZ_1M}$ are normal subgroups of S/N. Thus Lemma 8(iii) and the surjectivity of $\bar{\delta}$ imply that $|\langle M, M \rangle| \leq p^{a^2}$. This proves Theorem 2.

The proof of Theorem 1 for $n \geq 1$ is similar to that for $n = 1$. There is an induced precrossed module $\partial : M/PZ_nM \to G/(\partial PZ_nM)$. Observe that the induced action is well-defined since, for $a \in PZ_nM$ and $m \in M$, we have ${}^{\partial a}m = \langle a, m \rangle ama^{-1}$ and $\langle a, m \rangle \in PZ_nM$.

The above construction of the precrossed module $\delta : T^{\beta} \to S$ depends on the precrossed module $\partial : M \to G$. To emphasize this dependence let us write $T^{\beta}(M) = T^{\beta}$. Then for each sequence β of 0s and 1s, with $\beta_1 = 1$, we have a commutative triangle of group homomorphisms.

$$\begin{array}{ccc} T^{\beta}M & \longrightarrow & T^{\beta}(M/PZ_nM) \\ \downarrow{\scriptstyle\delta} & \swarrow{\scriptstyle\bar{\delta}} & \\ M^{\beta} & & \end{array}$$

The homomorphism $\bar{\delta}$ is induced by δ thanks to Lemmas 6 and 9.

Suppose that M/PZ_nM is finite. Then $T^{\beta}(M/PZ_nM)$ is finite by Proposition 8(ii). Lemma 10(i) implies that M^{β} is finite. So Lemma 10(ii) implies that $P\gamma_{n+1}M$ is finite, thus proving Theorem 1.

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Department of Mathematics
University College Galway
Ireland
E-mail: graham.ellis@ucg.ie

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