Can. J. Math., Vol. XXXVIII, No. 1, 1986, pp. 1-18

METRIZABILITY OF FINITE DIMENSIONAL SPACES WITH A BINARY CONVEXITY

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0. Introduction. A *convex structure* consists of a set X, together with a collection $\mathscr{C}$ of subsets of X, which is closed under intersection and under updirected union. The members of $\mathscr{C}$ are called *convex sets*, and $\mathscr{C}$ is a *convexity* on X. For A a subset of X, $h(A)$ denotes the (*convex*) *hull* of A. If A is finite, then $h(A)$ is called a *polytope*. $\mathscr{C}$ is called a *binary* convexity if each finite collection of pairwise intersecting convex sets has a nonempty intersection. See **[8]**, **[21]** for general references.

If X is also equipped with a topology, then the corresponding *weak topology* is the one generated by the convex closed sets. It is usually assumed that at least all polytopes are closed. $\mathscr{C}$ is called *normal* provided that for each two disjoint convex closed sets C, D there exist convex closed sets C', D', with

$$C' \cup D' = X, C \subset C' \setminus D', D \subset D' \setminus C'.$$

The main types of normal binary convexities are: superextensions **[10]**, trees **[21]**, and completely distributive lattices **[27]**.

It is assumed throughout that all singleton sets are convex. Our main result is the following one:

0.1. THEOREM. *Let X be a finite dimensional, connected space equipped with a normal binary convexity with compact polytopes. If X is separable, then X is also metrizable in its weak topology.*

With the same efforts, it can even be shown that the density of X equals the weight of the weak topology. Note that if X is compact, then the weak topology must be the original one. Theorem 0.1 includes as a particular case the equivalence of separability and metrizability in connected distributive lattices of finite dimension, **[19]**, and in compact trees, **[4]**.

By applying techniques from superextension theory, we are able to extend Theorem 0.1 in the following direction:

0.2. THEOREM. *Let Y be a connected space which can be embedded in a finite dimensional compact space with a normal binary convexity. Then weight and density of Y are equal.*

Received January 13, 1983 and in revised form July 12, 1984.

https://doi.org/10.4153/CJM-1986-001-3 Published online by Cambridge University Press

On the other hand, a technique from [**28**] allows us to extend these results in part to nonconnected spaces as follows:

0.3. THEOREM. *Let Y be a compact space which can be embedded in a finite dimensional compact space with a normal binary convexity. Then the weight of Y equals the maximum of: the density of Y, and, the weight of the decomposition space of Y.*

These results are derived in Section 2 below. In Section 1, we introduce a concept of directional degree for convex structures, and we derive some inequalities involving certain other invariants of convex structures as well. These results, in which the directional degree is of crucial importance, are used in the proof of Theorem 0.1, but we also derive some other consequences: the nonexistence of finite dimensional, normal binary continua of "weakly infinite rank" (answering a question of [**24**]), an estimation of the dimension of superextensions, and an equivalence between finiteness of the "rank" and finiteness of the "generating degree". These concepts are explained in Section 1 below.

The topological meaning of this new invariant is expressed by the result that for normal binary convexities on connected spaces, directional degree and dimension of the underlying space are equal. Similar ideas can be found in the work of E. Deák (see for instance [**1**]), where a notion of "directional structure" has been studied for topological spaces.

Certain problems with the directional degree relate to vertex colouring problems in graphs, which have a negative solution. We are indebted to Murray Bell for this observation.

1. Directional degree.

1.1. *Some additional concepts in convexity*. Let $\mathscr{C}$ be a convexity on X. A collection $\mathscr{S}$ of subsets of X *generates* $\mathscr{C}$ if $\mathscr{S} \subset \mathscr{C}$ and if $\mathscr{C}$ is the smallest convexity with this property. $\mathscr{S}$ is then called a *subbase* for $\mathscr{C}$.

Every family $\mathscr{S}$ of subsets of X generates a convexity $\mathscr{C}$. This convexity can be described explicitly as the family of all updirected unions of intersections, formed with the members of $\mathscr{S}$, [**8**]. On the other hand, a subbase $\mathscr{S}$ for a convexity $\mathscr{C}$ can easily be recognized by the following property: every nonempty $\mathscr{C}$-polytope is the intersection of a subfamily of $\mathscr{S}$.

If $\mathscr{S}$ is a collection of sets, such that among every $n + 1$ members of $\mathscr{S}$ there are two comparable ones (under inclusion), then $\mathscr{S}$ is said to be of degree $\leqq n$. By a theorem of Dilworth [**2**, 11], $\mathscr{S}$ can be written as the union of n (or less) totally ordered families. The *generating degree*, gen(X), of a convex structure X is defined as follows: $\text{gen}(X) \leqq n$ (where $n < \infty$) if and only if there is a subbase for X of degree $\leqq n$, [**24**].

If, among every $n + 1$ members of a family $\mathscr{S}$, there are two members which are either comparable, or disjoint, or supplementary (union equals

X), then $\mathscr{S}$ is said to have a directional degree $\leqq n$. The *directional degree*, $\mathrm{dir}(X)$, of a convex structure X is obtained as follows: $\mathrm{dir}(X) \leqq n$ (where $n < \infty$) if and only if there is a subbase for X of directional degree $\leqq n$. This provides a new concept in convexity theory, which is inspired by an informal discussion in [**25**] on "directions" of $\mathbf{R}^n$, relative to certain subconvexities of the ordinary convexity. Additional motivation can be got from certain results below.

One could think of a "direction in $\mathscr{S}$" as a subfamily of $\mathscr{S}$ of which the members are pairwise comparable, disjoint, or supplementary. There is no analogue of Dilworth's theorem for such "directions": see 1.6 below.

We start with an auxilliary result on "dir". Recall [**21**] that a *half-space* of a convex structure X is a convex set with a convex complement. Then [**8**] X is said to have the *separation property* S_3, if every convex set in X is the intersection of a family of half-spaces of X (equivalently, the half-spaces of X form a subbase); the *separation property* S_4, if every two disjoint convex sets extend to complementary half-spaces. A normal convexity is S_4 by [**21**, 2.2].

1.2. LEMMA. *Let X be a convex structure with the separation property S_3. Then the following are true*:

(1) $\mathrm{dir}(X)$ *equals the directional degree of some subbase consisting of half-spaces*;

(2) *if X is also binary, then* $\mathrm{dir}(X)$ *equals the directional degree of any subbase, consisting of half-spaces.*

Proof. Let $\mathscr{S}$ be a subbase for X. By [**24**, 2.1], the closure of $\mathscr{S}$ in the Cantor cube $\{0, 1\}^X = 2^X$ includes a subbase for X, consisting of certain half-spaces and, if X is binary, then this closure includes all half-spaces of X. It is easy to see that the directional degree of $\mathscr{S}$ is the same as the directional degree of its closure in 2^X. This degree does not increase when passing to a subfamily.

A finite set F in a convex structure X is called *interchangeable*, [**23**], provided for each $u \in F$, $h(F\setminus\{u\})$ is included in the union of $h(F\setminus\{x\})$, $x \in F$, $x \neq u$. The *exchange number*, $e(X)$, of X is determined as follows: $e(X) \leqq n$ if and only if each finite set with $n + 1$ or more points is interchangeable, [**18**], [**23**].

1.3. THEOREM. *Let X be a nonempty convex structure. Then*

$$\mathrm{dir}(X) \geqq e(X) - 1,$$

and equality holds if X is binary S_4.

Proof. If X is a one-point set, then $e(X) = 1$ and $\mathrm{dir}(X) = 0$ (since the empty family is a subbase; throughout, the intersection of the empty family in X is considered equal to X). We henceforth assume that X has more than one point, so that $e(X) \geqq 2$, $\mathrm{dir}(X) \geqq 1$.

First, assume $e(X) \geqq n + 1$, where $n \geqq 1$. Then there is a set $\{x_0, \dots, x_n\}$ with $n + 1$ points, which is not interchangeable, say:

$$x \in h\{x_1, \dots, x_n\} \setminus \bigcup_{i=1}^{n} h\{x_0, \dots, \hat{x}_i, \dots, x_n\}$$

(where the symbol "^" indicates that the element x_i is omitted). Let $\mathcal{S}$ be any subbase for X. Since nonempty polytopes equal subbasic intersections (see 1.1), we can for each $i \neq 0$ find an S_i in $\mathcal{S}$ with

$$h\{x_0, \dots, \hat{x}_i, \dots, x_n\} \subset S_i,\ x \notin S_i.$$

Note that also $x_i \notin S_i$, for otherwise

$$x \in h\{x_1, \dots, x_n\} \subset S_i.$$

For $i \neq j$ in $\{1, \dots, n\}$ we find that

$$x \notin S_i \cup S_j;\ x_0 \in S_i \cap S_j;\ x_j \in S_i \setminus S_j.$$

Hence the directional degree of $\mathcal{S}$ is at least n. As $\mathcal{S}$ is an arbitrary subbase for X, we conclude that $\mathrm{dir}(X) \geqq n$.

Let X now be binary S_4, and let $\mathrm{dir}(X) \geqq n \geqq 1$. By Lemma 1.2, there exist n half-spaces $H_1, \dots, H_n$ of X, no two of which are comparable, disjoint, or supplementary. For each $i \leqq n$ the convex sets

$$H_i,\ H'_j = X \setminus H_j \quad (j \neq i)$$

meet two by two, whence by binarity there is a point

$$x_i \in \bigcap_{j \neq i} H'_j \cap H_i.$$

As the sets H'_i $i \in n$, meet two by two, we also obtain a point

$$x_0 \in \bigcap_{i=1}^{n} H'_i.$$

By construction, the convex sets

$$h\{x_1, \dots, x_n\},\ H_i\ (i \leqq n)$$

meet two by two. Hence there is a point

$$x \in h\{x_1, \dots, x_n\} \cap \bigcap_{i=1}^{n} H_i.$$

By construction, we find for $i \neq 0$ that

(1) $\quad h\{x_0, \dots, \hat{x}_i, \dots, x_n\} \subset H'_i; \quad x \in H_i,$

and it follows that $h\{x_1, \dots, x_n\}$ is not covered by the left-hand sets of (1). So $\{x_0, \dots, x_n\}$ is not interchangeable, and $e(X) \geqq n + 1$.

The above result considerably simplifies the evaluation of dir(X) for certain convex structures. Henceforth, the term "dimension" of a binary topological convex structure X should be interpreted as the "convex dimension", [**22**]. By results in [**28**], convex dimension equals the small inductive dimension of X in its weak topology; if X is Lindelöf, then it also equals the Lebesgue covering dimension, and if X is even compact, then it also equals the large inductive dimension. For spaces with connected convex sets, these results were established previously in [**15**]. In the sequel we will no longer specify the term "dimension".

1.4. COROLLARY. *Let X be a normal binary convex structure with compact polytopes, where X is connected. Then* dir(X) *equals the dimension of X.*

Proof. By [**23**, 2.8], $e(X)$ equals dimension plus one. Then apply Theorem 1.3.

As a simple illustration of this, consider a tree X, that is: a connected and locally connected Hausdorff space in which any two points can be separated by a third one. As shown in [**21**, 2.9], the connected subsets of X form a normal binary convexity with compact polytopes. By the very definition of X, convex dimension equals one. This leads to the well-known conclusion [**9**, p. 130]; implicitly [**12**, 4.3], that every two connected subsets with a connected complement in X are either disjoint, or comparable, or supplementary.

Let X be a Hausdorff space, and let $\mathcal{S}$ be a closed (topological) subbase. Then $\mathcal{S}$ is called *normal* T_1, [**10**, 2.2.1], provided the following two conditions hold.

(1) for each $S \in \mathcal{S}$ and for each $x \in X\setminus S$ there is an $S' \in \mathcal{S}$ with $x \in S'$ and with $S \cap S' = \emptyset$;

(2) for each $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \emptyset$ there exist $S'_1, S'_2 \in \mathcal{S}$ with

$$S'_1 \cup S'_2 = X;\ S_1 \subset X\setminus S'_2;\ S_2 \subset X\setminus S'_1.$$

A *linked system* in $\mathcal{S}$ is a collection $\mathcal{L} \subset \mathcal{S}$ of pairwise intersecting sets. Let $\lambda(X, \mathcal{S})$ denote the set of all maximal linked systems in $\mathcal{S}$. We put

$$S^+ = \{\mathcal{L} \in \lambda(X, \mathcal{S}) : S \in \mathcal{L}\} \quad (S \in \mathcal{S});$$

$$\mathcal{S}^+ = \{S^+ : S \in \mathcal{S}\}.$$

The *superextension of X relative to $\mathcal{S}$* is the set $\lambda(X, \mathcal{S})$, equipped with the topology generated by the closed subbase $\mathcal{S}^+$. This construction was introduced by de Groot in [**7**], and it has been studied at length in [**29**], [**10**]. Many results have been obtained by using the convexity of $\lambda(X, \mathcal{S})$ which is generated by $\mathcal{S}^+$, [**11**], [**13**], [**20**]. This convexity is binary, and normal if $\mathcal{S}^+$ is normal T_1, and it has been an important source of inspiration for results in the area.

In our next application, we have also mentioned some standard results for later reference. These results can be found in [**29**].

1.5. Application. *Let X be a topological space, and let $\mathcal{S}$ be a normal T_1 subbase for X. Then $\lambda(X, \mathcal{S})$ is a compact space which is connected if and only if X is. Also*:

(1) *the density of* $\lambda(X, \mathcal{S})$ *is at most the density of* X;

(2) *if X is compact, then* $\lambda(X, \mathcal{S})$ *and X have equal weight*;

(3) dir $\lambda(X, \mathcal{S})$ *is at most the directional degree of* $\mathcal{S}$.

Proof. We are only concerned with (3). It is well-known, [**10**, p. 72], that for each two members S_1, S_2 of $\mathcal{S}$,

$$S_1 \subset S_2 \text{ if and only if } S_1^+ \subset S_2^+;$$

$$S_1 \cap S_2 = \emptyset \text{ if and only if } S_1^+ \cap S_2^+ = \emptyset$$

$$S_1 \cup S_2 = X \text{ if and only if } S_1^+ \cup S_2^+ = \lambda(X, \mathcal{S}).$$

It follows that $\mathcal{S}$ and $\mathcal{S}^+$ have the same directional degree, and (3) follows from the fact that $\mathcal{S}^+$ is a convexity subbase.

We have no example where dir $\lambda(X, \mathcal{S})$ differs from the directional degree of $\mathcal{S}$. The main difficulty in proving equality lies in the fact that $\mathcal{S}^+$ need not consist of half-spaces. It appears to us that $\mathcal{S}^+$ is quite "representative" as a convexity subbase for $\lambda(X, \mathcal{S})$.

1.6. Application. *There is no "Dilworth theorem" for directions.*

Proof. Consider the following subspace X of the 3-cube:

$$X = \bigcup_{i=1}^{4} A_i,$$

where

$$A_1 = \left\{(x, y, z):z = 0, 0 \leqq x \leqq \frac{1}{2}, 0 \leqq y \leqq 1\right\};$$

$$A_2 = \left\{ x, y, z):z = 0, \frac{1}{2} \leqq x \leqq 1, 0 \leqq y \leqq \frac{1}{2}\right\};$$

$$A_3 = \left\{(x, y, z):x = \frac{1}{2}, \frac{1}{2} \leqq y \leqq 1, 0 \leqq z \leqq 1\right\};$$

$$A_4 = \left\{(x, y, z):y = \frac{1}{2}, \frac{1}{2} \leqq x \leqq 1, 0 \leqq z \leqq 1\right\}.$$

This space is pictured in fig. 1. It corresponds to a "median stable" subset of the lattice $[0, 1]^3$, and hence the trace of the cubical convexity on X is normal and binary, [**16**, 3.4].

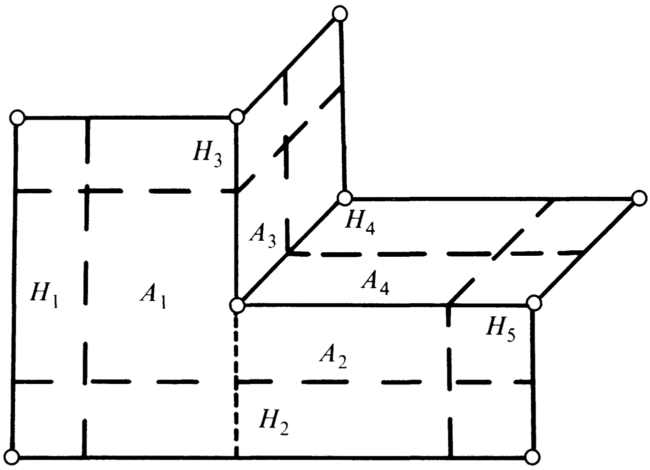

Figure 1

X is 2-dimensional, and hence $\text{dir}(X) = 2$ by 1.4. Let $\mathscr{H}$ be the collection of all traces on X of sets of type

(1) $\quad \pi_i^{-1}[0, t] \quad \text{or} \quad -\pi_i^{-1}[t, 1],$

where $t \in [0, 1]$, $\pi_i:[0, 1]^3 \to [0, 1]$ the i^{th} projection ($i = 1, 2, 3$). The sets (1) are closed half-spaces of the 3-cube, and hence $\mathscr{H}$ consists of closed half-spaces in X. Consequently, among every 3 members of $\mathscr{H}$, some two are comparable, disjoint, or supplementary. Assume

$$\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2,$$

where $\mathscr{H}_1$ and $\mathscr{H}_2$ are "directions". Consider the 5 members of $\mathscr{H}$, indicated in fig. 1. We assume $H_1 \in \mathscr{H}_1$. As H_2 and H_3 are not comparable with H_1, nor disjoint with H_1, nor supplementary with H_1, they must both be in $\mathscr{H}_2$. Then each of H_4, H_5 cannot be in $\mathscr{H}_2$, so they are in $\mathscr{H}_1$. But H_4 and H_5 are not disjoint, nor comparable, nor supplementary, a contradiction.

It is easy to see that $\mathscr{H}$ is the union of three directions. The above example X is taken from unpublished notes of the author, in which it was shown that X cannot be embedded (as a topological convex structure) in a product of two trees. It appears to us that each "direction" of half-spaces includes a description of a tree-factor (see also 1.4), used for a product in which to embed the space. We are led to ask the following.

1.7. *Question*. Let $\mathscr{S}$ be the collection of all half-spaces of a binary S_4 convex structure. If $\mathscr{S}$ has a finite directional degree n, can $\mathscr{S}$ then be covered with $f(n, \mathscr{S}) < \infty$ many directions? Is there a formula $f(n, \mathscr{S})$ which does not depend on the choice of $\mathscr{S}$?

As Murray Bell pointed out to me, the above question has a negative answer if $\mathcal{S}$ is allowed to be an arbitrary collection of sets. To such an $\mathcal{S}$, one can associate a graph (V, E), where the vertex set V consists of the members of $\mathcal{S}$, and where $\{S, T\} \in E$ (the set of edges) if and only if S, T are not comparable, nor disjoint, nor supplementary. Conversely, if (V, E) is a graph with more than two vertices, then one can construct a collection $\mathcal{S}$ of subsets of $V^{\leqq 2}$ (the collection of all subsets of V with one or two points) as follows. For $v \in V$, let S_v consist of all sets of type

$$\{v\}, \text{ or } \{v, w\}, \text{ where } \{v, w\} \in E.$$

The correspondence $v \leftrightarrow S_v$ is bijective, and $\{v, w\} \in E$ if and only if S_v and S_w are not disjoint, nor comparable, nor supplementary.

By [**5**, (4) p. 34], there exist finite graphs (V_n, E_n), $n \in \mathbf{N}$, which do not contain a triangle, and which cannot be painted with n colours. The corresponding collections $\mathcal{S}_n$ have a directional degree 2 by the non-existence of triangles, and as each colour corresponds to a direction, $\mathcal{S}_n$ cannot be covered with n or less directions. The disjoint sum of these graphs cannot even be painted with finitely many colours.

Note that this gives another proof that there is no "Dilworth theorem" for directions, though the argument is not directly involved with binary S_4 convex structures, or even with the half-spaces of an S_3 convex structure.

We turn to the second main result in this section.

1.8. THEOREM. *Let X be an S_3 convex structure, such that X is the hull of $k + 1$ points, where $k \geqq 0$. Then*

$$\text{gen}(X) \leqq 2k \text{ dir}(X).$$

Proof. Let $X = h(F)$, where

$$F = \{x_0, \ldots, x_k\}.$$

We may assume that $\text{dir}(X) = n < \infty$. Put $q = 2kn$, and let $H_1, \ldots, H_{q+1}$ be half-spaces of X with $\emptyset \neq H_j \neq X$ for all $j \leqq q + 1$. For $i = 1, \ldots, k$, we put

$$\mathcal{H}_i = \{H_j : x_0 \in H_j, x_i \notin H_j\};$$
$$\mathcal{H}^i = \{H_j : x_0 \notin H_j, x_i \in H_j\}.$$

If $x_0 \notin H_j$, then it follows from $H_j \neq \emptyset$ and from $X = h(F)$ that $x_i \in H_j$ for some i. If $x_0 \in H_j$, then it follows from $H_j \neq X = h(F)$ that $x_i \notin H_j$ for some i. Hence

$$\{H_1, \ldots, H_{q+1}\} = \bigcup_{i=1}^{k} \mathcal{H}_i \cup \bigcup_{i=1}^{k} \mathcal{H}^i.$$

It follows that one of the two summands

https://doi.org/10.4153/CJM-1986-001-3 Published online by Cambridge University Press

$$\bigcup_{i=1}^{k} \mathscr{H}_i, \quad \bigcup_{i=1}^{k} \mathscr{H}^i$$

has at least $(q/2) + 1 = kn + 1$ members. Hence for some $i \leqq k$, $\mathscr{H}_i$ or $\mathscr{H}^i$ has at least $n + 1$ members. As $\text{dir}(X) = n$, we find that some two members of $\mathscr{H}_i$ (or, of $\mathscr{H}^i$) are either comparable, or disjoint, or supplementary. By the construction of $\mathscr{H}_i$ and of $\mathscr{H}^i$, the second and third possibilities are excluded.

We have shown that among every $2kn + 1$ half-spaces of X there are two comparable ones. X being S_3, its half-spaces form a subbase. Hence $\text{gen}(X) \leqq 2kn$.

1.9. *Some examples*. (1) Let X be a compact tree with $k + 1$ endpoints. Then [**24**, 3.1] $\text{gen}(X) = k + 1$ if $k > 0$. On the other hand, $\text{dir}(X)$ and $\text{gen}(X)$ are 0 if $k = 0$ (then X is a one-point space), and $\text{dir}(X) = 1$ for $k > 0$. Note that gen does not attain its maximal possible value, $2k$, if $k > 1$.

(2) Let $X = \{0, 1\}^n$ be a (graph) n-cube, equipped with the "subcube" convexity. X is a binary S_4 convex structure, and its half-spaces are: $\emptyset$, X, and the $2n$ faces of "dimension" $n - 1$. These faces form a subbase, in which every two members are incomparable, showing that $\text{gen}(X) = 2n$. Clearly, X is the hull of two points, and

$$\text{dir}(X) = e(X) - 1 = n.$$

Hence the inequality in 1.8 is sharp for $k = 1$ and for any value of dir.

(3) Let $\lambda(n)$ denote the superextension of a (discrete) n-point set A. Note that for each $B \subset A$, B^+ is a half-space. Hence by 1.5 (3) and a remark following 1.5, dir $\lambda(n)$ equals the directional degree of the family 2^A. By [**25**, 2.6], there is a family $\mathscr{H}$ of mutually incomparable, non-disjoint, and non-supplementary subsets of A, such that $\mathscr{H}$ has

$$C\left(n - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right)$$

many members. Here, $[x]$ denotes the lower integer approximation to x, and $C(p, q)$ denotes the number of combinations of q points in a p-set. By the theorem of Erdös, Ko and Rado on linked antichains, this is the best possible result. Hence

$$\text{dir}\ \lambda(n) = C\left(n - 1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right).$$

The argument of 1.4 shows that $\mathscr{S} = 2^A$, and $\mathscr{S}^+$, have the same degree. For $\mathscr{S}$, this degree equals the maximal cardinality of an antichain in A. By the Sperner theorem, this cardinality equals

$$C\left(n, \left\lfloor \frac{n}{2} \right\rfloor\right).$$

By [**24**, 2.3], the generating degree of a binary S_4 convex structure equals the degree of any subbase, consisting of half-spaces. We conclude that

$$\text{gen } \lambda(n) = C\left(n, \left\lfloor \frac{n}{2} \right\rfloor\right).$$

Finally, $\lambda(n)$ is the hull of A, and each member of A is a singleton half-space. Thus, A is the best possible set.

Here are some concrete values:

n	gen	$2k$	dir
3	3	4	1
4	6	6	3
5	10	8	4
6	20	10	10
7	35	12	15

Again, the bound for gen, predicted by 1.8 is not attained.

Recall that a (finite or infinite) subset F of a convex structure X is *free* provided for each $x \in F$,

$$x \notin h(F \setminus \{x\}).$$

The rank, $d(X)$, of X is defined as follows: $d(X) \leqq n$ (where $n < \infty$) if and only if no finite set with $n + 1$ or more points is free, [**9**], [**24**].

When $d(X) = \infty$, it is still possible that no infinite subset of X is free. Then X is said to have a *weakly infinite rank*, and to have a *strongly infinite rank* otherwise. A (normal binary) example of the former phenomenon was given in [**24**, 2.4]. This example is an infinite-dimensional continuum, and until now, no such example was found in finite dimensions. The following result answers a question of [**24**]:

1.10. APPLICATION. *Let X be a finite dimensional continuum with a normal binary convexity. Then the rank of X is either finite, or strongly infinite.*

Proof. Suppose X does not have a strongly infinite rank. As X is connected and finite dimensional, it follows from [**24**, 4.6, 4.7] that each compact convex set in X must be a polytope (for finite rank, this is fairly easy, but the proof is much harder in case of a weakly infinite rank). In particular, we find that X is the hull of some $(k + 1)$-set, $k < \infty$. By 1.4, $\text{dir}(X) < \infty$, and hence by Theorem 1.8,

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$$\text{gen}(X) \leqq 2k \text{ dir}(X) < \infty.$$

For a convex structure with more than one point, it is known, [**24**, 2.2], that $d \leqq$ gen, showing that the above X has a finite rank.

The concept of generating degree was introduced in [**24**] to obtain an upper bound for the rank. No other relationship has been obtained yet. Both invariants tend to be equal on spaces with connected convex sets, and a (discrete) example was given in [**24**, 2.2] with $d = 2$ and gen $= 4$.

1.11. Application. *Let X be a binary S_4 convex structure. Then* gen(X) *is finite if and only if $d(X)$ is finite. In fact,*

(1) $\quad \text{gen}(X) \leqq 2 \cdot d(X)(d(X) - 1).$

If every polytope in X is the hull of at most $k + 1$ points, then

(2) $\quad \text{gen}(X) \leqq 2kd(X).$

Proof. We may assume that $d(X) = n < \infty$. Note that if $F \subset X$ is finite and has $n + 1$ or more points, then one of them is in the hull of the other ones. Hence each polytope of X is the hull of at most $k + 1$ points, where $k + 1 \leqq d(X)$. It is therefore sufficient to establish the inequality (2). Let

$$F = \{x_0, \ldots, x_p\}$$

be a set with $p + 1 > n + 2$ points. As $d(X) = n$, there exist two points of F which are in the hull of the remaining $p - 1$ points. Hence for each $i \leqq p$ there is a $j \neq i$ with

$$h\{x_0, \ldots, \hat{x}_j, \ldots, x_p\} = h(F)$$

(notation of 1.3), and it follows that F is interchangeable. Consequently, $e(X) \leqq n + 1$. Also, for each convex set $C \subset X$, $e(C) \leqq e(X)$.

Now assume gen$(X) > 2kn$. Then there exist $2kn + 1$ half-spaces H_i of X, and, for each $i \neq j$, a point $x_{ij} \in H_i \setminus H_j$. In the polytope

$$P = h\{x_{ij} : i \neq j\},$$

the traces $H_i \cap P$ are pairwise incomparable relative half-spaces, so by [**24**, 2.3], gen$(P) > 2kn$. By assumption, P is the hull of some set with at most $k + 1$ points. We conclude from Theorems 1.8, 1.3 that

$$\text{gen}(P) \leqq 2k \text{ dir}(P) = 2k(e(P) - 1) \leqq 2kn,$$

a contradiction.

For $X = \{0, 1\}^n$, we found in 1.9 (2) that gen$(X) = 2n$. It can be seen that $d(X) = n$ as follows: first, the set of all points with exactly one non-zero coördinate has n members, and it is a free set. Hence $d(X) \geqq n$. If $F \subset X$ is free and has $n + 1$ or more points, then fix an $x \in F$ and a half-space H with

$$F\backslash\{x\} \subset H, x \notin H.$$

Then H must be an $(n - 1)$-face of X, and a contradiction can be obtained through an inductive argument.

2. Results on metrizability, density, and weight.

2.1. *Binarity revisited.* Let X be a normal binary convex structure with compact polytopes. If $C \subset X$ is convex closed and nonempty, then [**21**, 2.9] there is a function

$$p = p_C:X \to C \subset X$$

with the following properties:

(1) for each $x \in X$,

$$h\{x, p(x)\} \cap C = \{p(x)\};$$

in particular, $p(x) = x$ for $x \in C$;

(2) p is continuous in the weak topology of X;

(3) p is *convexity preserving* (CP), that is: it inverts convex sets into convex sets.

This function p is called the *nearest-point map* of C. The term "nearest-point" is motivated by order-theoretic considerations, [**21**, 2.7].

2.2. *Proof of Theorem* 0.1. Let X be a normal binary convex structure with compact polytopes, such that X is connected, separable, and finite dimensional in its weak topology. Note that then each convex (closed) set is connected by 2.1 (1), (2).

First step: *each polytope of X is metrizable.* Let $P \subset X$ be a polytope. Then P is connected and its convex dimension does not exceed the one of X, [**22**]. By 1.4, P has a finite directional degree, and by 1.8, we conclude that $\text{gen}(P) < \infty$. By Dilworth's theorem, the collection of all closed half-spaces in P can be written in the form

(1) $\quad \mathscr{H}_1 \cup \ldots \cup \mathscr{H}_q \quad (q < \infty)$,

where each $\mathscr{H}_i$ is totally ordered under inclusion. Let $D \subset X$ be countable dense. If $p:X \to P$ denotes the nearest-point map, then $p(D) \subset P$ is countable and dense by 2.1 (1) and (2). For each $i = 1, \ldots, q$, the family

(2) $\quad \{H \cap p(D):H \in \mathscr{H}_i\}$

is totally ordered and hence countable. Then there is a countable $\mathscr{H}'_i \subset \mathscr{H}_i$ such that the sets of type $H \cap p(D)$, $H \in \mathscr{H}'_i$, build up the whole family (2). We show that the countable family

$$\mathscr{H}' = \bigcup_{i=1}^{q} \mathscr{H}'_i$$

is a closed subbase for the topology of P. To this end, note that by the compactness of P and by the normality of the X-convexity, the convex closed subsets of P already form a closed subbase for the P-topology. Therefore, it suffices to show that each convex closed set $C \subset P$ is the intersection of a subfamily of $\mathcal{H}'$. Take $x \in P \backslash C$. By [**16**, 2.1] there is a CP map (see 2.1)

$$f:P \rightarrow [0, 1] \quad ([0, 1] \text{ with the order-convexity})$$

with $C \subset f^{-1}(0)$, $x \in f^{-1}(1)$. Fix a sequence of points

$$1 > t_1 > t_2 > \ldots > t_{2q+1} > 0,$$

and for each $j = 1, \ldots, 2q + 1$, let

$$H(j) = f^{-1}[0, t_j].$$

Note that $H(j)$ is a closed half-space of P, and that there are proper inclusions

$$H(j + 1) \subset \text{int } H(j), \quad j \leqq 2q.$$

By (1), there must be an $i \in \{1, \ldots, q\}$ such that some three sets of type

$$H(j_1), H(j_2), H(j_3) \quad (j_1 > j_2 > j_3)$$

are in $\mathcal{H}_i$. Let $H \in \mathcal{H}'_i$ be such that

$$H \cap p(D) = H(j_2) \cap p(D).$$

Then $H \not\subset H(j_1)$, for otherwise the points of

$$(\text{int } H(j_2) \backslash H(j_1)) \cap p(D) \neq \emptyset$$

are not in H. Also, $H(j_3) \not\subset H$, for otherwise H includes the points of

$$(\text{int } H(j_3) \backslash H(j_2)) \cap p(D) \neq \emptyset.$$

As H, $H(j_1)$, and $H(j_3)$ are in the totally ordered collection $\mathcal{H}_i$, we find that

$$H(j_1) \subset H \subset H(j_3),$$

and hence that

$$C \subset H, x \notin H,$$

establishing the desired result.

Second step: *X is metrizable in the weak topology*. Enumerate the members of the countable dense set D:

$$D = \{d_n : n \in \mathbf{N}\}.$$

For each $n \in \mathbf{N}$ we have a nearest-point map

$$p_n:X \to h\{d_1, \ldots, d_n\}.$$

This leads us to a map

$$f = (p_n)_{n \in \mathbf{N}}:X \to \prod_{n \in \mathbf{N}} h\{d_1, \ldots, d_n\},$$

which is injective: for $x \neq x'$ in X, there is a closed half-space H of X with

$$x \in \text{int } H, \ x' \in X \setminus H.$$

Take $d_k \in D \cap \text{int } H$ and $d_l \in D \setminus H$. For $n = \max\{k, l\}$, we find that $p_n(x) \neq p_n(x')$. Indeed, the convex sets

$$h\{x, p_n(x)\}, \ h\{d_1, \ldots, d_n\}, \ H$$

meet two by two, and hence they have a point in common, which must be $p_n(x)$ by 2.2 (1). Hence $p_n(x) \in H$. Similarly, $p_n(x') \in X \setminus H$. By [**26**, 2.3 (3) (4)], the injective map f is an embedding of X (with its weak topology) in the product of polytopes. It follows that X is metrizable in its weak topology.

2.3. *Remark.* The above proof applies equally well to establish the following: let X be a normal binary convex structure with compact polytopes. If X is connected, then the density of X equals the weight of the weak topology.

2.4. COROLLARY (compare [**19**, 5.1, 5.2]). *Let X be a completely distributive lattice, equipped with its interval topology. If X is connected and finite dimensional, then X is metrizable if and only if it is separable. In fact, X has weight equal to density.*

Proof. The interval topology makes X into a compact Hausdorff space, [**6**, p. 318], and the collection of all order-convex sublattices of X yields a normal binary convexity on X, [**27**, 4.12].

2.5. *Remark.* In [**27**], we studied certain partial orderings, induced in an S_3 convexity. One consequence of this theory is that all intervals of a normal binary convex structure become completely distributive lattices under suitable orderings. This (nontrivial) fact, together with Corollary 2.4, allows for another proof of Theorem 0.1 as follows. Let X be as in 2.1, with a countable dense subset D. Use of the nearest-point map onto an interval shows the latter to be separable, and hence metrizable by Corollary 2.4. In the second part of the argument in 2.2, one could as well have used intervals $h\{d, d'\}$ with $d, d' \in D$, instead of polytopes. This leads to an embedding of X in a countable product of intervals.

This alternative proof relies on an "alien" completely distributive lattice structure on the intervals of X, which is hiding behind the structure of convexity. Also, when comparing our proof to the one of [**19**], it appears that the original argument in 2.2 is more direct.

We now work towards some improvements of Theorem 0.1. First, note that the connectedness assumption on X cannot just be dropped: the product of continually many copies of $\{0, 1\}$ is a normal binary convex structure of dimension 0, whose underlying space is separable by [**3**, 2.3.15]. Nevertheless, there is a method to weaken the connectedness assumption with the aid of a technique, developed in [**28**]. Let X be a compact space with a normal binary convexity. Let dX denote the decomposition space (space of components) of X. The following embedding theorem has been obtained in [**28**, 3.1]:

(1) dX can be equipped with a unique normal binary convexity making the decomposition map $d:X \to dX$ CP;

(2) if X is n-dimensional, then there is an n-dimensional connected quotient qX of X with a corresponding quotient map $q:X \to qX$, together with a unique normal binary convexity on qX making q CP. Also, the map

$$(q, d):X \to qX \times dX$$

is an isomorphism between X and its image in $qX \times dX$.

As a direct consequence, we obtain the following result from Theorem 0.1 and Remark 2.3:

2.6. COROLLARY. *Let X be a finite dimensional compact space with a normal binary convexity. Then the weight of X is the maximum of: the density of X, and, the weight of its decomposition space dX.*

Superextension theory provides us with another, still stronger improvement of 0.1:

2.7. COROLLARY. *Let X be a finite dimensional compact space with a normal binary convexity.*

(1) *If Y is a connected subspace of X, then the weight of Y equals its density*;

(2) *If Y is a compact subspace of X, then the weight of Y equals the maximum of: the density of Y, and, the weight of the decomposition space of Y.*

Proof. We first assume that Y is compact. Let $\lambda(Y)$ denote the superextension of Y relative to the subbase of all closed sets. Then [**14**, 2.6] the inclusion $Y \to X$ extends uniquely to a CP map

$$\lambda(Y) \to X.$$

Its image Y^* in X is a normal binary substructure of X, [**26**, 2.3 (4)]. By 1.5 (2), Y and $\lambda(Y)$ have the same weight. Since perfect maps do not raise the weight, [**3**, 3.7.19], we find that

(3) weight of Y = weight of Y^*.

Similarly, we obtain from 1.5 (1) that

(4) density of $Y \geqq$ density of Y^*.

We establish (1) as follows. If Y is compact and connected, then $\lambda(Y)$ is compact and connected, and hence Y^* is compact, connected, and finite dimensional. By Theorem 0.1, and by (3), (4), we find that

weight of $Y \leqq$ density of Y,

from which equality follows. If Y is not compact, then

weight of $Y \leqq$ weight of Y^- = density of $Y^- \leqq$ density of Y,

giving the desired result again.

For a proof of (2) we consider the natural maps

$$Y \to dY \to dX;$$

$$Y \to X \to dX.$$

These two compositions lead to the same function $Y \to dX$. By the uniqueness of induced CP maps, we obtain equal composed CP maps

(5) $\lambda(Y) \to \lambda(dY) \to dX$;

(6) $\lambda(Y) \to X \to dX$.

The image of (6) is the decomposition space $d(Y^*)$ of Y^*. On the other hand, dY is a compact subspace of dX, and following the above introduced *-notation, we let $(dY)^*$ denote the image of $\lambda(dY) \to dX$. Now $\lambda(Y) \to \lambda(dY)$ is onto since $Y \to dY$ is, and hence the image of (5) equals $(dY)^*$. Hence:

$$d(Y^*) = (dY)^*.$$

By applying (3) with Y replaced by dY, we find that

(7) weight of dY = weight of $d(Y^*)$.

Y^* being finite dimensional, we conclude from 2.6 that its weight equals the maximum of

(i) the density of Y^*;

(ii) the weight of $d(Y^*)$.

Substituting the (in)equalities of (3), (4), and (7) leads to the desired result on Y.

We have thereby established Theorems 0.2 and 0.3 from the introduction. It is somewhat surprising that non-embeddability of a space Y into a finite dimensional normal binary space can be decided by a simple check on some cardinal invariants of Y and of dY.

2.8. *Application*. It was shown in [**17**, 2.5] that a Hausdorff space X can be embedded in a compact tree if and only if X has a closed subbase $\mathcal{S}$ which has only one direction (in [**17**] the term "crossfree" is used). It follows that if such an X is connected or compact, then its weight equals

$$\max\{\text{density of } X, \text{ weight of } dX\}.$$

2.9. *Question*. Let X be a Hausdorff space and let $\mathcal{S}$ be a closed subbase for X which can be covered with n directions. Can X be embedded in a product of n trees? What if $\mathcal{S}$ is normal T_1?

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