Problem Corner

Solutions are invited to the following problems. They should be addressed to Chris Starr, (e-mail: czqstarr@gmail.com) c/o Kintail, Longmorn, Elgin IV30 8RJ and should arrive not later than 10th April 2026. Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address or e-mail and should be accompanied by solutions and any relevant background information.

109.E (Jacob Siehler)

Two walkers begin at opposite ends of a path of length four (thus, having five vertices). At each step, they both move to an adjacent vertex at random, stopping when they both arrive at the same vertex. Determine:

(a) the probability for each vertex to be the end of the walk,

(b) the expected number of steps before they meet.

109.F (Mihály Bencze)

Let ABC be a triangle with orthocentre H and circumcentre O, and $f : R \to R$ be a convex function. Prove that:

$$\sum f\left(\text{area } HAB\right) + 3f\left(\frac{sr}{3}\right) \geqslant 2\sum f\left(\text{area } OAB\right)$$

where s is the semiperimeter and r is the inradius.

109.G (Peter Shiu)

Find a formula for $f(n)$, the number of solutions (x_1, x_2, x_3) in non-negative integers to the equation

$$x_1 + 2x_2 + 5x_3 = n.$$

Hence, or otherwise (but without the use of a computing machine!) determine the number of ways of making up £1 using coins of values 1p, 2p, 5p, 10p, 20p, 50p and the £1 coin.

109.H (Toyesh Prakash Sharma)

Evaluate

$$\left(\int_0^\infty e^{-x^2}\cos(\ln x)\,dx\right)^2 + \left(\int_0^\infty e^{-x^2}\sin(\ln x)\,dx\right)^2.$$

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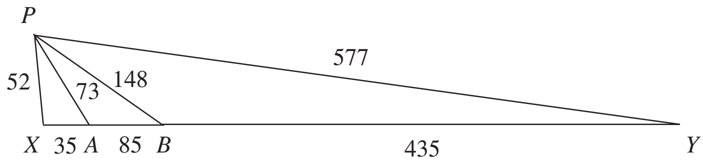
Solutions and comments on **108.I**, **108.J**, **108.K**, **108.L** (November 2024).

108.I (Chris Starr)

Consider triangle PXY, with $PX = 52$ cm, $PY = 577$ cm and $XY = 555$ cm. The lines PA and PB are constructed such that $XA = 35$ cm and $AB = 85$ cm. It may then be verified that triangle PXY is split into three triangles, each with integer side lengths, and the areas of triangles ΔPXA, ΔPAB and ΔPBY are all integer values.



(a) Can you find a triangle PXY that can be split into four triangles with integer side lengths and integer areas?

(b) Is there a value N such that PXY cannot be split into N triangles with integer side lengths and integer areas?

Answers: (a) Yes (b) No

Solution:

This was the last problem I submitted to Problem Corner, and I originally set up a diagram where $\angle PXY$ was obtuse, thereby making the diagram more misleading (Diagram Not To Scale!). However, in consultation with the former editor, Nick Lord, we agreed that the diagram be drawn more to scale as above. This immediately brought about a simple answer to part (a) spotted by Zoltan Retkes; if we drop a perpendicular from P to XY produced, we generate a fourth triangle with sides 20, 48, 52 which is similar to the Pythagorean triple (5, 12, 13). Stan Dolan also pointed out that four (3, 4, 5) triangles can be joined together to make a (6, 8, 10) triangle, but also provided a solution for the more general case.

Most solvers proceeded along these lines for part (a). Place X at the origin and Y on the x-axis and let the coordinates of A, B, C, Y, be $(a, 0)$, $(b, 0)$, $(c, 0)$, $(d, 0)$, respectively, with a, b, c, d integers. If we set the coordinates of P to be (p, q), where p is an integer, and q is an even number, then this guarantees that the areas of PXA, PAB, PBC, PCY are all integers.

We can use Pythagoras to find the lengths PX, PA, PB, PC, PY thus:

$$PX^2 = q^2 + p^2$$

$$PA^2 = q^2 + (p - a)^2$$

$$PB^2 = q^2 + (p - b)^2$$

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$$PC^2 = q^2 + (p - c)^2$$

$$PY^2 = q^2 + (p - d)^2$$

Then if PX, PA, PB, PC, PY are all required to be integers, they must form Pythagorean triples. In fact, the problem boils down to finding several Pythagorean triples with one side in common. Using the standard parametrisation $(2mn, m^2 - n^2, m^2 + n^2)$, since q is even, we can set $q = 2mn$, and choose q to have a sufficiently high number of factors to give 5 different Pythagorean triples. For example, $q = 96$ gives the following table:

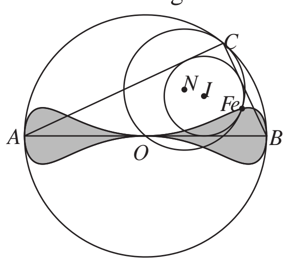
m	n	q	$m^2 - n^2$	$m^2 + n^2$
48	1	96	$2303 = p$	$2305 = PX$
24	2	96	$572 = p - a$	$580 = PA$
16	3	96	$247 = p - b$	$265 = PB$
12	4	96	$126 = p - c$	$160 = PC$
8	6	96	$28 = p - d$	$100 = PY$

We have therefore that PX, PA, PB, PC, PY are all integers and we can use the fourth column to work out a, b, c, d. In this case, $a, b, c, d = 1731, 2056, 2175, 2275$ respectively. Since these represent the x coordinates, we have $XA = 1731$, $AB = 325$, $BC = 119$, $CY = 100$. Furthermore, PX, PA, PB, PC and PY are given respectively by the numbers in column 5: 2305, 580, 265, 160 and 100.

For part (b), this method can clearly be generalised provided you pick an even number q with a sufficiently high number of factors, therefore there is no upper limit for N. James Mundie showed that a sufficient condition to guarantee N factor pairs is $m = p_1$, $n = p_2^{n-1}$, where p_1, p_2 are prime numbers.

108.J (Mark Hennings)

The points A and B lie on the diameter of a unit circle, and C is a third point on that circle, making a right-angled triangle ABC. The Feuerbach point Fe of a triangle is the point where the triangle's incircle (centre I) and the nine-point circle (centre N) are tangential to each other. The locus of the Feuerbach point as C varies forms an elegant bow-tie shape as below:



What is the area of the region enclosed by the locus?

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Answer: $2 - \frac{1}{2}\pi$

The key step is to determine an appropriate parametrisation for F_e, and solvers were ingenious and resilient in their approaches. The following is based on the solution provided by the proposer, Mark Hennings.

Let angle BAC be θ, where $0 < \theta < \frac{1}{2}\pi$ (so that C is in the upper semicircle), then the coordinates of A, B, C are $(-1, 0)$, $(1, 0)$, $(\cos 2\theta, \sin 2\theta)$ respectively. The semiperimeter of the rectangle is therefore $\cos\theta + \sin\theta + 1$ and the area of the triangle is given by:

$$\Delta = 2 \sin\theta \cos\theta \equiv (\cos\theta + \sin\theta)^2 - 1.$$

Hence, the inradius of triangle ABC is given by

$$r = \frac{\Delta}{s} = \frac{(\cos\theta + \sin\theta)^2 - 1}{\cos\theta + \sin\theta + 1} = \cos\theta + \sin\theta - 1.$$

The incentre I has coordinates $(-1 + AC - r, r)$, so combining this with the above results gives coordinates $I(\cos\theta - \sin\theta, \cos\theta + \sin\theta + 1)$.

The orthocentre of triangle ABC is C, and hence the nine-point centre N is the midpoint of OC, so has coordinates $\left(\frac{1}{2}\cos 2\theta, \frac{1}{2}\sin 2\theta\right)$. We therefore deduce, after some algebra, that $IN = \frac{1}{2}\left(3 - 2\cos\theta - 2\sin\theta\right)$.

It is known (see for example: [1]) that the nine-point circle and the incircle are tangential to each other at the Feuerbach point F_e. Therefore N, I, F_e are collinear (in that order). Since the incircle has radius $\frac{1}{2}$, we deduce that the position vector of F_e is given by

$$\overrightarrow{OF_e} = \overrightarrow{ON} + \frac{1}{2}.\frac{1}{|\overrightarrow{NI}|}\overrightarrow{NI} = \overrightarrow{ON} + \frac{1}{3 - 2\cos\theta - 2\sin\theta}\overrightarrow{NI}.$$

Since we have the coordinates of N and I, we can therefore use the above expression to find the coordinates of F_e. This can be simplified somewhat by introducing the change of variable $u = \theta - \frac{1}{4}\pi$, and after some simplification we obtain the coordinates $\left(X(u), Y(u)\right)$, where

$$X(u) = \frac{\sqrt{2}\sin u\left(2\cos^2 u - \sqrt{2}\cos u - 1\right)}{3 - 2\sqrt{2}\cos u},$$

$$Y(u) = \frac{2\sin^2 u\left(\sqrt{2}\cos u - 1\right)}{3 - 2\sqrt{2}\cos u}.$$

For the upper half of the locus, θ varies from 0 to $\frac{1}{2}\pi$, with $\theta = 0$ at B and $\theta = \frac{\pi}{2}$ at A, so that the range of θ goes from right to left along the x-axis. Therefore u varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$. Using c to represent $\cos u$ and A to represent the area of the "bow tie", we have (omitting the details):

$$-\frac{1}{2}A = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} Y(u)X'(u)du = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{2\left(1 - c^2\right)\left(1 - \sqrt{2}c\right)^2\left(10 - 5\sqrt{2}c - 14c^2 + 8\sqrt{2}c^3\right)}{\left(3 - 2\sqrt{2}c\right)^3}du$$

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$$= \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \left(-\frac{15}{16} + \frac{1}{4}\sqrt{2}c + \frac{17}{8}c^2 - \frac{3}{4}\sqrt{2}c^3 - 2c^4 + \frac{41}{64}Q_1 - \frac{5}{32}Q_2 + \frac{1}{64}Q_3 \right) du$$

where $Q_n = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{1}{(3 - 2\sqrt{2}c)^n}\, du, n = 1, 2, 3.$

The first five terms of the integrand may be evaluated by elementary means, and their contribution is $-\frac{11}{16} - \frac{5}{16}\pi$.

The final three terms can be dealt with by using the successive substitutions $t = \tan \frac{1}{2}u$ and $t = (\sqrt{2} - 1)^2 \tan \theta$ to give:

$$\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{1}{(3 - 2\sqrt{2}c)^n} du = 2 \int_0^{\frac{1}{4}\pi} \frac{1}{(3 - 2\sqrt{2}c)^n} du$$

$$= \frac{4}{(\sqrt{2} - 1)^{2(n-1)}} \int_0^{\frac{3}{8}\pi} \left(\cos^2 \theta + (\sqrt{2} - 1)^4 \sin^2 \theta \right)^{n-1} d\theta.$$

This can now be evaluated by elementary means for the values $n = 1, 2, 3$, and the total contribution of the last three terms is found to be $\frac{9}{16}\pi - \frac{5}{16}$. Combining this with the other result, and noting that this gives $-\frac{1}{2}A$ we finally obtain the result $A = 2 - \frac{1}{2}\pi$.

It is remarkable that this problem yields such a simple answer, and solvers were very appreciative of this. G. Howlett, who provided the reference [1] verified the result to 6 decimal places using a 5th order Romberg integration. It would be interesting to see if there were a more direct route to this solution.

Reference

1. G. Leversha, *The Geometry of the Triangle* (UKMT 2013) Chapter 21.

Correct solutions were received from: G. Howlett, J. A. Mundie, V. Schindler and the proposer, M. Hennings.

108.K (Toyesh Prakash Sharma)

Show that, in the acute-angled triangle ABC, the following inequality holds:

$$(\sin A)^{\cos A} (\sin B)^{\cos B} (\sin C)^{\cos C} \leqslant \left(\frac{27}{64} \right)^{1/4}.$$

This attractive problem was typically solved using Jensen's inequality, but it first had to be established that the function was concave. The following is based on the solution by H. Ricardo.

If we denote the left hand side of the inequality by S, then the problem may be recast as follows:

$$\ln S = \sum \cos A \ln (\sin A) \leqslant \frac{3}{4} \ln \left(\frac{3}{4} \right).$$

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If we let $f(x) = \cos x \ln(\sin x), 0 < x \leqslant \frac{1}{2}\pi$, then

$$f''(x) = -\cos x\left(3 + \ln(\sin x) + \cot^2 x\right) \leqslant 0$$

so that f is concave on that interval. We may now use Jensen's inequality to obtain:

$$\begin{aligned}\sum \cos A \ln\left(\sin A\right) &\leqslant 3\cos\left(\frac{A + B + C}{3}\right)\ln\left(\sin\left(\frac{A + B + C}{3}\right)\right) \\ &= 3\cos\frac{\pi}{3}\ln\left(\sin\frac{\pi}{3}\right) \\ &= \frac{3}{4}\ln\frac{3}{4}\end{aligned}$$

as required, with equality when $A = B = C$. As a development of the problem, readers may wish to consider whether the inequality is true if the triangle contains an obtuse angle.

Correct solutions were received from: H. Ricardo, S. Dolan, M. Hennings, J. A. Mundie, N. Curwen, P. F. Johnson and the proposer, T. P.Sharma.

108.L (Albert Natian)

You are invited to play the following two-stage game using a fair n-sided die labelled $1, 2, \dots, n$.

Stage 1: You roll the die to get a number, say x, which is the number of gold coins that you win, and the possession of which is subject to the outcome(s) in Stage 2.

Stage 2: Now you roll the die x times. You win, in gold coins, all numbers that come up in the x rolls, except if any number is x, in which case you lose all your winnings, including that of Stage 1.

Find an expression for the expected winnings $\mathrm{E}[W]$ in this game, and determine

$$\lim_{n \to \infty} \frac{\mathrm{E}[W]}{n^2}.$$

Solution:

$$E[W] = \frac{n - 1}{2n^n}\left[n^{n+1} - 2\left(n^2 - 2n - 1\right)(n - 1)^{n-1}\right], \lim_{n \to \infty} \frac{E[W]}{n^2} = \frac{1}{2} - \frac{1}{e}$$

Some careful counting techniques were employed by solvers, as in the following based on that by the proposer, Albert Natian.

If we denote the possible winnings in Stage 1 and Stage 2 separately as W_1, W_2 respectively, then the total combined winnings of the game will either be 0 or $W_1 + W_2$. If the number on the first roll is x, then $W_1 = x$.

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Let the x-tuple $(y_1, y_2, \ldots, y_x)$, where represents the outcome of the i-th roll denote the outcomes of x successive rolls in Stage 2. For a win in Stage 2, none of the y_i can take the value x, so there are $(n - 1)^x$ possible n-tuples. The total winnings for Stage 2 alone is therefore $W_2 = \sum_{i=1}^{x} y_i$, and since the probability of each outcome y_i has probability of occurring $\frac{1}{n}$, then the probability of any n-tuple occurring is $\frac{1}{n^x}$. Therefore the total winnings from both stages is $W_1 + W_2 = x + \sum_{i=1}^{x} y_i$ with probability $\frac{1}{n^{x+1}}$.

Each of the $n - 1$ x-tuples occurs the same number of times as each other in the aggregation of all possible $(n - 1)^x$ possible x-tuples, comprising a total of $x(n - 1)^x$ equally often repeated numbers. So each of the $n - 1$ values appears $x(n - 1)^{x-1}$ times, and their total will be

$$x(n - 1)^x\big((1 + 2 + \ldots + n) - x\big) = x(n - 1)^{x-1}\left[-x + \tfrac{1}{2}n(n + 1)\right].$$

Therefore, the overall total winnings will be

$$x(n - 1)^x + x(n - 1)^{x-1}\left[-x + \tfrac{1}{2}n(n + 1)\right].$$

We therefore have

$$E[W] = \sum_{x=1}^{n}\left(x(n - 1)^x + x(n - 1)^{x-1}\left[-x + \frac{1}{2}n(n + 1)\right]\right)\frac{1}{n^{x+1}}.$$

After some simplification we obtain:

$$E[W] = \frac{n^2 + 3n - 2}{2n(n - 1)}\sum_{x=1}^{n} x\left(\frac{n - 1}{n}\right)^x - \frac{1}{n(n - 1)}\sum_{x=1}^{n} x^2\left(\frac{n - 1}{n}\right)^x.$$

These series can be summed using differentiation of geometric series, and for reference they are included below:

$$\sum_{x=1}^{n} xr^x = \frac{r}{(1 - r)^2}\left[1 - (n + 1)r^n + nr^{n+1}\right],$$

$$\sum_{x=1}^{n} x^2r^x = \frac{r}{(1 - r)^3}\left[1 + r - (n + 1)^2r^n + \left(2n^2 + 2n - 1\right)r^{n+1} - n^2r^{n+2}\right].$$

Using these with $r = \dfrac{n - 1}{n}$ eventually gives

$$E[W] = \frac{n - 1}{2n^n}\left[n^{n+1} - 2\left(n^2 - 2n - 1\right)(n - 1)^{n-1}\right].$$

Finally,

$$\lim_{n\to\infty}\frac{E[W]}{n^2} = \lim_{n\to\infty}\left(\frac{n - 1}{2n} - \frac{n^2 - 2n - 1}{n^2}\left(1 - \frac{1}{n}\right)^n\right) = \frac{1}{2} - \frac{1}{e}.$$

Correct solutions were received from: G. Howlett, S. Dolan, M. Hennings, J. A. Mundie, N. Curwen, P. F. Johnson and the proposer, A. Natian.

10.1017/mag.2025.10099 C. STARR

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