FREE SUB-BANDS OF FINITELY PRESENTED BANDS

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Abstract. For each variety of bands $\mathcal{V}$, we give a formula for $\phi_{\mathcal{V}}(m,k)$, which is the largest integer such that for every band B in $\mathcal{V}$ generated by m generators and k relations, there is a subset of the generators of size $\phi_{\mathcal{V}}(m,k)$ which generates a (relatively) free sub-band of B as a basis. We also determine the semilattice structure of a finitely presented band.

1. Introduction and preliminaries. Let $\mathcal{V}$ be a variety of algebras. For integers $m > k$, we define $\phi_{\mathcal{V}}(m,k)$ to be the largest integer such that for every algebra B in $\mathcal{V}$ presented by m generators $a_1,\ldots,a_m$ and k relations, it is possible to choose $\phi_{\mathcal{V}}(m,k)$ elements from among $a_1,\ldots,a_m$ which generate a (relatively) free subalgebra of B as a basis.

The question of finding $\phi_{\mathcal{V}}$ for the variety of groups was posed by Lyndon [7, Question 5.29]. Romanovskii [8] showed that $\phi_{\mathcal{V}}(m,k) = m-k$ in this case, but the proof is not constructive. For the variety of semigroups Shneerson [10] gave a constructive proof, based on linear algebra, that $\phi_{\mathcal{V}}(m,k) = m–k$. Standard linear algebra methods, applied to abelian groups (see [5, Chapter 10]), shows that $\phi_{\mathcal{V}}(m,k) = m–k$ for the variety of abelian groups also. This implies, by an easy exercise, that $\phi_{\mathcal{V}}(m,k) = m–k$ for the variety of commutative semigroups also.

Let $\mathcal{V}$ be a variety of algebras. Let A be an alphabet and R a set of relations. Then $\langle A|R\rangle_{\mathcal{V}}$ will denote the algebra in $\mathcal{V}$ generated by A, subject to the relations of R. The subscript $\mathcal{V}$ is sometimes dropped when it is clear from context. We record the following properties of $\phi_{\mathcal{V}}$.

LEMMA 1.1. *For any variety of algebras $\mathcal{V}$,*

(i) $\phi_{\mathcal{V}}(m_1 + m_2, k_1 + k_2) \leq \phi_{\mathcal{V}}(m_1, k_1) + \phi_{\mathcal{V}}(m_2, k_2)$, and
(ii) $\phi_{\mathcal{V}}(m, k) \leq \phi_{\mathcal{V}}(m - 1, k - 1)$.

Proof. If $S_1 = \langle A_1|R_1\rangle_{\mathcal{V}}$ and $S_2 = \langle A_2|R_2\rangle_{\mathcal{V}}$ are two algebras defined with disjoint alphabets, then $A_0 \subseteq A_1 \cup A_2$ generates a relatively free subalgebra as a basis in $\langle A_1 \cup A_2|R_1 \cup R_2\rangle_{\mathcal{V}}$ only if $A_0 \cap A_i$ does so in S_i for $i=1,2$. Hence (i) follows.

Suppose $S = \langle A|R\rangle_{\mathcal{V}}$. Choose $a \in A$ and a letter b not in A. We prove (ii) by considering $\langle A,b|R,a=b\rangle_{\mathcal{V}}$, using similar reasoning to the above. $\square$

In this paper, we give a formula for $\phi_{\mathcal{V}}(m,k)$ for each variety of bands $\mathcal{V}$. Section 2 proves a combinatorial lemma which is the main ingredient of the proof of the formula in Section 3.

We shall use the following notation: the free semigroup over a set A is denoted by F_A. If $w \in F_A{}^1$ we define its *content* $C(w)$ to be the set of elements of A appearing in w.

2. A combinatorial lemma.

LEMMA 2.1. *Let A be a set of size m. Let $A_1, A_2, \ldots, A_k$ be subsets of A such that $|A_i| \geq l$ for each i. Then there exists a subset A' of A such that*

(i) $|A'| \leq \min\{\lfloor \frac{m+(l-1)k}{2l-1} \rfloor, k\}$,
(ii) $A' \cap A_i \neq \emptyset$ *for each* i.

Proof. Let $n = lk - m$. We shall deal with the case when $n \leq 0$ first, and then prove the lemma for $n > 0$ by induction.

If $n \leq 0$, then $m \geq lk$ and $\lfloor \frac{m+(l-1)k}{2l-1} \rfloor \geq k$. For each i choose $a_i \in A_i$. The lemma is proved by letting $A' = \{a_1, a_2, \ldots, a_k\}$.

Now suppose $n > 0$. Since $|A_i| \geq l$ for each i and $lk > m$, some $a_0 \in A$ is in (at least) two of the A_i's. In other words,

$$k' = |\{i | a_0 \notin A_i\}| \leq k - 2.$$

Since $lk' - (m-1) \leq lk - m - (2l-1) < n$, by induction there is a subset B of $A \setminus \{a_0\}$ such that

$$\begin{aligned}
|B| &\leq \min\{\lfloor \frac{(m-1)+(l-1)k'}{2l-1} \rfloor, k'\} \\
&\leq \min\{\lfloor \frac{(m+(l-1)k-(2l-1)}{2l-1} \rfloor, k-2\} \\
&\leq \min\{\lfloor \frac{m+(l-1)k}{2l-1} \rfloor, k\} - 1,
\end{aligned}$$

and $B \cap A_i \neq \emptyset$ if $a_0 \notin A_i$. The lemma is proved by letting $A' = \{a_0\} \cup B$. $\square$

Obviously, the bound on $|A'|$ can be attained when $m \geq lk$, and when $m = l$, the bound can be improved to $|A'| = 1$. I do not know the necessary and sufficient conditions for the bound in Lemma 2.1 to be sharp.

Using Lemma 2.1 in the case $l = 2$ allows us to prove the following combinatorial lemma, crucial to the proof of the formula in Section 3.

LEMMA 2.2. *Let A be a set of size m. Suppose $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ are subsets of A such that for each i, either*

(a) $|A_i| \geq 2$ *and* $|B_i| \geq 2$, *or*
(b) $|A_i| = 1$ *and* $A_i \subseteq B_i$.

Then there exists a subset A' of A such that

(i) $|A'| \leq k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}$,
(ii) $A' \cap A_i \neq \emptyset$ *and* $A' \cap B_i \neq \emptyset$ *for each* i.

Proof. By reordering we can assume $|A_i| = 1$ if and only if $i \in \{1, 2, \ldots, k_0\}$ for some $k_0 \leq k$. Let $C = A_1 \cup A_2 \cup \cdots \cup A_{k_0}$. Then by condition (b),

$$k' = |\{i | C \cap A_i = \emptyset\}| + |\{i | C \cap B_i = \emptyset\}| \leq 2(k - k_0) \leq 2(k - |C|).$$

Applying Lemma 2.1 with $l = 2$, we obtain a subset D of $A \setminus C$ so that

$$|D| \leq \min\{\lfloor \frac{(m - |C|) + k'}{3} \rfloor, k'\}$$

$$\leq \min\{\lfloor \frac{(m - |C| + 2(k - |C|)}{3} \rfloor, 2(k - |C|)\}$$

$$= (k - |C|) + \min\{\lfloor \frac{(m - |C|) - (k - |C|)}{3} \rfloor, k - |C|\}$$

$$\leq k + \min\{\lfloor \frac{m - k}{3} \rfloor, k\} - |C|,$$

$D \cap A_i \neq \emptyset$ if $C \cap A_i = \emptyset$, and $D \cap B_i \neq \emptyset$ if $C \cap B_i = \emptyset$. The lemma is proved by letting $A' = C \cup D$. □

Note that the proofs of the lemmas provide an algorithm for finding the required A'. The algorithm guarantees an upper bound on the size of A', but may not give the smallest possible A'.

3. The formula. In this section we determine a formula for $\phi_{\mathcal{V}}(m,k)$ for each variety of bands $\mathcal{V}$. We assume familiarity with the basic results from the theory of semigroups and bands, which can be found in [**6**].

THEOREM 3.1. *In any band B, $\mathcal{J}$ equals the minimum semilattice congruence on B, and B is a semilattice of rectangular bands.*

COROLLARY 3.2. *Let T be a sub-band of a band B. Then for all x, y in T, $x \mathcal{J} y$ in T if and only if $x \mathcal{J} y$ in B.*

Proof. If $x \mathcal{J} y$ in a band, then x and y lie in the same rectangular band. Therefore $xyx = x$ and $yxy = y$. □

The lattice **LB** of varieties of bands has been determined by Birjukov [**1**] Fennemore [**2**] and Gerhard [**3**] (see also [**4**]). Only the following fact will be needed in this paper ([**9**, Remark 3.11]).

Let $\mathbf{LB_0}$ be the lattice obtained from **LB** by taking away the four varieties of trivial, right zero, left zero, and rectangular bands. If $\mathcal{V} \in \mathbf{LB_0}$, then for a set A, two words in F_A represent the same element in the free band on A in $\mathcal{V}$ only if they have the same content.

We first treat the case of the subvarieties of rectangular bands.

THEOREM 3.3. *$\phi_{\mathcal{V}}(m,k) = m - k$ for the variety of left [right] zero semigroups. $\phi_{\mathcal{V}}(m - k) = \max\{m - 2k, 1\}$ for the variety of rectangular bands.*

Proof. If $w \in F_A$, denote its initial and terminal letters by $h(w)$ and $t(w)$ (*head* and *tail*) respectively.

Let $\mathcal{LZ}$ and $\mathcal{RZ}$ denote the varieties of left and right zero semigroups respectively. If $\mathcal{V} = \mathcal{LZ}$, then

$$\langle A|u_i = v_i, i \in I \rangle_{\mathcal{V}} \simeq \langle A|h(u_i) = h(v_i), i \in I \rangle_{\mathcal{V}},$$

and so the result is clear. The case of right zero semigroups is dual.

If $\mathcal{V}$ is the variety of rectangular bands, then it is easy to see that

$$\langle A|u_i = v_i, i \in I\rangle_{\mathcal{V}} \simeq \langle A|h(u_i)t(u_i) = h(v_i)t(v_i), i \in I\rangle_{\mathcal{V}}$$
$$\simeq \langle A|h(u_i) = h(v_i), i \in I\rangle_{\mathcal{LZ}} \times \langle A|t(u_i) = t(v_i), i \in I\rangle_{\mathcal{RZ}},$$

and the result follows. $\qquad\square$

For the rest of the section let $\mathcal{V}$ be a variety in $\mathbf{LB_0}$.

Let B be a band in the variety $\mathcal{V}$ defined by the presentation

$$B = \langle A|u_i = v_i, i \in I\rangle_{\mathcal{V}},$$

where I is some index set and u_i, $v_i \in F_A$ for each $i \in I$. Let $SL(B)$ be the semilattice defined by the presentation $\langle A|u_i = v_i, i \in I\rangle$. The $\mathcal{J}$ relation on B is simply denoted by $\mathcal{J}$.

THEOREM 3.4. *The semilattice $\mathcal{B}/\mathcal{J}$ is isomorphic to $SL(B)$.*

Proof. Since $\mathcal{V} \in \mathbf{LB_0}$, $\mathcal{V}$ contains the variety of semilattices and so the theorem follows from Theorem 3.1. $\qquad\square$

In particular, if B is the free band on A in $\mathcal{V}$, then $\mathcal{B}/\mathcal{J}$ is isomorphic to the free semilattice on A and so we have the following result.

COROLLARY 3.5. *Suppose that a subset A_0 of A generates a (relatively) free sub-band of B. Then A_0 generates a free sub-semilattice of $SL(B)$.*

Proof. Let ϕ denote the isomorphism from $\mathcal{B}/\mathcal{J}$ to $SL(B)$. Let T denote the sub-band generated by A_0, and $\mathcal{J}^T$ the $\mathcal{J}$ relation on T. Then

$$\phi(T) \simeq T/\mathcal{J}^T$$

by Corollary 3.2. Since $\phi(T)$ is precisely the sub-semilattice of $SL(B)$ generated by A_0, the result follows. $\qquad\square$

Before proving the formula for $\phi_{\mathcal{V}}$ we give an upper bound on $\phi_{\mathcal{V}}(m,k)$ in the case of semilattices.

LEMMA 3.6. *If $\mathcal{V}$ is the variety of semilattices, then for $m > k$,*

$$\phi_{\mathcal{V}}(m, k) \leq m - (k + \min\{\lfloor \frac{m-k}{3}\rfloor, k\}).$$

Proof. We first prove a special case, then use it to prove the lemma in general.

Let $E = \langle a,b,c,d|ab = cd\rangle_{\mathcal{V}}$. If $B \subset \{a, b, c, d\}$ has 3 elements, then either $\{a, b\} \subset B$ or $\{c, d\} \subset B$. Suppose without loss of generality that $B = \{a,b,c\}$. Since $abc = (ab)^2c = abcdc = abcd = ab$, B does not freely generate a sub-semilattice of E. Therefore $\phi_{\mathcal{V}}(4,1) \leq 2$.

Case 1. Suppose $\lfloor \frac{m-k}{3} \rfloor \geq k$; that is, $m = 4k + l$ for some $l \geq 0$. Then, by Lemma 1.1 (i),

$$\phi_{\mathcal{V}}(m, k) \leq \phi_{\mathcal{V}}(l, 0) + \sum_{i=1}^{k} \phi_{\mathcal{V}}(4, 1) \leq l + 2k = m - 2k.$$

Case 2. If $\lfloor \frac{m-k}{3} \rfloor \leq k$, then writing $m' = m - (k - \lfloor \frac{m-k}{3} \rfloor)$ and $k' = k - (k - \lfloor \frac{m-k}{3} \rfloor)$,

$$\phi_{\mathcal{V}}(m, k) \leq \phi_{\mathcal{V}}(m', k') \leq m' - 2k' = m - (k + \lfloor \frac{m-k}{3} \rfloor),$$

where the first inequality follows from Lemma 1.1 (ii), and the second inequality follows from Case 1, because $\lfloor \frac{m'-k'}{3} \rfloor = \lfloor \frac{m-k}{3} \rfloor = k'$. □

THEOREM 3.7. *Suppose* $\mathcal{V} \in \mathbf{LB_0}$. *Then for* $m > k$,

$$\phi_{\mathcal{V}}(m, k) = m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}).$$

Proof. Let A be an alphabet of size m and let

$$S = \langle A | u_1 = v_1, u_2 = v_2, \ldots, u_k = v_k \rangle_{\mathcal{V}}.$$

We need to show that there exists a subset A_0 of A of the correct size that generates a (relatively) free sub-band of S as a basis.

Case 1. Suppose that for each i, either

(a) $|C(u_i)| \geq 2$ and $|C(v_i)| \geq 2$, or
(b) $|C(u_i)| = 1$ and $C(u_i) \subseteq C(v_i)$.

By Lemma 2.2, we can obtain a subset A' of A such that $|A'| \leq k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}$, and if $A_0 = A \backslash A'$, then $u_i, v_i \notin F_{A_0}$ for any i. Therefore, as none of the relations can be applied, A_0 generates a (relatively) free sub-band of S as a basis. (Note that the hypothesis that $\mathcal{V} \in \mathbf{LB_0}$ is used here.)

Case 2. Suppose that one of the relations, say $u_k = v_k$, is of the form $a_0 = w$, where $a_0 \in A \backslash C(w)$. Let $u_i'[v_i']$ be the word in $F_{A \backslash \{a_0\}}$ obtained by substituting each occurrence of a_0 in $u_i[v_i]$ by w. Then it is easy to check that

$$S \simeq \langle A | u_1' = v_1', u_2' = v_2', \ldots, u_{k-1}' = v_{k-1}', a_0 = w \rangle_{\mathcal{V}} \simeq S',$$

where we put

$$S' = \langle A \backslash \{a_0\} | u_1' = v_1', u_2' = v_2', \ldots, u_{k-1}' = v_{k-1}' \rangle_{\mathcal{V}},$$

and $A_0 \subseteq A \backslash \{a_0\}$ generates a (relatively) free sub-band of S as a basis if it generates a (relatively) free sub-band of S' as a basis. Since

$$(m - 1) - ((k - 1) + \min\{\lfloor \frac{(m-1)-(k-1)}{3} \rfloor, k - 1\})$$
$$\geq m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}),$$

the problem can be reduced to the first case.

Hence we have shown that

$$\phi_{\mathcal{V}}(m, k) \geq m - (k + \min\{\lfloor \frac{m-k}{3} \rfloor, k\}).$$

Lemma 3.6 and Corollary 3.5 show that the reverse inequality also holds. Hence the proof is complete.   □

Note that the proof of the theorem, combined with the procedure in Section 2, gives a procedure to obtain a subset of A of size $\phi_{\mathcal{V}}(m,k)$ which generates a relatively free sub-band.

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