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Relative elegance and Cartesian cubes with one connection

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Abstract. We establish a Quillen equivalence between the Kan–Quillen model structure and a model structure, derived from a cubical model of homotopy type theory (HoTT), on the category of Cartesian cubical sets with one connection. We thereby identify a second model structure which both constructively models HoTT and presents ∞-groupoids, the first example being the equivariant Cartesian model of Awodey–Cavallo–Coquand–Riehl–Sattler.

1 Introduction

Homotopy type theory (HoTT) [Uni13] is said to be a language for reasoning in homotopical settings. The conjecture ("Awodey's proposal") goes that HoTT should have an interpretation in any $(\infty,1)$-category belonging to some class of "elementary $(\infty,1)$-topoi", indeed, that models of HoTT should be in correspondence with such $(\infty,1)$-categories. When one says that HoTT interprets in a given $(\infty,1)$-category, one typically means more precisely that it admits a 1-categorical *presentation* interpreting HoTT in a 1-categorical sense. These presentations have historically come in the form of *Quillen model categories*. As an example, Voevodsky's interpretation of HoTT [KL21] lands in the Kan–Quillen model structure on simplicial sets, which presents the $(\infty,1)$-category ∞-**Gpd** of $(\infty,1)$-groupoids. Shulman [Shu19] has now shown that every Grothendieck $(\infty,1)$-topos can be presented by a model category that interprets HoTT.

The interests of type theorists have thus led to new questions in homotopy theory; one avenue is through the search for *constructive* interpretations of HoTT. The first constructive model to be discovered, due to Bezem et al. [BCH13, BCH19], interprets HoTT in a category of *affine cubical sets*, presheaves over a certain *affine cube category* $\square_{\mathrm{aff}}$ whose objects are symmetric monoidal products of an interval object I. Subsequent constructions [CCHM15, OP18, LOPS18, AFH18, CMS20, ABCHFL21] use different cube categories to obtain better properties. With the exception of the BCH model, all employ presheaves over a cube category with *Cartesian* products, i.e., including degeneracy, diagonal, and permutation maps among its generators. While

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natural from a type-theoretic perspective, the presence of diagonals—and to a lesser degree, permutations—is not typical in the homotopy-theoretic literature on cubical structure.

Initially, none of these cubical models was shown to be compatible with a Quillen model structure; they were models of HoTT (or of *cubical type theories*) in the direct sense that they gave an interpretation of the type-theoretic judgments, though they certainly made use of model-categorical intuitions. The connection with model category theory is first made precise in [GS17, Sat17], where it is shown that structure patterned on Cohen et al.'s cubical set model [CCHM15]—in particular, a functorial cylinder with connections—gives rise to a Quillen model structure. These methods were adapted by Cavallo, Mörtberg, and Swan [CMS20] and Awodey [Awo23] to presheaves over Cartesian cube categories not necessarily supporting connections, producing model structures compatible with the type theories and interpretations of Angiuli et al. [AFH18, ABCHFL21]. Model structures in this lineage have been called *cubical-type model structures*.

It is now natural to ask which $(\infty,1)$-categories these model structures present. In particular, we would like to know if any present ∞-**Gpd**: such a presentation would be a constructive setting for standard homotopy theory equipped with a constructive interpretation of HoTT, and could serve as a base case for constructing further constructive models following Shulman [Shu19]. However, Buchholtz and Sattler determined in 2018 [Coq+18, Sat18] that almost all concrete cubical-type model structures considered up to that point present $(\infty,1)$-categories *in*equivalent to ∞-**Gpd**. The exception is the Sattler model structure $\widehat{\square}^{\mathrm{ty}}_{\wedge\vee}$ on presheaves on the *Dedekind cube category* $\square_{\wedge\vee}$, the cube category with Cartesian structure and both connections, whose status remains an open problem.

1.1 Cubes with one connection

The difficulty in analyzing the Dedekind cube category $\square_{\wedge\vee}$ is that it is not a *(generalized) Reedy category* [BM11], one in which each object is associated with an ordinal degree and any morphism factors as a degeneracy-like degree-lowering map followed by a face-like degree-raising map. Any presheaf over a Reedy category can be built up inductively by attaching cells drawn from a set of generators, namely, quotients of representables by automorphism subgroups. In the subclasses of *elegant* or *Eilenberg–Zilber (EZ)* categories, this cellular decomposition is moreover homotopically well-behaved with respect to any model structure in which the cofibrations are the monomorphisms: it exhibits any presheaf as the *homotopy* colimit of basic cells. The problem in $\square_{\wedge\vee}$ is the combination of connections and diagonals, exemplified the morphism $(x,y,z) \mapsto (x \vee y, y \vee z, x \wedge y)$ from the 3-cube to itself. This map has no (split epi, mono) factorization, a state of affairs forbidden in an elegant Reedy category.[1]

[1] A simpler map without a (split epi, mono) factorization in $\square_{\wedge\vee}$ is $(x,y) \mapsto (x, x \vee y)$, but this is an idempotent and so admits such a factorization in the idempotent completion $\overline{\square}_{\wedge\vee}$ (characterized in [Sat19, Theorem 2.1]). The aforementioned 3-cube endomap does not: it does have an (epi, mono) factorization in $\overline{\square}_{\wedge\vee}$, but the left map does not split. It is the idempotent completion that counts when we consider whether elegant Reedy techniques apply.

Thus, while Sattler [Sat19] and Streicher and Weinberger [SW21] have identified an adjoint triple of Quillen adjunctions relating $\widehat{\square}^{\mathrm{ty}}_{\wedge\vee}$ and $\widehat{\Delta}^{\mathrm{kq}}$, it is not known whether there is a Quillen equivalence. In particular, it is unclear how to prove that a round-trip composite $\widehat{\square}^{\mathrm{ty}}_{\wedge\vee} \to \widehat{\Delta}^{\mathrm{kq}} \to \widehat{\square}^{\mathrm{ty}}_{\wedge\vee}$ is weakly equivalent to the identity in the absence of an elegant Reedy structure on $\square_{\wedge\vee}$.

In this article, we consider an overlooked cube category: the category $\square_\vee$ of cubes with Cartesian structure and a *single* connection. (We arbitrarily choose the "max" or "negative" connection, but this choice plays no role.) Presheaves on this category satisfy conditions sufficient to obtain a cubical-type model structure $\widehat{\square}^{\mathrm{ty}}_{\vee}$ using existing techniques [CMS20, Awo23]. Moreover, the arguments used in [Sat19, SW21] adapt readily from $\square_{\wedge\vee}$ to $\square_\vee$, providing a Quillen adjoint triple relating $\widehat{\square}^{\mathrm{ty}}_{\vee}$ with $\widehat{\Delta}^{\mathrm{kq}}$.

Like the Dedekind cube category, $\square_\vee$ is not Reedy. In this case, the archetypical problematic map is $(x, y, z) \mapsto (x \vee y, y \vee z, z \vee x)$.[2] However, $\square_\vee$ does *embed* nicely in a Reedy category, namely, the category of finite inhabited join-semilattices: we have a functor $i\colon \square_\vee \to \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ sending the n-cube to the n-fold product of the poset $\{0 < 1\}$. While $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ is not itself elegant, it satisfies a relativized form of elegance with respect to the subcategory $\square_\vee$. Whereas elegance would require the Yoneda embedding $\text{ɣ}\colon \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}} \to \mathrm{PSh}(\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}})$ to preserve pushouts of spans of degeneracy maps, here it is the nerve $N_i \coloneqq i^*\text{ɣ}\colon \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}} \to \mathrm{PSh}(\square_\vee)$ that preserves such pushouts. We say that $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ is *elegant relative to i*, or that i is an *elegant embedding*.

We find that the useful properties of elegant Reedy categories can be extended, in an appropriately relativized form, to categories $\mathbf{C}$ with an elegant embedding $i\colon \mathbf{C} \to \mathbf{R}$ in a Reedy category. In particular, we show that any presheaf over $\mathbf{C}$ admits a homotopically well-behaved cellular decomposition whose cells are automorphism quotients of objects in the image of N_i. With these tools in hand, we are able to establish that the Quillen adjunctions relating $\widehat{\square}^{\mathrm{ty}}_{\vee}$ and $\widehat{\Delta}^{\mathrm{kq}}$ are Quillen equivalences. We thus identify a cubical-type model structure presenting ∞-**Gpd**, compatible with a constructive interpretation of either HoTT or of cubical type theory with one connection.

1.2 Outline

We begin in Section 2 with a brief review of model structures, Quillen equivalences, Reedy categories, and the Kan–Quillen model structure on simplicial sets. In Section 3, we present an improvement on the first part of [Sat17]: a series of increasingly specialized criteria under which candidate (cofibration, trivial fibration) and (trivial cofibration, fibration) factorization systems induce a model structure, culminating in a theorem tailored to models of type theory with universes.

In Section 4, we introduce the cube category $\square_\vee$ and its basic properties, construct the cubical-type model structure on $\mathrm{PSh}(\square_\vee)$ using the results of the previous section, and define a *triangulation* adjunction $\mathrm{T}\colon \mathrm{PSh}(\square_\vee) \rightleftarrows \mathrm{PSh}(\Delta) : N_{\boxslash}$. We moreover characterize the cube category's idempotent completion $\overline{\square}_\vee$. The categories of presheaves on $\square_\vee$ and $\overline{\square}_\vee$ are equivalent, but by working with the latter we can more easily compare with the simplex category, following [Sat19, SW21]. In particular,

[2] See Appendix A.1 for a proof that neither $\square_\vee$ nor its idempotent completion is Reedy.

we have an embedding $\blacktriangle\colon \Delta \to \overline{\square}_\vee$, thus an adjoint triple $\blacktriangle_! \dashv \blacktriangle^* \dashv \blacktriangle_*$ relating $\mathrm{PSh}(\Delta)$ and $\mathrm{PSh}(\overline{\square}_\vee)$; the triangulation adjunction corresponds to $\blacktriangle^* \dashv \blacktriangle_*$ along the equivalence $\mathrm{PSh}(\square_\vee) \simeq \mathrm{PSh}(\overline{\square}_\vee)$. In Section 4.4, we show that both $\blacktriangle_! \dashv \blacktriangle^*$ and $\blacktriangle^* \dashv \blacktriangle_*$ are Quillen adjunctions.

We focus on the adjunction $\blacktriangle_! \dashv \blacktriangle^*$. It is easy to see that its derived unit is valued in weak equivalences, as $\blacktriangle$ is fully faithful. To show its derived counit is valued in weak equivalences, we spend Section 5 developing a theory of relative elegance. In Section 6, we show that the functor $i\colon \square_\vee \to \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ is relatively elegant by way of a general analysis of Reedy categories of finite algebras. In Section 7, we use this result to complete the Quillen equivalence between $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$ and $\widehat{\Delta}^{\mathrm{kq}}$. We show first that $\blacktriangle_! \dashv \blacktriangle^*$ is a Quillen equivalence, then deduce that $\blacktriangle^* \dashv \blacktriangle_*$ is one as well, concluding with our main theorem as an immediate corollary:

Theorem 7.8 *The triangulation-nerve adjunction* $\mathrm{T}\colon \widehat{\overline{\square}}^{\mathrm{ty}}_\vee \rightleftarrows \widehat{\Delta}^{\mathrm{kq}} : N_{\not\square}$ *is a Quillen equivalence.*

As a final corollary, we show in Section 7.2 that $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$ coincides with Cisinski's *test model structure* on $\mathrm{PSh}(\square_\vee)$.

In Appendix A, we give proofs of some negative results concerning Reedy structures on Cartesian cube categories with connections. First, we check that neither $\square_\vee$ nor its idempotent completion supports a Reedy structure, justifying our recourse to relative elegance. Second, we prove that $\square_{\wedge\vee}$ does not embed elegantly in *any* Reedy category, showing that our techniques cannot be applied in the two-connection case.

1.3 Related work

1.3.1 Cartesian cubes

This work's closest relative is the *equivariant model structure* $\widehat{\overline{\square}}^{\mathrm{eq}}_\times$ on presheaves over the Cartesian cube category $\square_\times$ constructed by Awodey, Cavallo, Coquand, Riehl, and Sattler (ACCRS) [ACCRS24], which also classically presents ∞-**Gpd**. The ACCRS construction is a modification of earlier models in presheaves on $\square_\times$ [ABCHFL21, CMS20, Awo23]. Briefly, the definition of fibration involves lifting against maps $1 \to \mathbb{I}$ from the point to the interval, the definition of equivariant fibration involves lifting against maps $1 \to \mathbb{I}^n$ for all n and requires lifts stable under permutations of $\mathbb{I}^n$. Like our own model structure, $\widehat{\overline{\square}}^{\mathrm{eq}}_\times$ is compatible with a constructive interpretation of HoTT.

In $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$, equivariance does not appear explicitly but is still implicitly present: when the interval supports a connection operator, ordinary and equivariant lifting become interderivable (see Remark 4.25). Our model structure may thus be seen as an instance of the equivariant model structure construction applied in $\mathrm{PSh}(\square_\vee)$, one which happens to admit a simpler description.

1.3.2 Test category theory

Buchholtz and Morehouse [BM17] catalog a number of categories of cubical sets, specifically investigating cube categories used in models of HoTT, such as $\square_\times$, $\square_{\wedge\vee}$, and the De Morgan cube category. They observe that these categories are all *test categories*, thus that each supports a *test model structure* equivalent to $\widehat{\Delta}^{\mathrm{kq}}$ [Cis06]. To our

knowledge, however, none of these model structures is known to be compatible with a model of HoTT with the exception of the test model structure on $\square_\times$, which coincides with $\widehat{\square}^{\mathrm{eq}}_\times$ [ACCRS24, Theorem 6.3.6]. As a corollary of our Quillen equivalence, we check in Section 7.2 that $\widehat{\square}^{\mathrm{ty}}_\vee$ coincides with the test model structure on $\square_\vee$. Cisinski [Cis14] does show that the test model structure on any elegant strict (that is, non-generalized) Reedy category is compatible with a model of HoTT, but the strictness condition precludes application to any cube category with permutations.

1.3.3 Cubes with one connection

To our knowledge, the category of cubes with Cartesian structure and $\vee$-connections (or $\wedge$-connections) has not been studied before, except in passing by Buchholtz and Morehouse [BM17], though Cartesian cube categories with both $\vee$- and $\wedge$-connections have been used in interpretations of HoTT beginning with Cohen et al. [CCHM15].

On the other hand, subcategories without diagonals have seen use in classical homotopy theory. Indeed, Brown and Higgins use the cube category generated by faces, degeneracies, and $\vee$-connections in their seminal article introducing connections for cubical sets [BH81]. Isaacson [Isa11] studies the cube category with faces, degeneracies, symmetries, and $\wedge$-connections. Unlike $\square_\vee$, these are elegant Reedy categories [[Mal09, Remarque 5.6]]: connections are only problematic in combination with diagonals. They furthermore have useful properties compared to the minimal cube category (generated by faces and degeneracies). For one, they are *strict* test categories [Mal09; BM17, Theorem 3], meaning that the localization functor from the test model structures on these cubical sets to their homotopy categories preserves products.

It should be noted, however, that this particular distinction disappears in the Cartesian cases: any cube category with Cartesian structure is a strict test category, regardless of the presence of connections [BM17, Corollary 2]. For us, the convenient properties of $\square_\vee$ relative to $\square_\times$ are (1) the existence of an embedding from the simplex category into the idempotent completion of $\square_\vee$, which facilitates the comparison between their presheaf categories and (2) the existence of a contracting homotopy of each n-cube invariant under permutations, namely, $(x_1, \dots, x_n, t) \mapsto (x_1 \vee t, \dots, x_n \vee t) : [1]^n \times [1] \to [1]^n$.

1.3.4 Constructive simplicial models

Another line of work aims to reformulate the Kan–Quillen model structure and Voevodsky's simplicial model of HoTT so that these can be obtained constructively. Bezem, Coquand, and Parmann [BC15, BCP15, Par18] show that fibrations as usually defined[3] in $\widehat{\Delta}^{\mathrm{kq}}$ do not provide a model of HoTT constructively; in particular, they are not closed under pushforward along fibrations, which is necessary to interpret Π-types. These obstructions are avoided in the cubical models by working with *uniform fibrations*, which classically coincide with ordinary fibrations but provide

[3] [BC15, Par18] prove obstructions for a definition of fibration where lifting is treated as an operation, while [BCP15] considers fibrations requiring mere existence of a lift.

necessary extra structure in the constructive case. However, there are obstructions to constructing a universe classifying uniform fibrations in simplicial sets [BF22, Appendix D; Swa22, §8.4.1].

Henry [Hen25] discovered that the Kan–Quillen model structure can be constructivized by instead modifying the class of cofibrations, in particular taking a simplicial set to be cofibrant only when degeneracy of its cells is decidable. Alternative constructions of the same model structure were later presented by Gambino et al. [GSS22]. Gambino and Henry [GH22] exhibit a constructive form of Voevodsky's simplical model of HoTT using these ideas. The problem is not entirely settled, however: the *left adjoint splitting* coherence construction [LW15], applied to the classical simplicial model to obtain a strict model of type theory, does not apply constructively in this case [GH22, Remark 8.5]. There has since been progress on coherence theorems that do apply here [Boc22, GL21], but the question is not to our knowledge fully resolved. Separately, van den Berg and Faber [BF22] have identified and developed a theory of *effective fibrations* of simplicial sets, which are both closed under pushforward and support a classifying universe, but have not yet addressed the interpretation of univalence.

1.3.5 Constructivity

Though our interest in cubical-type model structures is motivated by constructive concerns, we work entirely and incautiously within a classical metatheory in this article, our goal being an equivalence with a classically defined model structure. Given that $\widehat{\square}^{\mathrm{ty}}_{\vee}$ is constructively definable, however, it is natural to wonder whether it is *constructively* equivalent with the ACCRS or constructive simplicial model structures. We leave this question for the future, referring to Shulman [Shu23] for further discussion of the constructive homotopy theory of spaces.

We note that the triangulation functor $\mathrm{T}\colon \mathrm{PSh}(\square_{\vee}) \to \mathrm{PSh}(\Delta)$ (Definition 4.35) is definitely *not* a left Quillen adjoint from $\widehat{\square}^{\mathrm{ty}}_{\vee}$ to Henry's simplicial model structure constructively, as it does not preserve cofibrations unless the excluded middle holds. The (triangulation, nerve) adjunction exhibits $\mathrm{PSh}(\Delta)$ as a reflective subcategory of $\mathrm{PSh}(\square_{\vee})$, so every simplicial set is the triangulation of some cubical set. But while every cubical set is cofibrant in $\widehat{\square}^{\mathrm{ty}}_{\vee}$, not every simplicial set is cofibrant in Henry's model structure. For example, given a subsingleton set P, the pushout of the span $\Delta^1 \leftarrow \Delta^1 \times P \to P$ is cofibrant if and only if P is decidable.

1.3.6 Reedy, non-Reedy, and Reedy-like categories

Campion [Cam23] studies the existence and non-existence of elegant Reedy structures on various cube categories, among them $\square_{\vee}$ (under the name $\square_{d,c^{\vee},s}$). A few observations are made independently in that article and our own; in particular, [Cam23, Proposition 8.3] is our Theorem 4.46, while [Cam23, Theorem 8.12(2)] follows from our Proposition A.1.

Shulman's *almost c-Reedy categories* [Shu15, Definition 8.8] generalize beyond generalized Reedy categories. These allow for non-isomorphisms that do not factor through a lower-degree object, so one may wonder if the aforementioned pathological

map $u\colon [1]^3 \to [1]^3$ in $\square_\vee$ (and $\overline{\square}_\vee$) defined by $(x, y, z) \mapsto (x \vee y, y \vee z, z \vee x)$ can be accommodated in this way. However, the class of degree-preserving maps not admitting a lower-degree factorization must be closed under composition [Shu15, Theorem 8.13(ii)]. While u factors through no lower-dimensional object, uu factors through the 1-cube. As such, this generalization is unlikely to be helpful here.

2 Background

2.1 Preliminaries

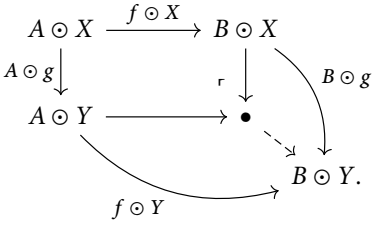
We begin by fixing a few notational conventions.

Notation 2.1 *We write* $[\mathbf{E}, \mathbf{F}]$ *for the category of functors from* $\mathbf{E}$ *and* $\mathbf{F}$*. We write* $\mathrm{PSh}(\mathbf{C}) \coloneqq [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ *for the category of presheaves on a category* $\mathbf{C}$ *and* $\text{よ}\colon \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$ *for the Yoneda embedding.*

Notation 2.2 *When regarding a functor as a diagram, we use superscripts for covariant indexing and subscripts for contravariant indexing. Thus, if* $F\colon \mathbf{D} \to \mathbf{E}$ *then we have* $F^d \in \mathbf{E}$ *for* $d \in \mathbf{D}$*, while if* $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{E}$ *then we have* $F_c \in \mathbf{E}$ *for* $c \in \mathbf{C}$*. We sometimes partially apply a multi-argument functor: given* $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{E}$ *and* $c \in \mathbf{C}$*,* $d \in \mathbf{D}$*, we have* $F_c \in \mathbf{D} \to \mathbf{E}$*,* $F^d \in \mathbf{C}^{\mathrm{op}} \to \mathbf{E}$*, and* $F^d_c \in \mathbf{E}$*.*

By a *bifunctor,* we mean a functor in two arguments. We make repeated use of the *Leibniz construction* [RV14, Definition 4.4], which transforms a bifunctor into an bifunctor on arrow categories.

Definition 2.3 Given a bifunctor $\odot\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ into a category $\mathbf{E}$ with pushouts, the *Leibniz construction* defines a bifunctor $\mathbin{\widehat{\odot}}\colon \mathbf{C}^{\to} \times \mathbf{D}^{\to} \to \mathbf{E}^{\to}$, with $f \mathbin{\widehat{\odot}} g$ defined for $f\colon A \to B$ and $g\colon X \to Y$ as the following induced map:



Example 2.4 If $\mathbf{E}$ is a category with binary products and pushouts, applying the Leibniz construction to the binary product functor $\times\colon \mathbf{E} \times \mathbf{E} \to \mathbf{E}$ produces the *pushout product* bifunctor $\mathbin{\widehat{\times}}\colon \mathbf{E}^{\to} \times \mathbf{E}^{\to} \to \mathbf{E}^{\to}$.

2.2 Model structures and Quillen equivalences

In the abstract, the force of our result is that a certain model category presents the $(\infty, 1)$-category of ∞-groupoids. Concretely, we work entirely in model-categorical terms, exhibiting a Quillen equivalence between this model category and another model category—simplicial sets—already known to present ∞-**Gpd**. We briefly fix the relevant basic definitions here but assume prior familiarity, especially with factorization systems; standard references include [Hov99, DHKS04].

Definition 2.5 A *model structure* on a category $\mathbf{M}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of morphisms in $\mathbf{M}$, called the *cofibrations*, *weak equivalences*, and *fibrations*, respectively, such that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems and $\mathcal{W}$ satisfies the 2-out-of-3 property. A *model category* is a finitely complete and cocomplete category equipped with a model structure. We use the arrow $\rightarrowtail$ for cofibrations, $\xrightarrow{\sim}$ for weak equivalences, and $\twoheadrightarrow$ for fibrations. Maps in $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$ are called *trivial* cofibrations and fibrations, respectively.

We say that a model structure on $\mathbf{M}$ *has monos as cofibrations* when its class of cofibrations is exactly the class of monomorphisms in $\mathbf{M}$.[4]

Definition 2.6 We say an object is *cofibrant* when $0 \to A$ is a cofibration, dually *fibrant* if $A \to 1$ is a fibration. The weak factorization system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ implies that for every object A, we have a diagram $0 \rightarrowtail A^{\text{cof}} \xrightarrow{\sim} A$ obtained by factorizing $0 \to A$; we say such an A^{cof} is a *cofibrant replacement* of A. Likewise, an object A^{fib} sitting in a diagram $A \xrightarrow{\sim} A^{\text{fib}} \twoheadrightarrow 1$ is a *fibrant replacement* of A.

Definition 2.7 We say an object X in a model category is *weakly contractible* when the map $X \to 1$ is a weak equivalence.

Note that given any two of the classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, we can reconstruct the third: $\mathcal{C}$ is the class of maps with left lifting against $\mathcal{F} \cap \mathcal{W}$, $\mathcal{F}$ is the class of maps with right lifting against $\mathcal{C} \cap \mathcal{W}$, and $\mathcal{W}$ is the class of maps that can be factored as a map with left lifting against $\mathcal{F}$ followed by a map with right lifting against $\mathcal{C}$. We will thus frequently introduce a model category by giving a description of two of its classes.

The two factorization systems are commonly generated by *sets* of left maps.

Definition 2.8 We say a weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathbf{E}$ is *cofibrantly generated* by some set $S \subseteq \mathcal{L}$ when $\mathcal{R}$ is the class of maps with the right lifting property against all maps in S. A model structure is cofibrantly generated when its component weak factorization systems are.

Now, we come to relationships between model categories.

Definition 2.9 A *Quillen adjunction* between model categories $\mathbf{M}$ and $\mathbf{N}$ is a pair of adjoint functors $F : \mathbf{M} \rightleftarrows \mathbf{N} : G$ such that F preserves cofibrations and G preserves fibrations.

Note that F preserves cofibrations if and only if G preserves trivial fibrations, while G preserves fibrations if and only if F preserves trivial cofibrations.

Definition 2.10 A Quillen adjunction $F : \mathbf{M} \rightleftarrows \mathbf{N} : G$ is a *Quillen equivalence* when

- for every cofibrant $X \in \mathbf{M}$, the *derived unit* $X \xrightarrow{\eta_X} GFX \xrightarrow{Gm} G((FX)^{\text{fib}})$ is a weak equivalence for some fibrant replacement $m : FX \xrightarrow{\sim} (FX)^{\text{fib}}$;
- for every fibrant $Y \in \mathbf{N}$, the *derived counit* $F((GY)^{\text{cof}}) \xrightarrow{Fp} FGY \xrightarrow{\varepsilon_Y} Y$ is a weak equivalence for some cofibrant replacement $p : (GY)^{\text{cof}} \xrightarrow{\sim} GY$.

Two model structures are *Quillen equivalent* when there is a zigzag of Quillen equivalences connecting them.

[4] Such a model structure which is also cofibrantly generated (see below) is called a *Cisinski model structure*, these being the subject of [Cis06].

2.3 Reedy categories and elegance

The linchpin of our approach is Reedy category theory, the theory of diagrams over categories whose morphisms factor into degeneracy-like and face-like components. As our base category of interest contains non-trivial isomorphisms, we work more specifically with the *generalized Reedy categories* introduced by Berger and Moerdijk [BM11].

Definition 2.11 A *(generalized) Reedy structure* on a category $\mathbf{R}$ consists of an orthogonal factorization system $(\mathbf{R}^-, \mathbf{R}^+)$ on $\mathbf{R}$ together with a *degree map* $|{-}|\colon \mathrm{Ob}\mathbf{R} \to \mathbb{N}$, compatible in the following sense: given $f\colon a \to b$ in $\mathbf{R}^-$ (resp. $\mathbf{R}^+$), we have $|a| \geq |b|$ (resp. $|a| \leq |b|$), with $|a| = |b|$ only if f is invertible.

We refer to maps in $\mathbf{R}^-$ as *lowering maps* and maps in $\mathbf{R}^+$ as *raising maps*, and we use the annotated arrows $\xrightarrow{-}$ and $\xrightarrow{+}$ to denote lowering and raising maps, respectively. The *degree* of a map is the degree of the intermediate object in its Reedy factorization. Note that this definition is self-dual: if $\mathbf{R}$ is a Reedy category, then $\mathbf{R}^{\mathrm{op}}$ is a Reedy category with the same degree function but with lowering and raising maps swapped.

Terminology 2.12 *We henceforth drop the qualifier generalized, as we are almost always working with generalized Reedy categories. Instead, we say a Reedy category is strict if any parallel isomorphisms are equal and it is skeletal, i.e., it is a Reedy category in the original sense.*

The prototypical strict Reedy category is the simplex category Δ: the degree of an n-simplex is n, while the lowering and raising maps are the degeneracy and face maps, respectively [GZ67, Section II.3.2].

A Reedy structure on a category $\mathbf{R}$ is essentially a tool for working with $\mathbf{R}$-shaped diagrams. For example, a weak factorization system on any category $\mathbf{E}$ induces *injective* and *projective Reedy weak factorization systems* on the category $[\mathbf{R}, \mathbf{E}]$ of $\mathbf{R}$-shaped diagrams in $\mathbf{E}$; likewise for model structures. Importantly for us, any diagram of shape $\mathbf{R}$ can be regarded as built iteratively from "partial" diagrams specifying the elements at indices up to a given degree. We are specifically interested in presheaves, i.e., $\mathbf{R}^{\mathrm{op}}$-shaped diagrams in **Set**. We refer to [DHKS04, Section 22; BM11; RV14; Shu15] for overviews of Reedy categories and their applications.

Berger and Moerdijk's definition of generalized Reedy category [BM11, Definition 1.1] includes one additional axiom. Following Riehl [Rie17], we treat this as a property to be assumed only where necessary.

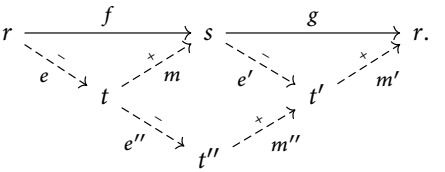
Definition 2.13 In a Reedy category $\mathbf{R}$, we say *isos act freely on lowering maps* when for any $e\colon r \xrightarrow{-} s$ and isomorphism $\theta\colon s \cong s$, if $\theta e = e$ then $\theta = \mathrm{id}$.

Note that any Reedy category in which all lowering maps are epic satisfies this property. The main results of this article are restricted to *pre-elegant* Reedy categories (Definition 5.28) for which this is always the case (Lemma 5.29); nevertheless, we try to record where only the weaker assumption is needed.

The following cancellation property will come in handy.

Lemma 2.14 *Let $f\colon r \to s$, $g\colon s \to t$ be maps in a Reedy category. If gf is a lowering map, then so is g. Dually, if gf is a raising map, then so is f.*

Proof We prove the first statement; the second follows by duality. Suppose gf is a lowering map. We take Reedy factorizations $f = me$, $g = m'e'$, and then $e'm = m''e''$:



This gives us a Reedy factorization $gf = (m'm'')(e''e)$. By uniqueness of factorizations, $m'm''$ must be an isomorphism; this implies $|t''| = |t'| = |r|$, so m' and m'' are also isomorphisms. Thus, $g \cong e'$ is a lowering map. ∎

Corollary 2.15 *Any split epimorphism in a Reedy category is a lowering map; dually, any split monomorphism is a raising map.*

When studying **Set**-valued presheaves over a Reedy category, it is useful to consider the narrower class of *elegant Reedy categories* [BM11, BR13].

Definition 2.16 A Reedy structure on a category $\mathbf{R}$ is *elegant* when

(a) any span $s \xleftarrow{e} r \xrightarrow{e'} s'$ consisting of lowering maps e, e' has a pushout;
(b) the Yoneda embedding $\yoneda : \mathbf{R} \to \mathrm{PSh}(\mathbf{R})$ preserves these pushouts.

We refer to spans consisting of lowering maps as *lowering spans*, likewise pushouts of such spans as *lowering pushouts*. Note that all the maps in a lowering pushout square are lowering maps, as the left class of any factorization system is closed under cobase change.

Intuitively, an elegant Reedy category is one where any pair of "degeneracies" $s \xleftarrow{-} r \xrightarrow{-} s'$ has a universal "combination" $r \xrightarrow{-} s \sqcup_r s'$, namely, the diagonal of their pushout. The condition on the Yoneda embedding asks that any r-cell in a presheaf is degenerate along (that is, factors through) both $r \xrightarrow{-} s$ and $r \xrightarrow{-} s'$ if and only if it is degenerate along their combination. Again, the simplex category is the prototypical elegant Reedy category [GZ67, Section II.3.2].

Remark 2.17 This definition is one of a few equivalent formulations introduced by Bergner and Rezk [BR13, Definition 3.5, Proposition 3.8] for strict Reedy categories. For generalized Reedy categories, Berger and Moerdijk [BM11, Definition 6.7] define *EZ categories*, which additionally require that $\mathbf{R}^+$ and $\mathbf{R}^-$ are exactly the monomorphisms and split epimorphisms, respectively. We make do without this restriction. It is always the case that the lowering maps in an elegant Reedy category are the split epis (see Remark 5.39 below), but the raising maps need not be monic. For example [Cam23, Example 4.3], any direct category (that is, any Reedy category with $\mathbf{R}^+ = \mathbf{R}^{\to}$) is elegant, but a direct category can contain non-monic arrows.

A presheaf $X \in \mathrm{PSh}(\mathbf{R})$ over any Reedy category can be written as the sequential colimit of a sequence of *n-skeleta* containing non-degenerate cells of X only up to degree n, with the maps between successive skeleta obtained as cobase changes of certain basic *cell maps*. When $\mathbf{R}$ is elegant, these cell maps are moreover monic. This

property gives rise to a kind of induction principle: any property closed under certain colimits can be verified for all presheaves on an elegant Reedy category by checking that it holds on basic cells. This principle is conveniently encapsulated by the following definition.

Definition 2.18 [Cis19, Definition 1.3.9] Let a category $\mathbf{E}$ be given. We say a replete class of objects $\mathcal{P} \subseteq \mathbf{E}$ is *saturated by monomorphisms* when

(a) $\mathcal{P}$ is closed under small coproducts.
(b) For every pushout square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle m}\downarrow & \ulcorner & \downarrow{\scriptstyle m'} \\ Y & \longrightarrow & Y' \end{array}$$

such that $X, X', Y \in \mathcal{P}$, we have $Y' \in \mathcal{P}$.
(c) For every diagram $X\colon \omega \to \mathbf{E}$ such that each object X^i is in $\mathcal{P}$ and each morphism $X^i \to X^{i+1}$ is monic, we have $\operatorname{colim}_{i<\omega} X^i \in \mathcal{P}$.

We note that when $\mathbf{E}$ is a model category with monos as cofibrations, these are all diagrams whose colimits agree with their *homotopy colimits*: we can compute their colimits in the $(\infty, 1)$-category presented by $\mathbf{E}$ by simply computing their 1-categorical colimits in $\mathbf{E}$, which is hardly the case in general. This fact is another application of Reedy category theory; see, for example, Dugger [Dug08, Section 14]. As a result, these colimits have homotopical properties analogous to 1-categorical properties of colimits. For example, recall that given a natural transformation $\alpha\colon F \to G$ between left adjoint functors $F, G\colon \mathbf{E} \to \mathbf{F}$, the class of $X \in \mathbf{E}$ such that α_X is an isomorphism is closed under colimits. If F, G are left *Quillen* adjoints and $\mathbf{E}, \mathbf{F}$ have monomorphisms as cofibrations, then the class of X such that α_X is a *weak equivalence* is saturated by monomorphisms. This particular fact will be key in Section 7.1.

For presheaves over an elegant Reedy category, the basic cells are the quotients of representables by automorphism subgroups.

Definition 2.19 Given an object X of a category $\mathbf{E}$ and a subgroup $H \leq \operatorname{Aut}_{\mathbf{E}}(X)$, their *quotient* is the colimit $X/H \coloneqq \operatorname{colim}(H \to \operatorname{Aut}_{\mathbf{E}}(X) \to \mathbf{E})$.

Proposition 2.20 *Let $\mathbf{R}$ be an elegant Reedy category. Let $\mathcal{P} \subseteq \operatorname{PSh}(\mathbf{R})$ be a class of objects such that*

- *for any $r \in \mathbf{R}$ and $H \leq \operatorname{Aut}_{\mathbf{R}}(r)$, we have $\yo r/H \in \mathcal{P}$;*
- *$\mathcal{P}$ is saturated by monomorphisms.*

Then, $\mathcal{P}$ contains all objects of $\operatorname{PSh}(\mathbf{R})$.

Proof [Cis19, Corollary 1.3.10] gives a proof for strict elegant Reedy categories; the proof for the generalized case is similar (and a special case of our Theorem 5.47). ∎

As described above, we will be studying a category $\square_\vee$ that is *not* a Reedy category. Thus, we will not use the previous proposition directly. Instead, our Section 5 establishes a generalization to categories that only *embed* in a Reedy category in a nice way.

2.4 Simplicial sets

To show that a given model category presents ∞-**Gpd**, it suffices to exhibit a Quillen equivalence to a model category already known to present ∞-**Gpd**. Here, our standard of comparison will be the classical Kan–Quillen model structure on simplicial sets [Qui67, Section II.3].

Definition 2.21 The *simplex category* Δ is the full subcategory of the category **Pos** of posets and monotone maps consisting of the finite inhabited linear orders $[n] \coloneqq \{0 < \cdots < n\}$ for $n \in \mathbb{N}$.

This is a strict Reedy category, in fact, an EZ category (see Remark 2.17). The raising and lowering maps are given by the *face* and *degeneracy maps*, defined as the injective and surjective maps of posets, respectively.

Definition 2.22 We define the usual generating maps of the simplex category:

- given $n \geq 0$ and $i \in [n]$, the *generating degeneracy map* $s_i : [n+1] \to [n]$ identifies the elements i and $i+1$ of $[n+1]$,
- given $n \geq 1$ and $i \in [n]$, the *generating face map* $d_i : [n-1] \to [n]$ skips over the element i of $[n]$.

Definition 2.23 Write $\Delta^n \in \mathrm{PSh}(\Delta)$ for the representable n-simplex $\yo[n]$. We define the following sets of maps in simplicial sets:

- For $n \geq 0$, the *boundary inclusion* $\partial\Delta^n \rightarrowtail \Delta^n$ is the union of the subobjects $\Delta^i \rightarrowtail \Delta^n$ given by a non-invertible face map $[i] \to [n]$.
- For $n \geq 1$ and $0 \leq k \leq n$, the *k-horn* $\Lambda^n_k \rightarrowtail \Delta^n$ is the union of the subobjects $\Delta^i \rightarrowtail \Delta^n$ given by a face map $d : [i] \to [n]$ whose pullback along $[n] - k \rightarrowtail [n]$ is non-invertible.

Proposition 2.24 (Kan–Quillen model structure) *There is a model structure on* $\mathrm{PSh}(\Delta)$ *with the following weak factorization systems:*

- *The weak factorization system (cofibration, trivial fibration) is cofibrantly generated by the boundary inclusions.*
- *The weak factorization system (trivial cofibration, fibration) is cofibrantly generated by the horn inclusions.*

We write $\widehat{\Delta}^{\mathrm{kq}}$ *for this model category.*

Proof This is Theorem 3 and the following Proposition 2 in [Qui67, Section II.3]. ■

Proposition 2.25 [GZ67, Section IV.2] *The weak factorization systems of* $\widehat{\Delta}^{\mathrm{kq}}$ *admit the following alternative descriptions:*

- *The cofibrations are the monomorphisms; the trivial fibrations are the maps right lifting against monomorphisms.*
- *The weak factorization system (trivial cofibration, fibration) is generated by pushout products* $d_k \mathbin{\widehat{\times}} m$ *of an endpoint inclusion* $d_k : 1 \to \Delta^1$ *with a monomorphism* $m : A \rightarrowtail B$.

3 Model structures from cubical models of type theory

As the cube category $\square_\vee$ is Cartesian, we may obtain our cubical-type model structure on $\mathrm{PSh}\square_\vee$ immediately by applying existing arguments [CMS20, Awo23], which build

on a criterion for recognizing model structures introduced in the first part of [Sat17]. We will instead take the opportunity to present an improvement on the latter criterion, hoping to give an idea of the character of these model structures along the way.

We begin in Section 3.1 with a set of conditions necessary and sufficient to determine when a *premodel structure*—essentially, all the ingredients of a model structure except 2-out-of-3 for weak equivalences—is in fact a model structure. In Section 3.2, we give a simplified set of conditions for the case where the premodel structure is equipped with a compatible adjoint functorial cylinder. Finally, in Section 3.3, we show that such a cylindrical premodel structure satisfies these conditions when all its objects are cofibrant and it satisfies the *fibration extension property*. We shall apply this result in Section 4.2 to obtain our model structure on $\mathrm{PSh}(\square_\vee)$; a reader who would prefer to take the existence of the model structure for granted may skip this section and read only Theorem 4.34 in Section 4.2.

3.1 Model structures from premodel structures

Definition 3.1 [Bar19, Definition 2.1.23] A *premodel structure* on a finitely complete and cocomplete category $\mathbf{M}$ consists of weak factorization systems $(\mathcal{C}, \mathcal{F}_t)$ (the *cofibrations* and *trivial fibrations*) and $(\mathcal{C}_t, \mathcal{F})$ (the *trivial cofibrations* and *fibrations*) on $\mathbf{M}$ such that $\mathcal{C}_t \subseteq \mathcal{C}$ (or equivalently $\mathcal{F}_t \subseteq \mathcal{F}$).

Remark 3.2 (Stability under (co)slicing) Given an object $X \in \mathbf{M}$, any weak factorization system on $\mathbf{M}$ descends to weak factorization systems on the slice over X and the coslice under X, with left and right classes created by the respective forgetful functor to $\mathbf{M}$. In the same fashion, any premodel structure on $\mathbf{M}$ descends to slices and coslices of $\mathbf{M}$.

As any two of the classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ defining a model structure determines the third, any premodel structure induces a candidate class of weak equivalences.

Definition 3.3 We say that a morphism in a premodel structure is a *weak equivalence* if it factors as a trivial cofibration followed by a trivial fibration; we write $\mathcal{W}(\mathcal{C}, \mathcal{F})$ for the class of such morphisms.

Remark 3.4 The above definition is only necessarily appropriate when examining when a premodel structure forms a model structure: there are premodel structures with a useful definition of weak equivalence not agreeing with $\mathcal{W}(\mathcal{C}, \mathcal{F})$. For example, there are various *weak model structures* on semisimplicial sets in which not all trivial fibrations are weak equivalences [Hen20, Remark 5.5.7].

For the remainder of this section, we fix a premodel category $\mathbf{M}$ with factorization systems $(\mathcal{C}, \mathcal{F}_t)$ and $(\mathcal{C}_t, \mathcal{F})$. The following two propositions are standard.

Proposition 3.5 *$\mathcal{C}_t = \mathcal{C} \cap \mathcal{W}(\mathcal{C}, \mathcal{F})$ and $\mathcal{F}_t = \mathcal{F} \cap \mathcal{W}(\mathcal{C}, \mathcal{F})$.*

Proof An immediate consequence of the retract argument [Hov99, Lemma 1.1.9]. ∎

In light of the above, we use the arrows $\rightarrowtail$ and $\twoheadrightarrow$ to denote trivial cofibrations and fibrations also in a premodel structure.

Corollary 3.6 *$(\mathcal{C}, \mathcal{W}(\mathcal{C},\mathcal{F}), \mathcal{F})$ forms a model structure if and only if $\mathcal{W}(\mathcal{C},\mathcal{F})$ satisfies 2-out-of-3.*

We now reduce the problem of checking 2-out-of-3 for $\mathcal{W}(\mathcal{C},\mathcal{F})$ to a reduced collection of special cases of 2-out-of-3 where some or all maps belong to $\mathcal{C}$ or $\mathcal{F}$.

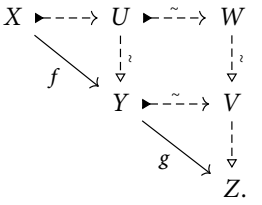
Definition 3.7 Given a wide subcategory $\mathcal{A} \subseteq \mathbf{E}$ of a category $\mathbf{E}$, we say $\mathcal{A}$ *has left cancellation in* $\mathbf{E}$ (or *among maps in* $\mathbf{E}$) when for every composable pair g, f in $\mathbf{E}$, if gf and g are in $\mathcal{A}$ then f is in $\mathcal{A}$. Dually, $\mathcal{A}$ has *right cancellation in* $\mathbf{E}$ when for all g, f with $gf, f \in \mathcal{A}$, we have $g \in \mathcal{A}$.

Theorem 3.8 *$\mathcal{W}(\mathcal{C},\mathcal{F})$ satisfies 2-out-of-3 exactly if the following hold:*

(A) *Trivial cofibrations have left cancellation among cofibrations and trivial fibrations have right cancellation among fibrations.*
(B) *Any (cofibration, trivial fibration) factorization or (trivial cofibration, fibration) factorization of a weak equivalence is a (trivial cofibration, trivial fibration) factorization;*
(C) *Any composite of a trivial fibration followed by a trivial cofibration is a weak equivalence.*

Note that each of these conditions is self-dual.

Proof Conditions A–C all follow by straightforward applications of 2-out-of-3 for $\mathcal{W}(\mathcal{C},\mathcal{F})$. Suppose conversely that we have A–C and let maps $g\colon Y \to Z$ and $f\colon X \to Y$ be given. Then, using the two factorization systems and condition C, we have the following diagram:



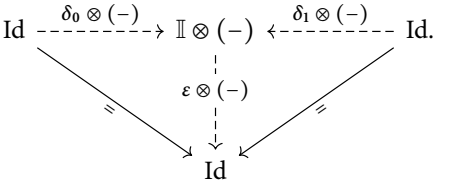
Suppose first that g and f are weak equivalences. Then, we may choose the factorizations of f and g such that the map $X \rightarrowtail U$ is a trivial cofibration and the map $V \twoheadrightarrow Z$ is a trivial fibration. Thus, gf factors as a trivial cofibration followed by a trivial fibration, i.e., is a weak equivalence.

Now suppose that f and gf are weak equivalences. We may choose the factorization of f such that the map $X \rightarrowtail U$ is a trivial cofibration. The composite $X \rightarrowtail W$ is then a trivial cofibration, so the composite $W \twoheadrightarrow Z$ is a trivial fibration by condition B. Then, the map $V \twoheadrightarrow Z$ is a trivial fibration by condition A. Hence, g is a weak equivalence. By the dual argument, if g and gf are weak equivalences then so is f. ∎

3.2 Cylindrical premodel structures

Now, we derive a simpler set of criteria for premodel structures equipped with a compatible *adjoint functorial cylinder*.

Definition 3.9 A *functorial cylinder* on a category $\mathbf{E}$ is a functor $\mathbb{I} \otimes (-) : \mathbf{E} \to \mathbf{E}$ equipped with *endpoint* and *contraction* transformations fitting in a diagram as shown:



An *adjoint functorial cylinder* is a cylinder such that $\mathbb{I} \otimes (-)$ is a left adjoint.

Notation 3.10 *Given a functorial cylinder in a finitely cocomplete category, we have induced boundary maps* $\partial \otimes X \coloneqq [\delta_0 \otimes X, \delta_1 \otimes X] : X \sqcup X \to \mathbb{I} \otimes X$.

There is a dual notion of *functorial path object* consisting of a functor $\mathbb{I} \oslash (-)$ and natural transformations $\delta_k \oslash (-) : \mathbb{I} \otimes (-) \to \mathrm{Id}$ and $\varepsilon \oslash (-) : \mathrm{Id} \to \mathbb{I} \otimes (-)$. By transposition, each adjoint functorial cylinder corresponds to an adjoint functorial path object.

Remark 3.11 (Stability under (co)slicing) Fix a functorial cylinder denoted as above and an object $X \in \mathbf{E}$. Then, $\mathbb{I} \otimes (-)$ lifts through the forgetful functor $\mathbf{E}/X \to \mathbf{E}$ to a functorial cylinder $\mathbb{I} \otimes_{\mathbf{E}/X} (-)$ on the slice over X. This crucially uses the contraction. For example, the action of $\mathbb{I} \otimes_{\mathbf{E}/X} (-)$ on $f : Y \to X$ is given by $(\varepsilon \otimes X)(\mathbb{I} \otimes f) : \mathbb{I} \otimes Y \to X$. Furthermore, $\mathbb{I} \otimes (-)$ lifts through the pushout functor $\mathbf{E} \to X/\mathbf{E}$ to a functorial cylinder $\mathbb{I} \otimes_{X/\mathbf{E}} (-)$ on the coslice under X. For example, the action of $\mathbb{I} \otimes_{X/\mathbf{E}} (-)$ on $f : X \to Y$ is given by the pushout of $\mathbb{I} \otimes f : \mathbb{I} \otimes X \to \mathbb{I} \otimes Y$ along $\varepsilon \otimes X$. In both cases, adjointness is preserved, and the corresponding functorial path object is given by performing the dual construction.

Definition 3.12 Write $@ : [\mathbf{E}, \mathbf{F}] \times \mathbf{E} \to \mathbf{F}$ for the application bifunctor defined by $F @ X \coloneqq F(X)$. Given a category $\mathbf{E}$ with a functorial cylinder and $f \in \mathbf{E}^{\to}$, we abbreviate $(\delta_k \otimes (-)) \mathbin{\widehat{@}} f \in \mathbf{E}^{\to}$ as $\delta_k \mathbin{\widehat{\otimes}} f$. We likewise write $\varepsilon \mathbin{\widehat{\otimes}} f$ for Leibniz application of the contraction. We write $\delta_k \mathbin{\widehat{\oslash}} (-)$ and $\varepsilon \mathbin{\widehat{\oslash}} (-)$ for the dual operations associated with a functorial path object.

Definition 3.13 Given a finitely cocomplete category $\mathbf{E}$ with a functorial cylinder, a weak factorization system $(\mathcal{L}, \mathcal{R})$ is *cylindrical* when $\partial \mathbin{\widehat{\otimes}} (-)$ preserves left maps.

Definition 3.14 Given $f : A \to B$ in a finitely cocomplete category with a functorial cylinder and $k \in \{0, 1\}$, we write $\mathrm{M}_k(f)$ for its *k-sided mapping cylinder*, defined as the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \delta_k \otimes A}\downarrow & \ulcorner & \downarrow{\scriptstyle \iota_1} \\ \mathbb{I} \otimes A & \xrightarrow[\iota_0]{} & \mathrm{M}_k(f). \end{array}$$

The *k-sided mapping cylinder factorization* of f is the factorization

$$A \xrightarrow{\iota_0(\delta_{1-k} \otimes A)} \mathrm{M}_k(f) \xrightarrow{[f(\varepsilon \otimes A), \mathrm{id}]} B.$$

Definition 3.15 A *cylindrical premodel structure* on a finitely complete and cocomplete category **E** consists of a premodel structure and adjoint functorial cylinder on **E** such that

- the (cofibration, trivial fibration) and (trivial cofibration, fibration) weak factorization systems are cylindrical;
- $\delta_k \mathbin{\widehat{\otimes}} (-)$ sends cofibrations to trivial cofibrations for $k \in \{0,1\}$.

Remark 3.16 The above conditions are transposed to equivalent dual conditions on the corresponding adjoint functorial path object. Like its constituent components, the notion of cylindrical premodel structure is thus self-dual: a cylindrical premodel structure on **E** is the same as a cylindrical premodel structure on $\mathbf{E}^{\mathrm{op}}$.

Remark 3.17 (Stability under (co)slicing) Continuing Remarks 3.2 and 3.11, a cylindrical premodel structure on **E** descends to cylindrical premodel structures on slices and coslices of **E**. We may exploit this to simplify arguments by, for example, working in a slice.

Fix once more a premodel category **M** with factorization systems $(\mathcal{C}, \mathcal{F}_t)$ and $(\mathcal{C}_t, \mathcal{F})$. We show that condition C is reducible to condition A when **M** is cylindrical by relating trivial fibrations with dual strong deformation retracts.

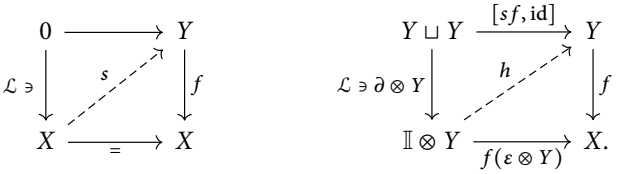
Definition 3.18 In a category with a functorial cylinder, we say $f\colon Y \to X$ is a *dual strong k-oriented deformation retract* for some $k \in \{0,1\}$ when we have a map $s\colon X \to Y$ such that $fs = \mathrm{id}$ and a homotopy $h\colon \mathbb{I} \otimes Y \to Y$ such that $h(\delta_k \otimes Y) = sf$, $h(\delta_{1-k} \otimes Y) = \mathrm{id}$, and fh is a constant homotopy. Equivalently (if the category is finitely cocomplete), f is a dual strong k-oriented deformation retract when we have a diagonal filler

$$\begin{array}{ccc} Y & \xrightarrow{=} & Y \\ {\scriptstyle \iota_0(\delta_{1-k} \otimes Y)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{M}_k(f) & \xrightarrow[[f(\varepsilon \otimes Y), \mathrm{id}]]{} & X. \end{array}$$

The following is a standard construction (see, e.g., [Qui67, Lemma I.5.1]).

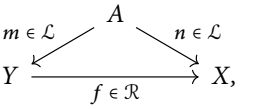
Lemma 3.19 *Let $(\mathcal{L}, \mathcal{R})$ be a cylindrical weak factorization system on a finitely cocomplete category with a functorial cylinder. Then, any $\mathcal{R}$-map between $\mathcal{L}$-objects is a dual strong k-oriented deformation retract for any $k \in \{0,1\}$.*

Proof Let $f\colon Y \to X$ be an $\mathcal{R}$-map between $\mathcal{L}$-objects. We solve two lifting problems in turn:



The maps s and h exhibit f as a dual strong 0-oriented deformation retract; we may similarly construct a 1-oriented equivalent. ■

Corollary 3.20 *Let $(\mathcal{L}, \mathcal{R})$ be a cylindrical weak factorization system on a category with a functorial cylinder. Then, in any diagram of the form*

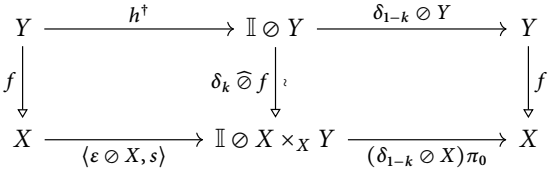


the horizontal map is a dual strong k-oriented deformation retract for any $k \in \{0,1\}$.

Proof By Lemma 3.19, applied in the coslice under A via Remark 3.17. ■

Lemma 3.21 *If* **M** *is cylindrical, then any fibration $f\colon Y \twoheadrightarrow X$ that is a dual strong k-oriented deformation retract for some $k \in \{0,1\}$ is a trivial fibration.*

Proof Let $s\colon X \to Y$ and $h\colon \mathbb{I} \otimes Y \to Y$ be as in the definition of dual strong k-oriented deformation retract. Then, the diagram

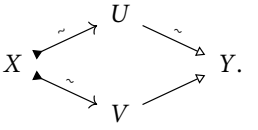


exhibits f as a retract of a trivial fibration. ■

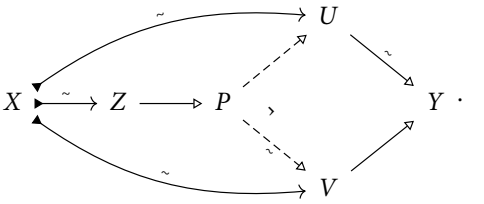
Lemma 3.22 *Suppose* **M** *is cylindrical. If trivial fibrations have right cancellation among fibrations, then any (trivial cofibration, fibration) factorization of a weak equivalence is a (trivial cofibration, trivial fibration) factorization.*

Dually, if trivial cofibrations have left cancellation among cofibrations, then any (cofibration, trivial fibration) factorization of a weak equivalence is a (trivial cofibration, trivial fibration) factorization.

Proof Suppose we have a weak equivalence $X \to Y$ factoring as a trivial cofibration followed by a fibration, thus a diagram of the following form:



We first take a pullback and factorize the induced gap map as a trivial cofibration followed by a fibration.



By Corollary 3.20, the composites $Z \rightarrowtriangle U$ and $Z \rightarrowtriangle V$ are dual strong deformation retracts, thus trivial fibrations by Lemma 3.21. Then, the composite $Z \rightarrowtriangle Y$ is a trivial fibration by composition, so $V \rightarrowtriangle Y$ is a trivial fibration by right cancellation. ∎

Theorem 3.23 *Suppose* **M** *is a cylindrical premodel structure. Then,* $\mathcal{W}(\mathcal{C}, \mathcal{F})$ *satisfies 2-out-of-3 exactly if the following hold:*

(A) *trivial cofibrations have left cancellation among cofibrations and trivial fibrations have right cancellation among fibrations;*
(C) *any composite of a trivial fibration followed by a trivial cofibration is a weak equivalence.*

Proof Theorem 3.8 combined with Lemma 3.22. ∎

Finally, we prove for reference below that the cancellation properties opposite of condition A always hold in a cylindrical premodel structure, though we will not need this fact.

Lemma 3.24 *Let* $(\mathcal{L}, \mathcal{R})$ *be a cylindrical weak factorization system on a category with a functorial cylinder. If* f *is a map between* $\mathcal{L}$*-objects, then the first factor of its k-sided mapping cylinder factorization is an* $\mathcal{L}$*-map.*

Proof The first factor $A \to \mathrm{M}_k(f)$ in the factorization of $f\colon A \to B$ decomposes as the composite

$$A \xrightarrow{\iota_0} A \sqcup B \xrightarrow{\cong} (A \sqcup A) \sqcup_A B \xrightarrow{(\partial \otimes A) \sqcup_A B} \mathbb{I} \otimes A \sqcup_A B.$$

The first map is a cobase change of $0 \to B$, thus an $\mathcal{L}$-map. The last map is a cobase change of $\partial \otimes A \cong \partial \mathbin{\widehat{\otimes}} (0 \to A)$, thus also an $\mathcal{L}$-map. ∎

Lemma 3.25 *If* **M** *is cylindrical, then any cofibration between trivially cofibrant objects is a trivial cofibration. Dually, any fibration between trivially fibrant objects is a trivial fibration.*

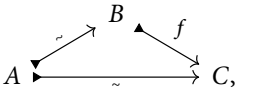
Proof Let $m\colon A \rightarrowtriangle B$ be a cofibration between trivially cofibrant objects. Consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow[\sim]{\iota_0(\delta_1 \otimes A)} & \mathrm{M}_0(m) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \delta_0 \mathbin{\widehat{\otimes}} m} \\ B & \xrightarrow[\delta_1 \otimes B]{} & \mathbb{I} \otimes B. \end{array}$$

The top horizontal map is a trivial cofibration by Lemma 3.24, while the right vertical map is a trivial cofibration by cylindricality. The bottom map is split monic, so m is a retract of a trivial cofibration and thus a trivial cofibration itself. ∎

Corollary 3.26 [Sat17, Lemma 4.5(iii)] *If* **M** *is cylindrical, then trivial cofibrations have right cancellation among cofibrations. Dually, trivial fibrations have left cancellation among fibrations.*

Proof Given a diagram



we apply Lemma 3.25 to f in the coslice under A via Remark 3.17. ∎

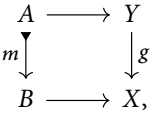
3.3 Model structures from the fibration extension property

We now narrow our attention to premodel structures satisfying properties common to cubical-type model structures: first, that all objects are cofibrant, and second, that fibrations *extend along trivial cofibrations*, the latter of which follows in particular from the existence of enough fibrant universes classifying fibrations. Note that our conditions cease to be self-dual at this point; moreover, the result is a criterion *sufficient* but not *necessary* to obtain a model structure.

Lemma 3.27 *Let* **M** *be a premodel category. Trivial fibrations have right cancellation in* **M** *if and only if the (cofibration, trivial fibration) factorization system is generated by cofibrations between cofibrant objects. Dually, trivial cofibrations have left cancellation in* **M** *if and only if the (trivial cofibration, fibration) factorization system is cogenerated by fibrations between fibrant objects.*

Proof Suppose trivial fibrations have right cancellation in **M** and let $p\colon Y \to X$ be a map lifting against cofibrations between cofibrant objects. We take a cofibrant replacement of Y, obtaining maps $0 \rightarrowtail Y' \xrightarrow{\sim\triangleright} Y$. By cancellation, it suffices to show the composite $p'\colon Y' \to X$ is a trivial fibration. We appeal to the retract argument: p' has the lifting property against the left part of its (cofibration, trivial fibration) factorization—this being a cofibration between cofibrant objects—so is a retract of the right part of its factorization. It is thus itself a trivial fibration.

The converse is an elementary exercise in lifting. Suppose the (cofibration, trivial fibration) factorization system is generated by cofibrations between cofibrant objects, let $f\colon Z \xrightarrow{\sim\triangleright} Y$ and $g\colon Y \to X$ be such that gf is a trivial fibration. Given a cofibration $m\colon A \rightarrowtail B$ between cofibrant objects and a lifting problem



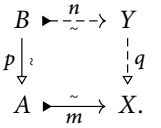
we solve lifting problems first against f and then against gf:

The composite fv is a lift for the original square. ∎

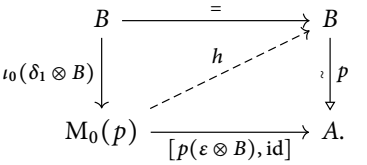
In particular, Lemma 3.27 tells us that trivial fibrations have right cancellation in any premodel structure where all objects are cofibrant. If the premodel structure is additionally cylindrical, then condition C is also always satisfied.

Lemma 3.28 *Let* **M** *be cylindrical and suppose that all objects are cofibrant. Then, any composite of a trivial fibration followed by a trivial cofibration is a weak equivalence.*

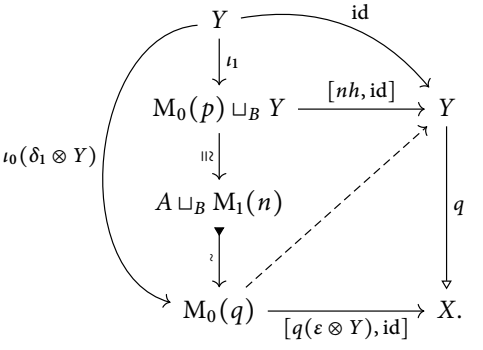
Proof Suppose we have $p\colon B \xrightarrow{\sim} A$ and $m\colon A \rightarrowtail\!\!\!\!\!\sim\; X$. We take their composite's (trivial cofibration, fibration) factorization:



We intend to show q is a trivial fibration. By Corollary 3.20 and the assumption that all objects are cofibrant, p has the structure of a dual strong 0-oriented deformation retract. Thus, we have a diagonal lift



Using that q is a fibration, we show that q is a dual strong deformation retract by solving a lifting problem of the form



The map $A \sqcup_B \mathrm{M}_1(n) \rightarrowtail\!\!\!\!\!\sim \mathrm{M}_0(q)$ is the following composite:

$$A \sqcup_B \mathrm{M}_1(n) \underset{m \sqcup_B \mathrm{M}_1(n)}{\rightarrowtail\!\!\!\!\!\sim} X \sqcup_B \mathrm{M}_1(n) \xrightarrow{\cong} X \sqcup_Y (\mathbb{I} \otimes B \sqcup_B (Y \sqcup Y)) \underset{X \sqcup_Y (\partial \mathbin{\widehat{\otimes}} n).}{\rightarrowtail\!\!\!\!\!\sim} \mathrm{M}_0(q)$$

The first map is a cobase change of the trivial cofibration m, while the final map is a cobase change of the trivial cofibration $\partial \mathbin{\widehat{\otimes}} n$; thus the composite is indeed a trivial cofibration. The diagonal lift exhibits q as a dual strong deformation retract, thus, a trivial fibration by Lemma 3.21. ∎

Thus, in a cylindrical premodel structure where all objects are cofibrant, the only non-trivial property necessary to apply Theorem 3.23 is left cancellation for trivial cofibrations among cofibrations. This we can further reduce to the following condition.

Definition 3.29 (FEP) We say a premodel category $\mathbf{M}$ has the *fibration extension property* when for any fibration $f\colon Y \twoheadrightarrow X$ and trivial cofibration $m\colon X \rightarrowtail\!\!\!\!\!\sim X'$, there exists a fibration $f'\colon Y' \twoheadrightarrow X'$ whose base change along m is f:

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} \\ X & \underset{m}{\rightarrowtail\!\!\!\!\!\sim} & X'. \end{array}$$

Lemma 3.30 *Suppose* $\mathbf{M}$ *is a premodel category with the fibration extension property. Then, trivial cofibrations have left cancellation in* $\mathbf{M}$.

Proof By Lemma 3.27, it suffices to show the (trivial cofibration, fibration) factorization system is cogenerated by fibrations between fibrant objects. Suppose $g\colon A \to B$ is a map with the left lifting property against all fibrations between fibrant objects. Let $f\colon Y \twoheadrightarrow X$ be an arbitrary fibration. Its codomain X has a fibrant replacement $m\colon X \rightarrowtail\!\!\!\!\!\sim X^{\mathrm{fib}}$; by the fibration extension property, there is some $f'\colon Y' \twoheadrightarrow X^{\mathrm{fib}}$ whose pullback along m is f. By assumption g lifts against f', and this lift induces a lift for g against f via the usual argument that right maps of a weak factorization system are closed under base change. ∎

Theorem 3.31 *Let* $\mathbf{M}$ *be a cylindrical premodel category in which:*

(D) *all objects are cofibrant;*
(E) *the fibration extension property is satisfied.*

Then, the premodel structure on $\mathbf{M}$ *defines a model structure.*

Proof By Theorem 3.23. Condition C is satisfied by Lemma 3.28. Trivial cofibrations have left cancellation by Lemma 3.30, while trivial fibrations have right cancellation by Lemma 3.27. ∎

The fibration extension property can, in particular, be obtained from the existence of fibrant classifiers for fibrations, i.e., fibrant universes of fibrations. We do not generally expect to have a single classifier for *all* fibrations, only those below a certain size. Thus, we now consider a setup where a premodel category sits inside a larger category containing classifiers for its fibrations.

Lemma 3.32 *Let* **E** *be a category, and let* **M** *be a subcategory of* **E** *equipped with a premodel structure. Say that a map in* **E** *is a fibration if it has the right lifting property against all trivial cofibrations in* **M**. *Suppose we have a class* $\mathcal{U} \subseteq \mathbf{E}^{\to}$ *of fibrations between fibrant objects that classifies fibrations in* **M**, *in following sense:*

(a) *every fibration in* **M** *is a pullback of some fibration in* $\mathcal{U}$;
(b) *if* $p\colon E \twoheadrightarrow U$ *is a map in* $\mathcal{U}$ *and* $y\colon X \to U$ *is a map with* $X \in \mathbf{M}$, *then there exists a map in* **M** *which is the pullback of* p *along* y:

$$\begin{array}{ccc} \bullet & \dashrightarrow & E \\ \mathbf{M} \ni \vdots & \lrcorner & \downarrow p \\ X & \xrightarrow[y]{} & U. \end{array}$$

Then, **M** *has the fibration extension property.*

Proof Let a fibration $f\colon Y \twoheadrightarrow X$ in **M** and trivial cofibration $m\colon X \rightarrowtail\!\!\!\!\!\sim\; X'$ in **M** be given. Then, f is the pullback of some fibration between fibrant objects $p\colon E \twoheadrightarrow U$ in **E** along some map $y\colon X \to U$. As U is fibrant, y extends along m to some $y'\colon X' \to U$. By assumption, we can choose a pullback $f'\colon Y' \twoheadrightarrow X'$ of p along y' belonging to **M**. By the pasting law for pullbacks, f is the pullback of f' along m. ■

Corollary 3.33 *Let* **E** *be a category, and let* **M** *be a subcategory of* **E** *equipped with a premodel structure. Suppose that* **M** *is cylindrical and the following conditions are satisfied:*

(D) *all objects of* **M** *are cofibrant;*
(F) *there is a class of fibrations between fibrant objects in* **E** *that classifies fibrations in* **M** *in the sense of Lemma* 3.32.

Then, the premodel structure on **M** *defines a model structure.*

Proof By Theorem 3.31 and Lemma 3.32. ■

Remark 3.34 In applications, one usually starts with a set (or category, when working with algebraic weak factorization systems) of *generating* trivial cofibrations that defines the class of fibrations via lifting. We can then consider an "extension" **E** of **M** large enough to build a classifier for fibrations in **M** (for example, by passing from presheaves to "large" presheaves as in Section 4.2). Fibrancy of the classifier is shown by extending fibrations along generating trivial cofibrations.

In such settings, there is also an alternative approach that directly moves from fibration extension along generating trivial cofibrations to general fibration extension. For a set of generating trivial cofibrations with representable codomain, this is described in [Sat17, Section 7]. It involves exhibiting trivial cofibrations as codomain retracts of cell complexes of the generators using the small object argument; fibration extension along such a cell complex is then obtained inductively. In the model structure we construct in Section 4.2, we instead have a *category* of generating trivial cofibrations with representable codomain (Definition 4.16). However, the same technique still applies, using an analysis of the algebraic small object argument [Sat23].

4 Semilattice cubical sets

4.1 The semilattice cube category

We now introduce this article's main character: the (join-)semilattice cube category $\square_\vee$ generated by an interval object, finite Cartesian products, and a binary connection operator. Like other Cartesian cube categories, it is a (single-sorted) *Lawvere theory* [Law63]: a finite product category in which every object is a finite power of some distinguished object.

Definition 4.1 The theory of *(join-)semilattices* consists of an associative and commutative binary operator $\vee$ for which all elements are idempotent, which we call the *join*. This means the following laws:

$$(x \vee y) \vee z = x \vee (y \vee z), \qquad x \vee y = y \vee x, \qquad x \vee x = x.$$

The theory of *01-bounded (join-)semilattices* consists, in addition to the above, of two constants $0, 1$ and the following laws:

$$0 \vee x = x, \qquad 1 \vee x = 1.$$

The *(join-)semilattice cube category* $\square_\vee$ is the Lawvere theory of 01-bounded semilattices. Concretely, the objects of $\square_\vee$ are of the form T^n for $n \in \mathbb{N}$, and the morphisms $T^m \to T^n$ are n-ary tuples of expressions over $0, 1, \vee$ in m variables modulo the equations above. We write $\mathbf{T}_\vee$ for the Lawvere theory of semilattices.

Remark 4.2 As a bicategory, $\mathbf{T}_\vee$ can be identified with the subcategory of the bicategory of onto (decidable) relations between finite sets. Equivalently, these are jointly injective spans in finite sets whose second leg is surjective. This can be strictified to a 1-category by replacing relations with Boolean-valued matrices.

Recall that the *category of algebras* $\mathrm{Alg}(\mathbf{T}) \coloneqq [\mathbf{T}, \mathbf{Set}]_{\mathrm{fp}}$ of a Lawvere theory $\mathbf{T}$ is the category of finite-product-preserving functors from $\mathbf{T}$ to **Set**, which supports an "underlying set" functor $U\colon [\mathbf{T}, \mathbf{Set}]_{\mathrm{fp}} \to \mathbf{Set}$ given by evaluation at the distinguished object T^1. This functor has a left adjoint $F\colon \mathbf{Set} \to \mathrm{Alg}(\mathbf{T})$ which produces the *free* $\mathbf{T}$*-algebra* on a set, and the covariant Yoneda embedding restricts to an embedding $\mathbf{T}^{\mathrm{op}} \to \mathrm{Alg}(\mathbf{T})$ sending T^n to the free algebra on n elements. We write **SLat** and **01SLat** for the categories of algebras of $\mathbf{T}_\vee$ and $\square_\vee$, respectively. Concretely, these are the categories of sets equipped with the operations described in Definition 4.1 and operation-preserving morphisms between them.

It can also be useful to take an order-theoretic perspective on **SLat** and **01SLat**, identifying them as subcategories of the category **Pos** of posets and monotone maps. Recall that the operator $\vee$ induces a poset structure on any semilattice, with $x \leq y$ when $x \vee y = y$.

Proposition 4.3 **SLat** *is equivalent to the subcategory of* **Pos** *consisting of posets with finite non-empty joins (that is, least upper bounds) and monotone maps that preserve said joins.* **01SLat** *is equivalent to the further (non-full) subcategory of posets that also have a minimum and maximum element and monotone maps that also preserve them.*

Remark 4.4 Any finite linear order is a semilattice, and it is 01-bounded if it is inhabited. Moreover, any monotone map between linear orders preserves joins. Thus, the inclusion $\Delta \to \mathbf{Pos}$ factors through a fully faithful inclusion $\Delta \to \mathbf{SLat}$.

In particular, the interval $[1] \in \mathbf{Pos}$ is a 01-bounded semilattice.

Proposition 4.5 *The interval is a dualizing object for a duality between the categories of finite semilattices and finite 01-bounded semilattices, which is to say that we have the following categorical equivalence:*

$$\mathbf{SLat}_{\mathrm{fin}}^{\mathrm{op}} \underset{\mathbf{01SLat}(-,[1])}{\overset{\mathbf{SLat}(-,[1])}{\rightleftarrows}} \mathbf{01SLat}_{\mathrm{fin}}.$$

Proof By a slight variation on the argument that $\mathbf{0SLat}_{\mathrm{fin}}^{\mathrm{op}} \simeq \mathbf{0SLat}_{\mathrm{fin}}$ indicated in [Joh82, Sections VI3.6 and VI.4.6(b)]. ∎

Given a semilattice A, the 01-bounded semilattice structure on $\mathbf{SLat}(A, [1])$ is defined pointwise from that on $[1]$; likewise $\mathbf{01SLat}(B, [1])$ has a pointwise semilattice structure for any $B \in \mathbf{01SLat}$. This extends the duality between the augmented simplex category and the category of finite intervals (i.e., finite bounded linear orders and bound-preserving monotone maps) observed by Joyal [Joy97, Section 1.1; Wra93].

By way of this duality, we have in particular an embedding of $\square_\vee$ in the category of finite semilattices, induced by the embedding of its opposite in its category of models:

$$\square_\vee \xrightarrow{\text{よ}} \mathbf{01SLat}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{SLat}_{\mathrm{fin}}.$$

Here, we use that the free semilattice on a finite set of generators is a finite semilattice. Unpacking, this embedding sends T^n to $\mathbf{01SLat}(F(n), [1]) \cong \mathbf{Set}(n, U[1]) \cong [1]^n$.

Notation 4.6 *Henceforth, we regard $\square_\vee$ as a subcategory of* **SLat***, in particular writing $[1]^n$ rather than T^n for its objects.*

We can also describe the cubes in **SLat** as free semilattices on posets. Given a poset A, write $1 \star A$ for the poset obtained by adjoining a minimum element $\bot$ to A. For any set S, we have a monotone map $\eta_n : 1 \star S \to [1]^S$ sending $\bot$ to $\bot$ and $i \in S$ to the element of $[1]^S$ with 1 at its ith component and 0 elsewhere.

Proposition 4.7 *For any $S \in \mathbf{Set}_{\mathrm{fin}}$, the map η_S exhibits $[1]^S$ as the free semilattice on the poset $1 \star S$. That is, for any $A \in \mathbf{SLat}$ and monotone map $f : 1 \star S \to A$, there is a unique semilattice morphism $f^\dagger : [1]^S \to A$ such that $f = f^\dagger \eta_S$.*

4.2 Cubical-type model structure on semilattice cubical sets

We now define our model structure on $\mathrm{PSh}(\square_\vee)$ using Corollary 3.33. That our case satisfies the corollary's hypotheses is essentially an application of existing work, namely, [CMS19] or [Awo23], so we do not give many proofs, only enough of an outline to guide an unfamiliar reader through the appropriate references. We point to [GS17; Sat17; AGH24, §8) for further details on constructing model structures of this kind and to [LOPS18] for the definition of the universe in particular.

Assumption 4.8 *For simplicity, we work with a single universe: we assume a strongly inaccessible cardinal κ and define a model structure on the category $\mathrm{PSh}_\kappa(\square_\vee)$ of κ-small presheaves. Outside of this section, we suppress the subscript κ. As described in Remark 3.34, it is possible to eliminate the use of universes at the cost of some complication; alternatively, one can assume that every fibration belongs to some universe to obtain a model structure on all of $\mathrm{PSh}(\square_\vee)$.*

Notation 4.9 *We write $\mathbb{I} \coloneqq \yo[1] \in \mathrm{PSh}(\square_\vee)$ for the representable 1-cube. We write $\delta_k : 1 \to [1]$ for the endpoint inclusion picking out $k \in \{0,1\}$ and write ε for the unique degeneracy map $[1] \to 1$.*

4.2.1 Factorization systems

As analyzed by Gambino and Sattler [GS17], a key feature of cubical-type model structures is that their fibrations are characterized by a *uniform lifting property*. This characterization is used to obtain the model structure's factorization systems constructively and to define fibrant universes of fibrations. We avoid formally introducing algebraic weak factorization systems [GT06, Gar09] for the sake of concision, but these form the conceptual backbone of Gambino and Sattler's results.

Definition 4.10 (Uniform lifting) Let $u : \mathbf{I} \to \mathbf{E}^{\to}$ be a functor. A *right u-map* is a map $f : Y \to X$ in $\mathbf{E}$ equipped with

- for each $i \in \mathbf{I}$ and filling problem

$$\begin{array}{ccc} A_i & \xrightarrow{h} & Y \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \xrightarrow[k]{} & X, \end{array}$$

  a diagonal filler $\varphi(i,h,k) : B_i \to Y$;
- such that for each $\alpha : j \to i$ and diagram

$$\begin{array}{ccccc} A_j & \xrightarrow{a} & A_i & \xrightarrow{h} & Y \\ {\scriptstyle u_j}\downarrow & u_\alpha & \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle f} \\ B_j & \xrightarrow[b]{} & B_i & \xrightarrow[k]{} & X, \end{array}$$

  we have $\varphi(i,h,k)b = \varphi(j, ha, kb)$.

When u is a subcategory inclusion, we may instead say that f is a *right* $\mathbf{I}$*-map*.

Notation 4.11 *Given a category $\mathbf{E}$, write $\mathbf{E}^{\to}_{\mathrm{cart}} \subseteq \mathbf{E}^{\to}$ for the category of arrows in $\mathbf{E}$ and Cartesian squares between them.*

Write $\mathcal{M}$ for the full subcategory of $\mathrm{PSh}_\kappa(\square_\vee)^{\to}_{\mathrm{cart}}$ consisting of monomorphisms.

Definition 4.12 We say a map in $\mathrm{PSh}_\kappa(\square_\vee)^{\to}_{\mathrm{cart}}$ is a *uniform trivial fibration* when it is a right $\mathcal{M}$-map.

Remark 4.13 If working constructively, one must replace $\mathcal{M}$ with the full subcategory $\mathcal{M}_{\mathrm{dec}}$ of levelwise decidable monomorphisms, i.e., those $m : A \rightarrowtail B$ such that m_I is

isomorphic to a coproduct inclusion for all $I \in \square_\vee$. This restriction is used (see, e.g., Orton and Pitts [OP18, Theorem 8.4]) in the proof of the *realignment* property, which is important to the construction of fibrant universes.

The following proposition lets us characterize the trivial fibrations (and later, the fibrations) as the maps with uniform right lifting against a *small* category.

Proposition 4.14 [GS17, Proposition 5.16] *Let* $\mathbf{C}$ *be a small category and* $\mathbf{I}$ *be a full subcategory of* $\mathrm{PSh}(\mathbf{C})^{\to}_{\mathrm{cart}}$ *closed under base change to representables, i.e., such that* $x^* f \in \mathbf{I}$ *for any* $f : Y \to X$ *in* $\mathbf{I}$ *and* $x : \yo a \to X$. *Write* $\mathbf{I}^{\yo}$ *for the full subcategory of* $\mathbf{I}$ *consisting of maps with representable codomain. Then, a map in* $\mathrm{PSh}(\mathbf{C})$ *is a right* $\mathbf{I}$*-map if and only if it is a right* $\mathbf{I}^{\yo}$*-map.*

Proposition 4.15 (Uniform trivial fibrations) *We have a weak factorization system* $(\mathcal{M}, \mathcal{F}_t)$, *where* $\mathcal{F}_t$ *is the class of uniform trivial fibrations.*

Proof By [GS17, Theorem 9.1], which goes through Garner's algebraic small object argument [Gar09], we have a factorization system $(\mathcal{C}, \mathcal{F}_t)$, where $\mathcal{F}_t$ is the class of uniform trivial fibrations. Here, we need that the right $\mathcal{M}$-maps coincide with the right $\mathcal{M}^{\yo}$-maps and that $\mathcal{M}^{\yo}$ is a small category. That the algebraic small object argument is constructive in this case is explained in [GS17, Remark 9.4] (see also [Hen20, Appendix C]).

An alternative construction of the factorization using partial map classifiers is described in [GS17, Remark 9.5] and used by Awodey et al. [AGH24, Awo23], while Swan [Swa18, Section 6] describes a construction using W-types with reductions. The partial map classifier factorization factors any map as a mono followed by a trivial fibration. By the retract argument, any map in $\mathcal{C}$ is then a retract of a mono and hence itself monic, so $\mathcal{C} = \mathcal{M}$. ∎

Definition 4.16 Define $u_\delta : \{0,1\} \times \mathcal{M}^{\yo} \to \mathrm{PSh}_\kappa(\square_\vee)^{\to}$ by $u_\delta(k, -) \coloneqq \delta_k \mathbin{\widehat{\times}} (-)$. A *uniform fibration* is a right u_δ-map.

Proposition 4.17 (Uniform fibrations) *There exists a weak factorization system* $(\mathcal{C}_t, \mathcal{F})$ *such that* $\mathcal{F}$ *is the class of uniform fibrations.*

Proof By [GS17, Theorem 7.5], using the algebraic small object argument. Again, see [GS17, Remark 9.4] for discussion of constructivity. ∎

Though the algebraic/uniform description is important to constructively establish the *existence* of these weak factorization systems, we can also—still constructively—recognize that $\mathcal{F}_t$ and $\mathcal{F}$ are classes of maps with lifting properties in the non-algebraic sense.

Proposition 4.18 *Let* $f : Y \to X$ *in* $\mathrm{PSh}_\kappa(\square_\vee)$. *Then,*

- *f is a right* $\mathcal{M}$*-map if and only if it has the right lifting property against all monomorphisms;*
- *f is a right* u_δ*-map if and only if it has the right lifting property with respect to* $\delta_k \mathbin{\widehat{\times}} m$ *for all* $k \in \{0,1\}$ *and monomorphisms m.*

Proof By [GS17, Theorem 9.9]. ∎

With the two factorization systems in hand, it is straightforward to verify the following.

Proposition 4.19 *$(\mathcal{C}_t, \mathcal{F})$ and $(\mathcal{M}, \mathcal{F}_t)$, together with the adjoint functorial cylinder $\mathbb{I} \times (-) \dashv (-)^{\mathbb{I}}$, constitute a cylindrical premodel structure.*

4.2.2 Unbiased fibrations

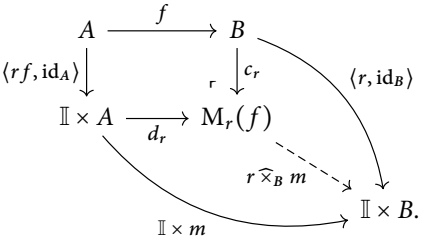
In order to apply Corollary 3.33, we must check that we have a fibration between fibrant objects in $\mathrm{PSh}(\square_\vee)$ classifying fibrations in $\mathrm{PSh}_\kappa(\square_\vee)$. This follows from work on cubical models of type theory, specifically the interpretation of universes. Our cube category falls within the ambit of [ABCHFL21], which describes a universe $p_{\mathrm{fib}} \colon \widetilde{U}_{\mathrm{fib}} \to U_{\mathrm{fib}}$ with fibration structures on p_{fib} and U_{fib} in type-theoretic terms; Awodey gives a construction of the same in categorical language [Awo23, Sections 6–8].

However, the fibrations used in these models are not *a priori* the fibrations we defined in the previous section: they are what Awodey [Awo23] calls *unbiased fibrations*, which lift not only against (pushout products with) endpoint inclusions $\delta_k \colon 1 \to \mathbb{I}$ but against generalized points on the interval. To see that $\widehat{\square}^{\mathrm{ty}}_\vee$ is compatible with this model of type theory, we check here that biased (i.e., ordinary) and unbiased fibrations coincide in the presence of a connection.

Definition 4.20 Given $r \colon B \to \mathbb{I}$ and $f \colon A \to B$, their *unbiased mapping cylinder* is the following pushout:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \langle rf, \mathrm{id}_A \rangle}\downarrow & & \downarrow{\scriptstyle c_r} \\ \mathbb{I} \times A & \xrightarrow[d_r]{} & \mathrm{M}_r(f). \end{array}$$

Note that $\mathrm{M}_{\delta_k !_B}(f)$ is the ordinary k-sided mapping cylinder (Definition 3.14). We write $r \mathbin{\widehat{\times}_B} m \colon \mathrm{M}_r(m) \to \mathbb{I} \times B$ for the unique map fitting in the diagram

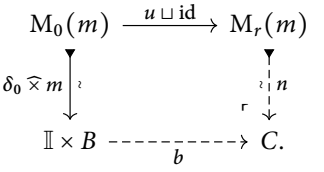


This is the pushout product in the slice over B of $\langle r, \mathrm{id}_B \rangle \colon \mathrm{id}_B \to \varepsilon \times B$ and $m \colon m \to \mathrm{id}_B$, hence the notation. Note that $(\delta_k !_B) \mathbin{\widehat{\times}_B} f$ is the ordinary pushout product $\delta_k \mathbin{\widehat{\times}} f$.

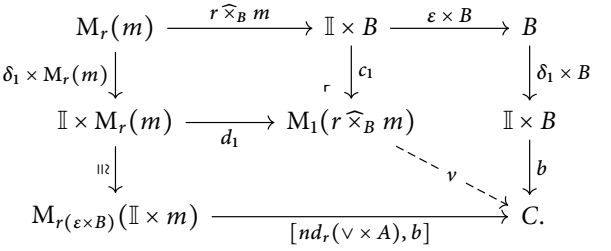
Definition 4.21 We say $f \colon Y \to X$ is an *unbiased fibration* when it has the right lifting property against $r \mathbin{\widehat{\times}_B} m$ for all $r \colon B \to \mathbb{I}$ and $m \colon A \rightarrowtail B$.

Lemma 4.22 *$r \mathbin{\widehat{\times}_B} m$ is a trivial cofibration for any $r \colon B \to \mathbb{I}$ and $m \colon A \rightarrowtail B$.*

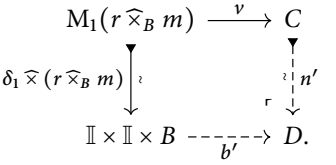
Proof Define $u(i,a) \coloneqq (i \vee r(m(a)), a) : \mathbb{I} \times A \to \mathbb{I} \times A$. Take a pushout of $\delta_0 \mathbin{\widehat{\times}} m$:



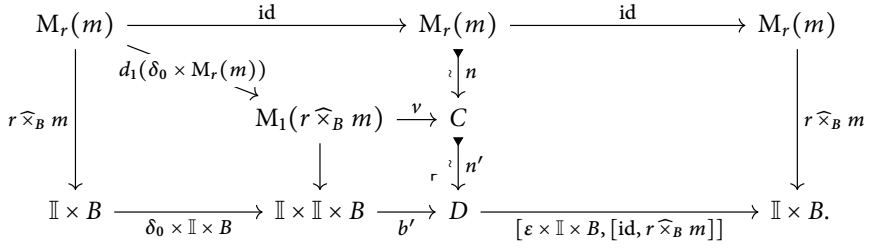
Define a map $v : \mathrm{M}_1(r \mathbin{\widehat{\times}}_B m) \to C$ like so:



Take the pushout of $\delta_1 \mathbin{\widehat{\times}} (r \mathbin{\widehat{\times}}_B m)$ by this map:



Then, we can exhibit $r \mathbin{\widehat{\times}}_B m$ as a retract of $n'n$:



As a retract of a trivial cofibration, $r \mathbin{\widehat{\times}}_B m$ is thus a trivial cofibration. ∎

Corollary 4.23 *A map is a fibration in $\widehat{\square}^{\mathrm{ty}}_{\vee}$ if and only if it is an unbiased fibration.*

Proof If $f : Y \to X$ is an unbiased fibration, then lifting against any $\delta_k \mathbin{\widehat{\times}} m$ is obtained as lifting against $(\delta_k !_B) \mathbin{\widehat{\times}}_B m$. The converse is Lemma 4.22. ∎

Remark 4.24 For the reader more comfortable with cubical type theories, we give the type-theoretic analog to the proof of Corollary 4.23. The ABCHFL type theory

equips types with a composition operator of the following form:

$$\frac{\begin{array}{c} i\colon\mathbb{I} \vdash A\ \mathsf{type} \qquad \varphi\ \mathsf{cof} \qquad r,s\colon\mathbb{I} \\ i\colon\mathbb{I},\varphi \vdash M\colon A \qquad M_0\colon A[r/i] \qquad \varphi \vdash M[r/i] = M_0\colon A[r/i] \end{array}}{\mathsf{com}_{i.A}^{r\to s}\ [\varphi \mapsto i.M]\ M_0\colon A[s/i]}$$

$$\mathsf{com}_{i.A}^{r\to r}\ [\varphi \mapsto i.M]\ M_0 = M_0\colon A[r/i]$$

$$\varphi \vdash \mathsf{com}_{i.A}^{r\to s}\ [\varphi \mapsto i.M]\ M_0 = M[s/i]\colon A[s/i].$$

In the presence of a connection, we can derive a term satisfying the equations required of $\mathsf{com}_{i.A}^{r\to s}\ [\varphi \mapsto i.M]\ M_0$ using only composition $\varepsilon \to s$, where $\varepsilon \in \{0,1\}$, namely, the term Q below.

$$P(k) \coloneqq \mathsf{com}_{j.A[r\vee j/i]}^{0\to k}\ \left[\ \varphi \ \mapsto\ j.M[r\vee j/i]\ \right]\ M_0$$

$$Q \coloneqq \mathsf{com}_{j.A[s\vee j/i]}^{1\to 0}\ \left[\begin{array}{lcl} \varphi & \mapsto & j.M[s\vee j/i] \\ r\equiv s & \mapsto & j.P(j) \end{array}\right]\ P(1).$$

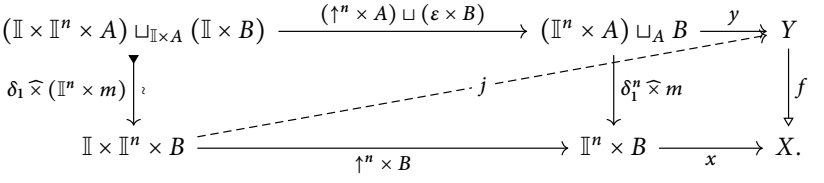
Remark 4.25 We can also use $\vee$ to show that any fibration is an *equivariant* fibration in the sense of the ACCRS model structure [ACCRS24]. For simplicity, let us restrict attention to lifting along $\delta_1^n\colon 1 \to \mathbb{I}^n$, which is the simplest case; we leave it as an exercise to formulate and derive unbiased equivariant lifting by combining the proof of Lemma 4.22 with the following sketch. A more complete proof (for simplicial sets rather than semilattice cubical sets, but with the same argument) is in [ACCRS24, Proposition 6.1.7].

Write $\Sigma \coloneqq \mathrm{Core}(\square_\vee)$ for the wide subcategory of isomorphisms of $\square_\vee$. We have a functor $\delta\colon \Sigma \to \mathrm{PSh}(\square_\vee)^{\to}$ sending $[1]^n$ to $\delta_1^n\colon 1 \to \mathbb{I}^n$ and $\sigma\colon [1]^n \cong [1]^n$ to $(\mathrm{id},\sigma)\colon \delta_1^n \to \delta_1^n$. Take $u_{\delta\Sigma}$ to be the composite

$$\Sigma \times \mathcal{M}^{\mathbb{2}} \xrightarrow{\delta \times \mathcal{M}^{\mathbb{2}}} \mathrm{PSh}(\square_\vee)^{\to} \times \mathcal{M}^{\mathbb{2}} \xrightarrow{\ \widehat{\times}\ } \mathrm{PSh}(\square_\vee)^{\to}.$$

A *uniform equivariant 1-fibration* is a right $u_{\delta\Sigma}$-map.

Suppose $f\colon Y \to X$ is a uniform fibration and let $m\colon A \twoheadrightarrow B$ and a lifting problem $(y,x)\colon \delta_1^n \mathbin{\widehat{\times}} m \to f$ be given. We have a map $\uparrow_n\colon [1] \times [1]^n \to [1]^n$ sending $(t,i_1,\dots,i_n) \mapsto (t\vee i_1,\dots,t\vee i_n)$ which we use to form a lifting problem against $\delta_1 \mathbin{\widehat{\times}} (\mathbb{I}^n \times m)$:



The composite tikz69 is our desired lift, while the uniformity conditions follow from those on f and the fact that $\uparrow_n \circ (\mathbb{I}\times\sigma) \cong \sigma \circ \uparrow_n$ for any $\sigma\colon [1]^n \cong [1]^n$.

4.2.3 Universe

To define a universe classifying fibrations, we use a theorem of Licata et al. [LOPS18]. The cardinal κ provides a Grothendieck universe in **Set**, from which Hofmann and Streicher's construction produces a universe $p_U : \widetilde{U} \to U$ in $\mathrm{PSh}(\square_\vee)$ classifying κ-small maps [HS97, Str05, Awo24]. Our classifier for κ-small fibrations shall be a subuniverse of p_U. The key property of $\mathrm{PSh}(\square_\vee)$ is that the cocylinder $(-)^{\mathbb{I}}$ has a right adjoint, i.e., that $\mathbb{I}$ is *internally tiny*: we have $(-)^{\mathbb{I}} \cong ((-) \times [1])^*$ and therefore $(-)^{\mathbb{I}} \dashv \sqrt[\mathbb{I}]{-} \coloneqq ((-) \times [1])_*$. This property is common to cube categories but fails, for example, in simplicial sets. We refer to Swan [Swa22] for a deeper analysis.

Given a κ-small map $f : Y \to X$ with characteristic map $A : X \to U$, we define a family $X^{\mathbb{I}} \to U$ whose sections correspond to fibration structures on A. To do so, it is convenient to work in the internal extensional type theory of the universe p_U in the style of Orton and Pitts [OP18].[5] Writing $\top : 1 \to \Omega$ for the subobject classifier in $\mathrm{PSh}(\square_\vee)$, the maps $!_\Omega : \Omega \to 1$ and $\top$ are both classified by p_U,[6] so appear as a closed type $\cdot \vdash \Omega : U$ and type family $\varphi : \Omega \vdash [\varphi] : U$, respectively. The interval likewise appears as a closed type $\cdot \vdash \mathbb{I} : U$ with inhabitants $\cdot \vdash 0, 1 : \mathbb{I}$.

Definition 4.26 Given a type $A : U$, define its type of trivial fibration structures $\mathrm{TFib}\, A : U$ as follows:

$$\mathrm{TFib}\, A \coloneqq \Pi \varphi : \Omega.\ \Pi v : [\varphi] \to A.\ \Sigma a : A.\ \Pi \alpha : [\varphi].\ v(\alpha) = a.$$

Definition 4.27 Given $k \in \{0, 1\}$ and $A : X \to U$, define the pullback exponential $(\delta_k \mathrel{\widehat{\Rightarrow}} A) : (\Sigma p : X^{\mathbb{I}}.\ A(p(k))) \to U$ internally as follows:

$$(\delta_k \mathrel{\widehat{\Rightarrow}} A)(p, a) \coloneqq \Sigma q : (\Pi i : \mathbb{I}.\ A(p(i))).\ q(k) = a.$$

Definition 4.28 Given $A : X \to U$, define $\mathrm{Fib}_k\, A : X^{\mathbb{I}} \to U$ for $k \in \{0, 1\}$ and then $\mathrm{Fib}\, A : X^{\mathbb{I}} \to U$ as follows:

$$(\mathrm{Fib}_k\, A)(p) \coloneqq \Pi a : A(p(k)).\ \mathrm{TFib}((\delta_k \mathrel{\widehat{\Rightarrow}} A)(p, a))$$
$$(\mathrm{Fib}\, A)(p) \coloneqq (\mathrm{Fib}_0\, A)(p) \times (\mathrm{Fib}_1\, A)(p).$$

Proposition 4.29 *Let $f : Y \to X$ be given with classifying map $A : X \to U$. Then, f is a uniform fibration if and only if the type $\Pi p : X^{\mathbb{I}}.\ (\mathrm{Fib}\, A)(p)$ is inhabited.*

Proof See [AGH24, Corollary 8.7]. ∎

Using the right adjoint to $(-)^{\mathbb{I}}$, we carve out the subuniverse of p_U corresponding to families $A : X \to U$ for which $\Pi p : X^{\mathbb{I}}.\ (\mathrm{Fib}\, A)(p)$ is inhabited. For this step, we return to working externally, as $\sqrt[\mathbb{I}]{-}$ does not straightforwardly internalize; Licata et al. [LOPS18] use a global sections modality to axiomatize $\sqrt[\mathbb{I}]{-}$ internally, while Riley [Ril24] has recently proposed a type theory which directly represents $\sqrt[\mathbb{I}]{-}$ as a modality. The following definition and proposition constitute Theorem 5.2 of [LOPS18].

[5] We refer to [AGH24] for a detailed translation between external and internal constructions in presheaf categories and to [Awo23, Section 6] for a fully externalized argument.

[6] If working predicatively, one should replace Ω with the classifier for levelwise decidable subobjects.

Definition 4.30 Define $p_{\mathrm{fib}} : \widetilde{U}_{\mathrm{fib}} \to U_{\mathrm{fib}}$ by pullback as follows:

$$\begin{array}{ccc} \widetilde{U}_{\mathrm{fib}} & \xrightarrow{\widetilde{\pi}_1} & \widetilde{U} \\ {\scriptstyle p_{\mathrm{fib}}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_U} \\ U_{\mathrm{fib}} & \xrightarrow{\pi_1} & U \\ {\scriptstyle \pi_0}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{Fib}\,\mathrm{id}_U)^\dagger} \\ \sqrt[\mathbb{I}]{\widetilde{U}} & \xrightarrow[\sqrt[\mathbb{I}]{p_U}]{} & \sqrt[\mathbb{I}]{U}. \end{array}$$

Proposition 4.31 [LOPS18, Theorem 5.2] *If $f : Y \to X$ is the pullback of p_U along some $A : X \to U$, then f is a uniform fibration if and only if A factors through $\pi_1 : U_{\mathrm{fib}} \to U$.*

Corollary 4.32 *The map p_{fib} is a uniform fibration.*

Proof p_{fib} is the pullback of p_U along π_1, which of course factors through itself. ∎

Finally, we need a fibrancy structure on the universe U itself. This is the most technically involved argument; we defer to prior work.

Proposition 4.33 *The object U_{fib} is uniform fibrant.*

Proof A fibrancy structure on U_{fib} is described in type-theoretic language in [ABCHFL21, Section 2.12], while Awodey [Awo23, Section 8] gives an external categorical construction. ∎

Theorem 4.34 (Cubical-type model structure on semilattice cubical sets) *There is a model structure on $\mathrm{PSh}_\kappa(\square_\vee)$ in which:*

- *The cofibrations are the monomorphisms;*
- *The fibrations are those maps with the right lifting property against all pushout products $\delta_k \mathbin{\widehat{\times}} m$ of an endpoint inclusion with a monomorphism.*

We write $\widehat{\square}^{\mathrm{ty}}_\vee$ for this model category.

Proof By Corollary 3.33 applied with $\mathrm{PSh}_\kappa(\square_\vee)$ inside $\mathrm{PSh}(\square_\vee)$ and the factorization systems $(\mathcal{M}, \mathcal{F}_t)$ and $(\mathcal{C}_t, \mathcal{F})$ defined in this section. Clearly all objects are cofibrant, and every fibration in $\mathrm{PSh}_\kappa(\square_\vee)$ is classified by $p_{\mathrm{fib}} : \widetilde{U}_{\mathrm{fib}} \to U_{\mathrm{fib}}$, which is a fibration (Corollary 4.32) between fibrant objects (Proposition 4.33). ∎

Our question now is whether $\widehat{\square}^{\mathrm{ty}}_\vee$ presents ∞-**Gpd**. More narrowly, we can ask whether the following comparison adjunction evinces a Quillen equivalence between $\widehat{\square}^{\mathrm{ty}}_\vee$ and $\widehat{\Delta}^{\mathrm{kq}}$.

Definition 4.35 (Triangulation) Define $\boxslash : \square_\vee \to \mathrm{PSh}(\Delta)$ to be the functor sending the n-cube $[1]^n$ to the n-fold product $(\Delta^1)^n$ of the 1-simplex, with the evident

functorial action. The *triangulation* functor $\mathrm{T}\colon \mathrm{PSh}(\square_\vee) \to \mathrm{PSh}(\Delta)$ is the left Kan extension of $\boxslash$:

$$\begin{array}{ccc} [1]^n & \xmapsto{\boxslash} & (\Delta^1)^n \\ \square_\vee & \longrightarrow & \mathrm{PSh}(\Delta). \\ \text{よ}\downarrow & \nearrow_{\mathrm{T}} & \\ \mathrm{PSh}(\square_\vee) & & \end{array}$$

Triangulation has a right adjoint, the nerve functor $N_\boxslash\colon \mathrm{PSh}(\Delta) \to \mathrm{PSh}(\square_\vee)$ defined by $N_\boxslash X \coloneqq \mathrm{PSh}(\Delta)(\boxslash -, X)$.

4.3 Idempotent completion

Although the triangulation adjunction $\mathrm{T} \dashv N_\boxslash$ is the most immediate means of comparing $\widehat{\square}_\vee^{\mathrm{ty}}$ and $\widehat{\Delta}^{\mathrm{kq}}$, it is not the most convenient. Ideally, we would like to have a comparison on the level of the base categories, some functor $i\colon \Delta \to \square_\vee$ or vice versa, in which case we would obtain an adjoint *triple* $i_! \dashv i^* \dashv i_*$ on their presheaf categories. This is too much to hope for, but we *can* define an embedding from Δ into the *idempotent completion* of $\square_\vee$, following the strategy used by Sattler [Sat19] and Streicher and Weinberger [SW21] to relate Δ and $\square_{\wedge\vee}$. The category of presheaves on any category $\mathbf{C}$ is equivalent to the category of presheaves on its idempotent completion $\overline{\mathbf{C}}$, the closure of $\mathbf{C}$ under splitting of idempotents [BD86]. We shall exhibit an embedding $\blacktriangle\colon \Delta \to \overline{\square}_\vee$; by composing the triple $\blacktriangle_! \dashv \blacktriangle^* \dashv \blacktriangle_*$ with the adjoint equivalence $\blacksquare^*\colon \mathrm{Psh}(\overline{\square}_\vee) \underset{\iota}{\rightleftarrows} \mathrm{PSh}(\square_\vee) :\blacksquare_!$, we obtain a triple relating $\mathrm{PSh}(\Delta)$ and $\mathrm{PSh}(\square_\vee)$.

We then observe that $\mathrm{T} \cong \blacktriangle^*\blacksquare_!$ (Lemma 4.48); thus, the upshot of this detour is that T is also a right adjoint. It will, however, be easier to study the adjunction $\blacktriangle_! \dashv \blacktriangle^*$ than $\mathrm{T} \dashv N_\boxslash$, in particular because both $\blacktriangle_!$ and $\blacktriangle^*$ are left Quillen adjoints (Corollary 4.53 and Lemma 4.54). We will first show in Section 7.1 that $\blacktriangle_! \dashv \blacktriangle^*$ is a Quillen equivalence, then deduce formally that $\blacktriangle^* \dashv \blacktriangle_*$ and $\mathrm{T} \dashv N_\boxslash$ are also Quillen equivalences.

Definition 4.36 An *idempotent* in a category $\mathbf{C}$ is a morphism $f\colon A \to A$ such that $ff = f$. A *splitting* for an idempotent is a section–retraction pair (s, r) such that $f = sr$.

The splitting of an idempotent is unique up to isomorphism if it exists: s is the equalizer of the pair $f, \mathrm{id}\colon A \to A$, while r is the coequalizer of the same. We say that $\mathbf{C}$ is *idempotent complete* if every idempotent splits.

Definition 4.37 An *idempotent completion* of a category $\mathbf{C}$ is a fully faithful functor $i\colon \mathbf{C} \to \overline{\mathbf{C}}$ such that $\overline{\mathbf{C}}$ is idempotent complete and every object in $\overline{\mathbf{C}}$ is a retract of iA for some $A \in \mathbf{C}$.

Equivalently, an idempotent completion is a universal (in a bicategorical sense) fully faithful functor $\mathbf{C} \to \overline{\mathbf{C}}$ into an idempotent complete category. We shall only need the following consequence of this characterization.

Proposition 4.38 (essentially [BD86, Theorem 1]) *Given an idempotent completion $i\colon \mathbf{C} \to \overline{\mathbf{C}}$, the induced substitution functor $i^*\colon \mathrm{PSh}(\overline{\mathbf{C}}) \to \mathrm{PSh}(\mathbf{C})$ is an equivalence of categories.*

We can describe the idempotent completion of $\square_\vee$ concretely as a full subcategory of **SLat**.

Definition 4.39 Write $\overline{\square}_\vee$ for the full subcategory of **SLat** consisting of finite inhabited distributive lattices. This subcategory contains all of $\square_\vee$; we write $\blacksquare\colon \square_\vee \to \overline{\square}_\vee$ for the inclusion.

Remark 4.40 Any finite inhabited lattice is bounded, with $\top$ and $\bot$ obtained as the join and meet of all elements, respectively. Moreover, a finite lattice is distributive if and only if it is a Heyting algebra, i.e., supports an implication operator $\Rightarrow$. Note however that we do not require the morphisms of $\overline{\square}_\vee$ to preserve $\wedge$, $\bot$, $\top$, or $\Rightarrow$, only binary (i.e., non-empty finite) joins.

We show that $\blacksquare\colon \square_\vee \to \overline{\square}_\vee$ is an idempotent completion using the following observations of Horn and Kimura.

Proposition 4.41 [HK71, Theorem 1.1] *A morphism in* **SLat** *is epic if and only if it is surjective.*

Proposition 4.42 [HK71, Corollaries 2.9 and 5.4] *Recall that an object in a category is injective if maps into it extend along monomorphisms, and dually projective if maps out of it lift along epimorphisms. A finite semilattice $A \in \mathbf{SLat}_{\mathrm{fin}}$ is*

- *injective if and only if A is a distributive lattice;*
- *projective if and only if $1 \star A$ is a distributive lattice.*

Corollary 4.43 *$\overline{\square}_\vee$ is closed under retracts in* **SLat**.

Proof A retract of an inhabited finite semilattice is clearly inhabited and finite, and the class of injective objects is closed under retracts in any category. ■

Corollary 4.44 *$\overline{\square}_\vee$ is idempotent complete.*

Proof Note that **SLat** is idempotent complete because it has limits. The claim follows from this using Corollary 4.43. ■

Lemma 4.45 *Any $A \in \overline{\square}_\vee$ is a retract of $[1]^n$ for some $n \in \mathbb{N}$.*

Proof For any $A \in \overline{\square}_\vee$, we have a poset map $p\colon 1 \star UA \to A$ sending $\bot$ to $\bot$ and $a \in UA$ to a. Per Proposition 4.7, this induces a surjective semilattice map $p^\dagger\colon [1]^{UA} \to A$. This is epic by Proposition 4.41. As A is distributive, so too is $1 \star A$, so A is projective. Thus, the identity on A factors through $p^\dagger$, exhibiting A as a retract of $[1]^{UA}$. ■

Theorem 4.46 *$\blacksquare\colon \square_\vee \to \overline{\square}_\vee$ is an idempotent completion.*

Proof By Corollary 4.44 and Lemma 4.45. ■

Recall from Remark 4.4 that we have an embedding $\Delta \to \mathbf{SLat}$. The induced lattice structure on a simplex is distributive, so this embedding factors through $\overline{\square}_\vee$.

Notation 4.47 *We write $\blacktriangle\colon \Delta \to \overline{\square}_\vee$ for the inclusion of the simplices among the finite inhabited distributive semilattices.*

We can now decompose the triangulation functor.

Lemma 4.48 *We have* $\blacktriangle^* \blacksquare_! \cong \mathrm{T} \colon \mathrm{PSh}(\square_\vee) \to \mathrm{PSh}(\Delta)$.

Proof As both functors are left adjoints and thus cocontinuous, it suffices to exhibit a natural isomorphism between their restrictions to representables, i.e., show that $\blacktriangle^* \text{よ} \blacksquare \cong \boxslash$. Both $\blacktriangle^* \text{よ} \blacksquare$ and $\boxslash$ preserve products, and $\blacktriangle^* \text{よ} \blacksquare[1] \cong \Delta^1 \cong \boxslash[1]$ by inspection. ■

4.4 Two Quillen adjunctions

In light of the equivalence $\mathrm{PSh}(\overline{\square}_\vee) \simeq \mathrm{PSh}(\square_\vee)$, it now suffices to compare $\widehat{\Delta}^{\mathrm{kq}}$ with the induced model structure $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$ on $\mathrm{PSh}(\overline{\square}_\vee)$, which again has monomorphisms for cofibrations and fibrations generated by pushout products $\delta_k \mathbin{\widehat{\times}} m$. We begin by observing that both $\blacktriangle_!$ and $\blacktriangle^*$ are left Quillen adjoints.

Lemma 4.49 $\blacktriangle_!$ *preserves monomorphisms.*

Proof Write Δ_{a} for the augmented simplex category, the full subcategory of **Pos** consisting of the objects $[n]$ for $n \in \mathbb{N}$ as well as $[-1] \coloneqq \varnothing$. Write $\overline{\square}_{\vee \mathrm{a}}$ for the category of finite distributive semilattices, which similarly freely extends $\overline{\square}_\vee$ with an initial object (the empty semilattice). The inclusion $\blacktriangle$ extends to an inclusion between the augmented categories:

$$\begin{array}{ccc} \Delta & \xrightarrow{\blacktriangle} & \overline{\square}_\vee \\ {\scriptstyle \iota}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ \Delta_{\mathrm{a}} & \xrightarrow[\blacktriangle_{\mathrm{a}}]{} & \overline{\square}_{\vee \mathrm{a}}. \end{array}$$

The square is a pullback and the vertical maps are (discrete) Grothendieck opfibrations, so the square is exact in the sense that the canonical map $v^*(\blacktriangle_{\mathrm{a}})_! \to \blacktriangle_! \iota^*$ is invertible [nLa24, Proposition 5.2 and Corollary 3.1]. This is also straightforward to check directly: the functors are cocontinuous, so it suffices to check on representables, and ι^* and v^* preserve all representables except for the initial representable, which they send to an initial object. Since ι is fully faithful, this gives $\blacktriangle_! \cong \blacktriangle_! \iota^* \iota_* \cong v^*(\blacktriangle_{\mathrm{a}})_! \iota_*$. Therefore, it suffices to prove that $(\blacktriangle_{\mathrm{a}})_!$ preserves monomorphisms.

Just like in simplicial sets, the monomorphisms in augmented simplicial sets form the left class of a weak factorization system generated by boundary inclusions (of augmented simplices). As $(\blacktriangle_{\mathrm{a}})_!$ is a left adjoint, it therefore suffices to show that it sends boundary inclusions to monomorphisms. The boundary inclusion $\partial\Delta^n \rightarrowtail \Delta^n$ is the joint image of the non-identity face maps $\Delta^k \xrightarrow{+} \Delta^n$. The joint image of a set of maps $(f_i : A_i \to B)_{i \in I}$ in any pretopos is computed as the coequalizer of

$$\coprod_{i,j \in I} A_i \times_B A_j \rightrightarrows \coprod_{i \in I} A_i.$$

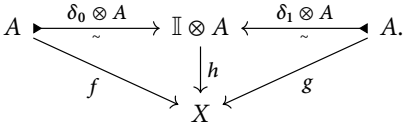
It, therefore, suffices to check that $(\blacktriangle_{\mathrm{a}})_!$ sends face maps to monomorphisms and preserves pullbacks of cospans whose legs are face maps. As face maps are monic, the latter condition implies the former. For the latter condition, as face maps go between representables and Δ_{a} has these pullbacks, it suffices to check that $\blacktriangle_{\mathrm{a}}$ preserves

pullbacks of cospans whose legs are face maps. In fact, $\blacktriangle_a$ creates such pullbacks, as any subposet of a linear poset is again linear. ∎

The following statements can be phrased more generally at the level of cylinder objects in a model category. They also have evident dual version in terms of path objects with fibrancy assumptions instead.

Lemma 4.50 *In a cylindrical model category, let maps $f, g\colon A \to X$ be related by a homotopy $h\colon \mathbb{I} \otimes A \to X$. If A is cofibrant, then f is a weak equivalence exactly if g is.*

Proof The top maps in the following diagram are trivial cofibrations because A is cofibrant:



The claim follows using 2-out-of-3. ∎

In a cylindrical model category, a *homotopy retract* is a pair of maps $s\colon X \to Y$, $r\colon Y \to X$ equipped with a homotopy $h\colon \mathbb{I} \otimes X \to X$ from rs to id_X.

Corollary 4.51 *In a cylindrical model category, any cofibrant homotopy retract of a weakly contractible object is weakly contractible.*

Proof Let a homotopy retract $s\colon X \to Y$, $r\colon Y \to X$, $h\colon \mathbb{I} \otimes X \to X$ from rs to id_X be given with X cofibrant and Y weakly contractible. By Lemma 4.50, rs is a weak equivalence. Since Y is weakly contractible, any endomorphism on Y is a weak equivalence by 2-out-of-3. As the two binary sub-composites of the ternary composite $X \xrightarrow{s} Y \xrightarrow{r} X \xrightarrow{s} Y$ are weak equivalences, both r and s are weak equivalences by 2-out-of-6 [Rie14, Remark 2.1.3]. ∎

Lemma 4.52 *Consider a model category $\mathbf{M}$ and a left adjoint $L\colon \widehat{\Delta}^{\mathrm{kq}} \to \mathbf{M}$ that preserves cofibrations. Then, L is a left Quillen adjoint exactly if it sends representables to weakly contractible objects.*

Proof For the non-trivial direction, assume that L sends representables to weakly contractible objects. Given $n \geq 1$ and $I \subseteq [n]$, write Λ^n_I for the union of the subobjects $d_i\colon \Delta^{n-1} \rightarrowtail \Delta^n$ over $i \in I$. We check by induction that L sends $\Lambda^n_I \rightarrowtail \Delta^n$ to a trivial cofibration for $n \in \mathbb{N}$ and $\varnothing \subsetneq I \subsetneq [n]$. When $|I| = 1$, Λ^n_I is the representable Δ^{n-1}, so the claim holds by assumption and 2-out-of-3. Otherwise, choose some $i \in I$. We have the following pushout square, which is preserved by L:

$$\begin{array}{ccc} \Lambda^{n-1}_{d_i^{-1}(I)} & \longrightarrow & \Lambda^n_{I\setminus\{i\}} \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow[d_i]{} & \Lambda^n_I. \end{array}$$

By induction hypothesis, L sends the left vertical map to a trivial cofibration. As trivial cofibrations are closed under cobase change, L then also sends the right vertical

map to a trivial cofibration. By induction hypothesis, L sends $\Lambda^n_{I\setminus\{i\}} \mapsto \Delta^n$ to a trivial cofibration. By 2-out-of-3, we conclude that L sends $\Lambda^n_{I\setminus\{i\}} \mapsto \Delta^n$ to a trivial cofibration. For $I = [n]\backslash k$, we obtain that L sends the horn inclusion $\Lambda^n_k \to \Delta^n$ to a trivial cofibration. This makes L a left Quillen adjoint. ∎

The combinatorics of the above proof have a conceptual explanation in terms of the pushout join in augmented simplicial sets, which produces boundary inclusions and horn inclusions starting from the maps $\varnothing \to 1$ and $\Delta^{-1} \to 1$.

Corollary 4.53 (cf. [Sat19, Proposition 3.6]) *$\blacktriangle_!$ is a left Quillen adjoint $\widehat{\Delta}^{\mathrm{kq}} \to \widehat{\square}^{\mathrm{ty}}_\vee$.*

Proof By Lemma 4.49, $\blacktriangle_!$ preserves monomorphisms. Using Lemma 4.52, it suffices to show that $\blacktriangle_!\Delta^n \cong \bigstar[n]$ is weakly contractible for $n \in \mathbb{N}$. For this, we observe that $\bigstar[n]$ is a homotopy retract of 1 for each $n \in \mathbb{N}$ via the homotopy $(t, i) \mapsto (t \vee i)\colon [1] \times [n] \to [n]$ and apply Corollary 4.51. ∎

Lemma 4.54 (cf. [Sat19, Section 3.3]) *$\blacktriangle^*$ is a left Quillen adjoint $\widehat{\square}^{\mathrm{ty}}_\vee \to \widehat{\Delta}^{\mathrm{kq}}$.*

Proof $\blacktriangle^*$ preserves monomorphisms because it is a right adjoint. As it is also a left adjoint, it also preserves pushout products, so $\blacktriangle^*(\delta_k \mathbin{\widehat{\times}} m) \cong \blacktriangle^*\delta_k \mathbin{\widehat{\times}} \blacktriangle^* m \cong d_{1-k} \mathbin{\widehat{\times}} \blacktriangle^* m$ is a trivial cofibration for any $k \in \{0,1\}$ and $m\colon A \rightarrowtail B$. ∎

We quickly see that $\blacktriangle_! \dashv \blacktriangle^*$ is a Quillen coreflection in the following sense.

Lemma 4.55 *The derived unit $X \xrightarrow{\eta_X} \blacktriangle^*\blacktriangle_! X \to \blacktriangle^*((\blacktriangle_! X)^{\mathrm{fib}})$ is valued in weak equivalences.*

Proof It is equivalent to prove the unit η is valued in weak equivalences: any fibrant replacement map $\blacktriangle_! X \xrightarrow{\sim} (\blacktriangle_! X)^{\mathrm{fib}}$ is a trivial cofibration, so is mapped to a trivial cofibration by the left Quillen adjoint $\blacktriangle^*$. But $\blacktriangle$ is fully faithful, so the unit is valued in isomorphisms. ∎

5 Relatively elegant Reedy categories

To show that the adjunction $\blacktriangle_! \dashv \blacktriangle^*$ defined in Section 4.4 is a Quillen equivalence, it remains to check that its counit is valued in weak equivalences, that is, that $\varepsilon_X\colon \blacktriangle_!\blacktriangle^* X \to X$ is a weak equivalence for every fibrant $X \in \mathrm{PSh}(\overline{\square}_\vee)$. We noted earlier (Proposition 2.20) that for an elegant Reedy category $\mathbf{R}$, we have a convenient set of objects—the automorphism quotients of representables—that generate the whole of $\mathrm{PSh}(\mathbf{R})$ upon saturation by monomorphisms. We will see later on (Corollary 7.3) that the class of $X \in \mathrm{PSh}(\overline{\square}_\vee)$ for which ε_X is a weak equivalence is saturated by monomorphisms, so if $\overline{\square}_\vee$ were an elegant Reedy category we would have a line of attack. Unfortunately, this is not the case. Indeed, $\overline{\square}_\vee$ is not a Reedy category at all (Proposition A.1).

We, therefore, require a generalization of elegant Reedy theory. We consider categories $\mathbf{C}$ equipped with a fully faithful functor $i\colon \mathbf{C} \to \mathbf{R}$ into a Reedy category $\mathbf{R}$ that has pushouts of lowering spans and is *elegant relative to i*: such that $N_i \coloneqq i^*\bigstar\colon \mathbf{R} \to \mathrm{PSh}(\mathbf{C})$ preserves lowering pushouts. In this case, the objects of $\mathrm{PSh}(\mathbf{C})$ are generated upon saturation by monos from the set of automorphism quotients of objects in the image of N_i. When $i = \mathrm{id}$, we recover the original theorem for elegant

Reedy categories. In Section 6, we shall see that $\overline{\square}_\vee$ embeds elegantly in the category of inhabited finite semilattices.

In Section 5.1, we review Reedy monomorphisms and the construction of cellular presentations for maps between presheaves over a Reedy category. In Section 5.2, we narrow our focus to what we call *pre-elegant* Reedy categories, those having pushouts of lowering spans. The Reedy monic presheaves are, in this case, characterized as those sending lowering pushouts to pullbacks. This sets the stage for Section 5.3, where we define and study elegance relative to an embedding $i\colon \mathbf{C} \to \mathbf{R}$.

5.1 Cellular presentations and Reedy monomorphisms

For the theory of cellular presentations of diagrams over Reedy categories, we follow Riehl and Verity [RV14, Rie17]. Almost none of the content in this section is novel. For simplicity, we restrict our attention throughout to presheaves, though much of the theory generalizes to functors from a Reedy category into any category.

5.1.1 Weighted colimits

Riehl and Verity observe that many arguments in Reedy category theory are naturally phrased in terms of *weighted (co)limits*. While more fundamental to enriched category theory, these can have a clarifying role even in ordinary (i.e., **Set**-enriched) category theory.

Definition 5.1 Let $\mathbf{E}$ be a category. Let a functor $W\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ (the *weight*) and a diagram $F\colon \mathbf{C} \to \mathbf{E}$ be given. A *weighted colimit* for this data is an object $W \circledast_{\mathbf{C}} F \in \mathbf{E}$, equipped with a natural transformation $W \to \mathbf{E}(F-, W \circledast_{\mathbf{C}} F)$, such that for any $X \in \mathbf{E}$ the induced map

$$\mathbf{E}(W \circledast_{\mathbf{C}} F, X) \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}](W, \mathbf{E}(F-, X))$$

of sets is an isomorphism.

Informally, the weight W specifies how many "copies" of each object in the diagram F to include in the weighted colimit $W \circledast_{\mathbf{C}} F$.

Example 5.2 The ordinary colimit of a diagram $F\colon \mathbf{C} \to \mathbf{E}$ can be described as $1 \circledast_{\mathbf{C}} F$, a colimit weighted by the terminal presheaf $1 \in \mathrm{PSh}(\mathbf{C})$. Conversely, any weighted colimit $W \circledast_{\mathbf{C}} F$ admits a characterization as an ordinary colimit over the category of elements of W:

$$W \circledast_{\mathbf{C}} F \cong \mathrm{colim}\left(\mathbf{el}W \xrightarrow{\pi} \mathbf{C} \xrightarrow{F} \mathbf{E}\right).$$

In particular, any cocomplete category has weighted colimits.

Example 5.3 Recall that a *tensor* of a set $S \in \mathbf{Set}$ and object $X \in \mathbf{E}$ is an object $S * X$ such that morphisms $S * X \to Y$ correspond to objects $\mathbf{Set}(S, \mathbf{E}(X, Y))$, i.e., families of morphisms $f_s\colon X \to Y$ for $s \in S$. In ordinary category theory, this is simply the S-ary coproduct $\coprod_{s \in S} X$, so can be expressed as the weighted colimit $1 \circledast_S \Delta X$ of the constant diagram $\Delta X\colon S \to \mathbf{E}$. Alternatively, we can encode the tensor as the S-weighted colimit $S \circledast_{\mathbf{1}} X$ of the diagram $X\colon \mathbf{1} \to \mathbf{E}$ over the terminal category. We can characterize any

weighted colimit $W \otimes_{\mathbf{C}} F$ as a coend of tensors:

$$W \otimes_{\mathbf{C}} F \cong \int^{c \in \mathbf{C}} W_c * F^c.$$

We will always be working in cocomplete categories. For a given $\mathbf{C}$, weighted colimits over $\mathbf{C}$ are then functorial in both the weight and the diagram, giving a bifunctor $\otimes_{\mathbf{C}} : [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] \times [\mathbf{C}, \mathbf{E}] \to \mathbf{E}$. This functoriality will be an essential tool. In particular, we will often take a family of weighted colimits over a family of weights.

Notation 5.4 *Given a family of weights $W : \mathbf{D} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and $F : \mathbf{C} \to \mathbf{E}$, we write $W \otimes_{\mathbf{C}} F : \mathbf{D} \to \mathbf{E}$ for the result of calculating the weighted colimit pointwise, that is, $(W \otimes_{\mathbf{C}} F)^d \coloneqq W^d \otimes_{\mathbf{C}} F$.*

Remark 5.5 From the characterization in terms of ordinary colimits, it follows that weighted colimits in presheaf categories are computed pointwise. Thus, for $W : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and $F : \mathbf{C} \times \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$, we have $(W \otimes_{\mathbf{C}} F)_d \cong W \otimes_{\mathbf{C}} F_d$, where on the left we regard F as a functor $\mathbf{C} \to \mathrm{PSh}(\mathbf{D})$.

It follows quickly from the universal property defining weighted colimits that the bifunctor $\otimes_{\mathbf{C}}$ preserves colimits in both arguments. It is therefore determined by its behavior on representable weights, which is simply characterized.

Proposition 5.6 *Naturally in $c \in \mathbf{C}$ and $X : \mathbf{C} \to \mathbf{E}$, we have $\text{よ} c \otimes_{\mathbf{C}} X \cong X^c$.*

Corollary 5.7 *Naturally in $W : \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$, $V : \mathbf{D} \times \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and $F : \mathbf{C} \to \mathbf{E}$, we have $(W \otimes_{\mathbf{D}} V) \otimes_{\mathbf{C}} F \cong W \otimes_{\mathbf{D}} (V \otimes_{\mathbf{C}} F)$.*

Proof By cocontinuity, it suffices to check the case where W is representable. ∎

Notation 5.8 *In this section, we use the notation $\text{よ}\mathbf{C} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ for the hom-bifunctor $\mathbf{C}(-,-)$. Thus, the representable functor for $c \in \mathbf{C}$, written $\text{よ} c$ in our usual notation, may now be written as $\text{よ}\mathbf{C}^c$, while we also have the co-representable $\text{よ}\mathbf{C}_c : \mathbf{C} \to \mathbf{Set}$. With our notation for parameterized weighted colimits, Proposition 5.6 then tells us that $\text{よ}\mathbf{C} \otimes_{\mathbf{C}} X \cong X$ for any $X \in \mathrm{PSh}(\mathbf{C})$. We have an analogous equation in the second argument: $X \otimes_{\mathbf{C}} \text{よ}\mathbf{C} \cong X$.*

5.1.2 Cellular presentations of presheaves

A central theorem of Reedy theory is the existence of cellular presentations: when $\mathbf{R}$ is a Reedy category, any $\mathbf{R}$-indexed diagram is a sequential colimit of maps that successively attach cells of increasing degree. Likewise, any natural transformation between $\mathbf{R}$-indexed diagrams decomposes as a transfinite composite of such maps. In the Riehl–Verity style, the intermediate objects and maps are obtained by taking (Leibniz) weighted colimits of the input diagram. As $X \cong \text{よ}\mathbf{R} \otimes_{\mathbf{R}^{\mathrm{op}}} X$ for any diagram X, one can exhibit a cellular presentation for X by constructing a cellular presentation for $\text{よ}\mathbf{R}$ and then applying the cocontinuous functor $(-) \otimes_{\mathbf{R}^{\mathrm{op}}} X$.

For the remainder of this section, we fix a Reedy category $\mathbf{R}$.

Definition 5.9 For each $n \in \mathbb{N}$, define $\partial\mathbf{R} : \mathrm{sk}_{<n}\mathbf{R} \nrightarrow \text{よ}\mathbf{R}$ to be the subfunctor of arrows of degree less than n.

Definition 5.10 For any $n \in \mathbb{N}$, write $\mathbf{R}[n]$ for the subcategory of $\mathbf{R}$ consisting of objects of degree n and isomorphisms between them. We introduce the following notation for restrictions of $ょ\mathbf{R}$ where one argument or the other is required to have a given degree:

$$\begin{array}{ccccc} \mathbf{R}[n] \times \mathbf{R} & \rightarrowtail & \mathbf{R}^{\mathrm{op}} \times \mathbf{R} & \leftarrowtail & \mathbf{R}^{\mathrm{op}} \times \mathbf{R}[n] \\ & \searrow & \downarrow & \swarrow & \\ & & ょ\mathbf{R} & & \\ & & \downarrow & & \\ & & \mathbf{Set} & & \end{array}$$

We similarly introduce notation for the corresponding restrictions of the skeleton bifunctor $\mathrm{sk}_{<n}\mathbf{R} \colon \mathbf{R}^{\mathrm{op}} \times \mathbf{R} \to \mathbf{Set}$:

$$\begin{array}{ccccc} \mathbf{R}[n] \times \mathbf{R} & \rightarrowtail & \mathbf{R}^{\mathrm{op}} \times \mathbf{R} & \leftarrowtail & \mathbf{R}^{\mathrm{op}} \times \mathbf{R}[n] \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{sk}_{<n}\mathbf{R} & & \\ & & \downarrow & & \\ & & \mathbf{Set} & & \end{array}$$

Finally, we write $ə_n\mathbf{R} \colon \partial_n\mathbf{R} \mathrel{↣} ょ_n\mathbf{R}$ and $ə^n\mathbf{R} \colon \partial^n\mathbf{R} \mathrel{↣} ょ^n\mathbf{R}$ for the restrictions of the inclusion $ə\mathbf{R} \colon \mathrm{sk}_{<n}\mathbf{R} \mathrel{↣} ょ\mathbf{R}$.

Notation 5.11 *For $r \in \mathbf{R}$ of degree n, we abbreviate $\partial_r\mathbf{R} \coloneqq (\mathrm{sk}_{<n}\mathbf{R})_r \colon \mathbf{R} \to \mathbf{Set}$ and $\partial^r\mathbf{R} \coloneqq (\mathrm{sk}_{<n}\mathbf{R})^r \colon \mathbf{R}^{\mathrm{op}} \to \mathbf{Set}$. Likewise, we write $ə_r\mathbf{R} \coloneqq (ə\mathbf{R})_r \colon \partial_r\mathbf{R} \mathrel{↣} ょ\mathbf{R}_r$ and $ə^r\mathbf{R} \coloneqq (ə\mathbf{R})^r \colon \partial^r\mathbf{R} \mathrel{↣} ょ\mathbf{R}^r$.*

Definition 5.12 For any $f \colon X \to Y$ in $\mathrm{PSh}(\mathbf{R})$ and $n \in \mathbb{N}$, the *$<n$-skeleton map* for f is the Leibniz weighted colimit

$$(\mathrm{sk}_{<n}\mathbf{R} \rightarrowtail ょ\mathbf{R}) \mathbin{\widehat{\circledast}_{\mathbf{R}^{\mathrm{op}}}} f.$$

We write $\mathrm{sk}_{<n} f \in \mathrm{PSh}(\mathbf{R})$ for the domain of this map, which we call the *$<n$-skeleton* of f; its codomain is Y. For $Y \in \mathrm{PSh}(\mathbf{R})$, we write $\mathrm{sk}_{<n} Y$ for the n-skeleton of the map $0 \mathrel{↣} Y$.

Note that the <0-skeleton map is $(0 \mathrel{↣} ょ\mathbf{R}) \mathbin{\widehat{\circledast}_{\mathbf{R}^{\mathrm{op}}}} f \cong ょ\mathbf{R} \mathbin{\circledast_{\mathbf{R}^{\mathrm{op}}}} f \cong f$. For each $m \leq n \in \mathbb{N}$, the inclusion $\mathrm{sk}_{<m}\mathbf{R} \mathrel{↣} \mathrm{sk}_{<n}\mathbf{R}$ induces a morphism $\mathrm{sk}_{<m} f \to \mathrm{sk}_{<n} f$ by functoriality of weighted colimits, and the fact that $ょ\mathbf{R}$ is the union of the subfunctors $\mathrm{sk}_{<n}\mathbf{R}$ implies that $Y \cong \mathrm{colim}_{n \in \mathbb{N}} \mathrm{sk}_{<n} f$. Thus, we have a natural decomposition of f as the transfinite composite $\mathrm{sk}_{<0} f \to \mathrm{sk}_{<1} f \to \mathrm{sk}_{<2} f \to \cdots$ where we may compute $\mathrm{sk}_{<n} f \cong X \sqcup_{\mathrm{sk}_{<n} X} \mathrm{sk}_{<n} Y$. The chain of skeleta may be further decomposed in terms of *latching maps*.

Definition 5.13 Given $f \colon X \to Y$ in $\mathrm{PSh}(\mathbf{R})$ and $r \in \mathbf{R}$, define the *latching map* $\widehat{\ell}_r f \in \mathbf{Set}^{\to}$ *for f at r* by the Leibniz weighted colimit

$$\widehat{\ell}_r f \coloneqq ə_r\mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}^{\mathrm{op}}}} f.$$

The codomain of this map is Y_r; we write $L_r f$ for its domain and call this the *latching object for f at r*.

We write $\widehat{\ell}_r Y$ and $L_r Y$ for the latching map and object of $0 \rightarrowtail Y$ at r. For general $f : X \to Y$, we can calculate that $L_r f \cong X_r \sqcup_{L_r X} L_r Y$ and $\widehat{\ell}_r f \cong [f_r, L_r f]$. It is convenient to have notation for the collected $\mathbf{R}[n]$-sets of latching maps at a given degree.

Definition 5.14 Given $f : X \to Y$ and $n \in \mathbb{N}$, we define the *nth latching map of* f by $\widehat{\ell}_n f \coloneqq \eth_n \mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}^{\mathrm{op}}}} f$. We write $L_n f \in \mathrm{PSh}(\mathbf{R}[n])$ for its domain and $f_n \in \mathrm{PSh}(\mathbf{R}[n])$ for its codomain.

These maps are assembled from the latching maps at the individual objects of degree n: we have $(\widehat{\ell}_n f)_r \cong \widehat{\ell}_r f$ for each $r \in \mathbf{R}[n]$.

We can now exhibit the maps between successive $<n$-skeleta as pushouts of Leibniz weighted colimits of boundary inclusions and latching maps. The induced decomposition of a map f into a sequential colimit of pushouts of basic maps is what we mean by a *cellular presentation* of f.

Proposition 5.15 [Rie17, Corollary 4.21] *For any $f : X \to Y$ and $n \in \mathbb{N}$, we have a pushout square of the following form:*

$$\begin{array}{ccc} \bullet & \xrightarrow{\eth^n \mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}[n]^{\mathrm{op}}}} \widehat{\ell}_n f} & \bullet \\ \downarrow & & \ulcorner \downarrow \\ \mathrm{sk}_{<n} f & \longrightarrow & \mathrm{sk}_{<n+1} f. \end{array}$$

We refer to the maps $\eth^n \mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}[n]^{\mathrm{op}}}} \widehat{\ell}_n f$ as cell maps.

Proof By applying $(-) \mathbin{\widehat{\circledast}_{\mathbf{R}^{\mathrm{op}}}} f$ to a pushout square in $\mathbf{R}^{\mathrm{op}} \times \mathbf{R} \to \mathbf{Set}$; see [Rie17, Theorem 4.15]. ∎

Corollary 5.16 *Every $f : X \to Y$ in $\mathrm{PSh}(\mathbf{R})$ has a cellular presentation by maps of the form $\eth^n \mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}[n]^{\mathrm{op}}}} \widehat{\ell}_n f$.*

For our purposes, namely, working with properties saturated by monomorphisms, it is important to know when the cell maps are monic.

Definition 5.17 A map $f : X \to Y$ in $\mathrm{PSh}(\mathbf{R})$ is a *Reedy monomorphism* when $\widehat{\ell}_r f$ is monic in **Set** for all $r \in \mathbf{R}$.

Here and in the following, we are specializing the theory of *Reedy cofibrations* to the (mono, epi) weak factorization system on **Set**. To see when Reedy monomorphisms have monic cell maps, we use the following lemma. Recall that a map is *epi-projective* if it has the left lifting property against all epimorphisms.

Proposition 5.18 *Let $\mathbf{C}$ be a small category, $f \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]^{\to}$, and $g \in [\mathbf{C}, \mathbf{Set}]^{\to}$. If f is epi-projective and g is monic, then $f \mathbin{\widehat{\circledast}_{\mathbf{C}}} g$ is monic.*

Proof By [Rie17, Lemma 3.13 and Corollary 3.17] applied to the (mono, epi) weak factorization system on **Set**. ∎

Lemma 5.19 *If isos act freely on lowering maps in $\mathbf{R}$, then $\eth^n \mathbf{R}_r$ is epi-projective in $\mathbf{R}[n] \to \mathbf{Set}$.*

Proof A given morphism from r to an object of degree n is either a lowering map or has degree less than n. This induces the following coproduct decomposition in $\mathbf{R}[n] \to \mathbf{Set}$:

$$\begin{array}{ccc} 0 & \longrightarrow & \partial^n \mathbf{R}_r \\ \downarrow & & \downarrow \partial^n \mathbf{R}_r \\ \mathbf{R}^-(r,-) & \longrightarrow & \not\exists^n \mathbf{R}_r. \end{array}$$

Since epi-projective is the left class in a weak factorization system, it is stable under cobase change. It thus suffices to show that $\mathbf{R}^-(r,-)$ is epi-projective. Since isos act freely on $\mathbf{R}^-(r,-)$, it is the left Kan extension along some functor $A \to \mathbf{R}[n]$ of some $F\colon A \to \mathbf{Set}$ with A a set. Recall that epimorphisms are characterized levelwise in **Set**-valued functors. By adjoint transposition, it thus suffices to show that F is epi-projective. Since A is a set, this just means that F is levelwise epi-projective. And in **Set**, every object is epi-projective. ∎

Corollary 5.20 *Suppose that isos act freely on lowering maps in* $\mathbf{R}$*. Given a Reedy monic* $f \in \mathrm{PSh}(\mathbf{R})^{\to}$*, the map* $\partial^n \mathbf{R} \mathbin{\widehat{\circledast}}_{\mathbf{R}[n]^{\mathrm{op}}} \widehat{\ell}_n f$ *is monic for all* $n \in \mathbb{N}$*.*

Proof We have $(\partial^n \mathbf{R} \mathbin{\widehat{\circledast}}_{\mathbf{R}[n]^{\mathrm{op}}} \widehat{\ell}_n f)_r = \partial^n \mathbf{R}_r \mathbin{\widehat{\circledast}}_{\mathbf{R}[n]^{\mathrm{op}}} \widehat{\ell}_n f$ for every $r \in \mathbf{R}$. We know $\partial^n \mathbf{R}_r$ is epi-projective by Lemma 5.19, and $\widehat{\ell}_n f$ is monic by assumption, so their Leibniz weighted colimit is monic by Proposition 5.18. ∎

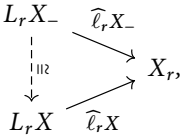
5.1.3 EZ decompositions

The Reedy monomorphisms with initial domain can be characterized more simply: an object X is Reedy monic exactly if every element of X writes uniquely up to isomorphism as a degeneracy of a non-degenerate element of X. We are not aware of a proof of this precise statement (Lemma 5.24) in the literature, though we would be surprised if it were unknown. We use Cisinski's term "EZ decomposition" [Cis06, Proposition 8.1.13] for what Berger and Moerdijk call *standard decompositions*.

Definition 5.21 Let $X \in \mathrm{PSh}(\mathbf{R})$. We say that $x \in X_r$ is *non-degenerate* when every lowering map $e\colon r \xrightarrow{-} s$ admitting an $x' \in X_s$ with $x'e = x$ is an isomorphism. An *EZ decomposition* of $x \in X_r$ is a pair (e, x'), where $x' \in X_s$ is non-degenerate, $e\colon r \to s$ is a lowering map, and $x = x'e$. We regard two EZ decompositions (e_0, x_0') and (e_1, x_1') of x as *isomorphic* when there exists an isomorphism $\theta\colon s_0 \cong s_1$ in $\mathbf{R}$ such that $x_0'\theta = x_1'$ and $e_0 = e_1\theta$. We say X has *unique EZ decompositions* when any two EZ decompositions of any element of X are isomorphic.

Remark 5.22 Every element of a presheaf admits at least one EZ decomposition: for any $x \in X_r$, there exists a minimal $n \in \mathbb{N}$ such that x factors though a lowering map to an object of degree n, and any such factorization is an EZ decomposition.

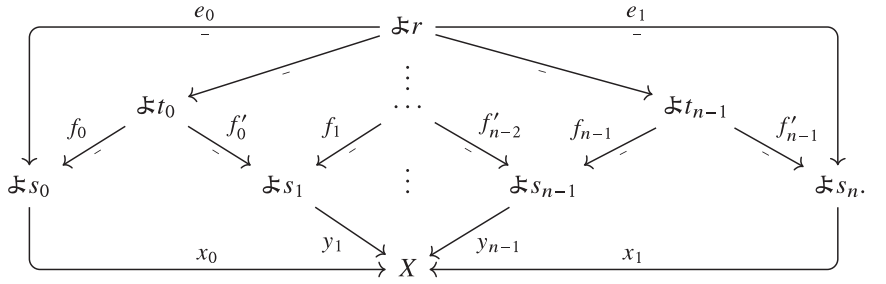
Proposition 5.23 [RV14, Observation 3.23] *Given $X \in \mathrm{PSh}(\mathbf{R})$ and $r \in \mathbf{R}$, we have an isomorphism*



where $X_- \in \mathrm{PSh}(\mathbf{R}^-)$ is the restriction of X along the Reedy category inclusion $\mathbf{R}^- \to \mathbf{R}$.

Lemma 5.24 *A presheaf $X \in \mathrm{PSh}(\mathbf{R})$ is Reedy monic if and only if it has unique EZ decompositions.*

Proof Suppose that X is Reedy monic. We show that any two EZ decompositions of any $x \in X_r$ are isomorphic by induction on $|r|$. Let two such factorizations (e_0, x_0), (e_1, x_1) be given. If either of e_0 or e_1 is an isomorphism, then the other must be as well, in which case the factorizations are trivially isomorphic; thus, we can assume that each e_k strictly decreases degree. Then, (e_0, x_0) and (e_1, x_1) belong to L_rX_-; because X is Reedy monic, they are moreover equal therein. By the concrete characterization of colimits in **Set**, we have a finite sequence of lowering spans $s_i \xleftarrow{f_i} t_i \xrightarrow{f_i'} s_{i+1}$ for $0 \le i < n$, always with $|s_i|, |t_i| < |r|$, together with elements $y_i \colon \text{よ}s_i \to X$ for each $i \le n$, such that $y_0 = x_0$, $y_n = x_1$, and $y_i f_i = y_{i+1} f_i'$:



By taking an EZ decomposition of each y_i and absorbing the lowering map into f_i', f_{i+1}, we can assume without loss of generality that each y_i is non-degenerate. Then, for each i, the equation $y_i f_i = y_{i+1} f_i'$ makes (y_i, f_i) and (y_{i+1}, f_i') EZ decompositions of the same element of X_{t_i}. As $|t_i| < |r|$, it follows by induction hypothesis that they are isomorphic. Chaining these isomorphisms, we conclude that (e_0, x_0) and (e_1, x_1) are isomorphic.

Now, suppose conversely that X has unique EZ decompositions. By Proposition 5.23, it suffices to show the map $L_rX_- \to X_r$ is monic. The elements of L_rX_- are pairs $(e \colon r \xrightarrow{-} s, x \in X_s)$, where e is a strictly lowering map, quotiented by the relation $(fe, x) = (e, xf)$ for any $f \in \mathbf{R}^-$; the latching map sends (e, x) to $xe \in X_r$. Let $(e_0, x_0), (e_1, x_1) \in L_rX_-$ be given such that $x_0 e_0 = x_1 e_1$. Without loss of generality, we may assume that these are EZ decompositions, in which case they are isomorphic and thus equal as elements of L_rX_-. ∎

5.1.4 Saturation by monomorphisms

Now, we check that the class of Reedy monic presheaves is contained in the saturation by monos of the set of automorphism quotients of representables, assuming isos act freely on lowering maps in $\mathbf{R}$.

Lemma 5.25 *For any $X \in \mathrm{PSh}(\mathbf{R}[n])$, the presheaf $ﾖ^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} X$ is a coproduct of automorphism quotients of representables.*

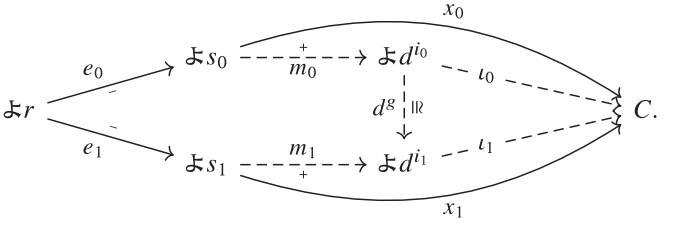
Proof Write $\mathbf{R}[n]$ as a coproduct of groups $\mathbf{R}[n] \cong \coprod_i G_i$. Using the characterization of orbits as quotients by stabilizer groups, we may decompose X as a coproduct of orbits $X \cong \coprod_{i,j} ﾖ r_i/H_{ij}$, where $r_i \in \mathbf{R}$ is the point of G_i. By cocontinuity of $ﾖ^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} (-)$, we then have

$$ﾖ^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} X \cong \coprod_{i,j} (ﾖ^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} ﾖ r_i)/H_{ij} \cong \coprod_{i,j} ﾖ r_i/H_{ij}$$

as desired. ∎

Lemma 5.26 *Any colimit of a groupoid of representables in $\mathrm{PSh}(\mathbf{R})$ is Reedy monic.*

Proof Let a groupoid $\mathbf{G}$ and $d\colon \mathbf{G} \to \mathbf{R}$ be given. Set $C \coloneqq \mathrm{colim}_{i \in \mathbf{G}} ﾖ d^i$. We show that C has unique EZ decompositions. Let two EZ decompositions (e_0, x_0) and (e_1, x_1) of the same element of C be given. As colimits are computed pointwise, each x_k factors as $x_k = \iota_k m_k$ through some leg $\iota_k\colon ﾖ d^{i_k} \to C$ of the coproduct and we have an arrow $g\colon i_0 \cong i_1$ in $\mathbf{G}$ making the following diagram commute:



Each m_k must be a raising map because x_k is non-degenerate. By uniqueness of Reedy factorizations, we have an isomorphism $\theta\colon s_0 \cong s_1$ fitting in the diagram above. ∎

Theorem 5.27 *Let $\mathbf{R}$ be a Reedy category in which isos act freely on lowering maps. Let $\mathcal{P} \subseteq \mathrm{PSh}(\mathbf{R})$ be a class of objects such that*

- *for any $r \in \mathbf{R}$ and $H \leq \mathrm{Aut}_{\mathbf{R}}(r)$, we have $ﾖ r/H \in \mathcal{P}$;*
- *$\mathcal{P}$ is saturated by monomorphisms.*

Then, $\mathcal{P}$ contains every Reedy monic presheaf.

Proof First, we show by induction on n that $\mathrm{sk}_{<n} X \in \mathcal{P}$ for any Reedy monic presheaf X. It then follows that $X \cong \mathrm{colim}_{n \in \mathbb{N}} \mathrm{sk}_{<n} X \in \mathcal{P}$ by saturation.

In the base case, $\mathrm{sk}_{<0}X$ is the empty coproduct and thus belongs to $\mathcal{P}$ by saturation. For any $n \in \mathbb{N}$, we have the following pushout square by Proposition 5.15:

$$\begin{array}{ccc} \bullet & \xrightarrow{\eth^n\mathbf{R} \mathbin{\widehat{\circledast}}_{\mathbf{R}[n]^{\mathrm{op}}} \widehat{\ell_n} X} & \bullet \\ \downarrow & & \downarrow \\ \mathrm{sk}_{<n}X & \longrightarrow & \mathrm{sk}_{<n+1}X. \end{array}$$

The upper horizontal map is monic by Corollary 5.20, the lower by closure of monos in $\mathrm{PSh}(\mathbf{R})$ under cobase change. We have $\mathrm{sk}_{<n}X \in \mathcal{P}$ by induction hypothesis. The upper-right corner is $\text{Ⱶ}^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} X_n$, which belongs to $\mathcal{P}$ by Lemma 5.25. Finally, the upper-left corner is by definition the following pushout object:

$$\begin{array}{ccc} \partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} L_nX & \longrightarrow & \text{Ⱶ}^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} L_nX \\ \downarrow & & \downarrow \\ \partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} X_n & \longrightarrow & \bullet. \end{array}$$

The upper horizontal map is monic by Proposition 5.18 and Lemma 5.19, as we can write it as the pushout product $\eth^n\mathbf{R} \mathbin{\widehat{\circledast}}_{\mathbf{R}[n]^{\mathrm{op}}} (\varnothing \rightarrowtail L_nX)$. The object $\text{Ⱶ}^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} L_nX$ is in $\mathcal{P}$ by Lemma 5.25. Using Corollary 5.7, we have

$$\partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} F \cong (\mathrm{sk}_{<n}\mathbf{R} \circledast_{\mathbf{R}^{\mathrm{op}}} \text{Ⱶ}^n\mathbf{R}) \circledast_{\mathbf{R}[n]^{\mathrm{op}}} F \cong \mathrm{sk}_{<n}\mathbf{R} \circledast_{\mathbf{R}^{\mathrm{op}}} (\text{Ⱶ}^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} F)$$

for any F. The objects $\partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} L_nX$ and $\partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} X_n$ thus belong to $\mathcal{P}$ by Lemmas 5.25 and 5.26 and the induction hypothesis. By saturation, the upper-left corner of our original pushout diagram now belongs to $\mathcal{P}$. For the same reason, we conclude that $\mathrm{sk}_{<n+1}X$ belongs to $\mathcal{P}$. ∎

5.2 Pre-elegant Reedy categories

We next consider the subclass of Reedy categories in which any span of lowering maps has a pushout. This restriction has some simplifying consequences (e.g., that all lowering maps are epic), and we can characterize the Reedy monic presheaves over such categories as those preserving lowering pushouts.

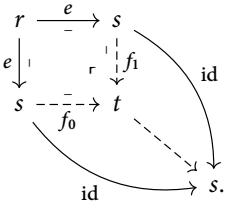
Definition 5.28 A Reedy category is *pre-elegant* when it has pushouts of lowering spans.

Intuitively, this means that any pair of lowering maps from the same object has a universal combination, the diagonal of their pushout. Of course, any elegant Reedy category is pre-elegant, so Δ is one example. Our motivating example is the (surjective, mono) Reedy structure on the category of finite inhabited semilattices, which is pre-elegant but not elegant. In Section 6, we see this is an instance of a general class of examples: the (surjective, mono) Reedy structure on the category $\mathrm{Alg}(\mathbf{T})_{\mathrm{fin}}$ of finite algebras for a Lawvere theory $\mathbf{T}$ is always pre-elegant, but not necessarily elegant.

The following lemma generalizes the fact that any lowering map in an elegant Reedy category is split epic, with essentially the same proof as Bergner and Rezk's Proposition 3.8(3) [BR13].

Lemma 5.29 *Let* **R** *be a pre-elegant Reedy category. Then, any lowering map is epic.*

Proof Consider a lowering map $e : r \to s$. We take the pushout of e with itself, then use its universal property to see that the legs of the pushout are split monic:

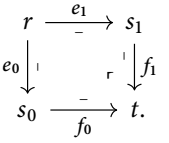


Any split mono is a raising map (Corollary 2.15), so f_0, f_1 are isomorphisms. Thus, e is epic. ■

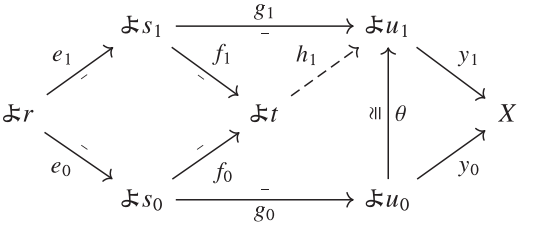
Corollary 5.30 *If* **R** *is a pre-elegant Reedy category, then isos act freely on lowering maps in* **R**.

Lemma 5.31 *Let* **R** *be a Reedy category in which isos act freely on lowering maps. If* $X \in \mathrm{PSh}(\mathbf{R})$ *is Reedy monic, then X sends pushouts of lowering spans (should they exist) to pullbacks.*

Proof Let a pushout square of lowering maps be given like so:



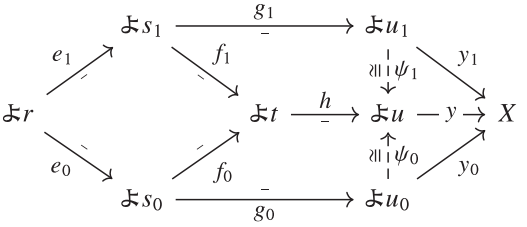
Suppose we have $x_0 \in X_{s_0}$ and $x_1 \in X_{s_1}$ such that $x_0 e_0 = x_1 e_1$; we show this data determines a unique element of X_t restricting to x_0 and x_1. For each $k \in \{0,1\}$, take an EZ decomposition (g_k, y_k) of x_k. Then, $(g_0 e_0, y_0)$ and $(g_1 e_1, y_1)$ are EZ decompositions of the same map, so by Lemma 5.24, they are isomorphic via some $\theta : u_0 \cong u_1$. The universal property of the pushout in **R** then provides a map $h_1 : t \to u_1$ like so:



This gives our desired element $y_1 h_1 \in X_t$ restricting to x_k along each f_k. Note that h_1 is a lowering map by Lemma 2.14.

To see that this element is unique, suppose we have $x \in X_t$ such that $x f_k = x_k$ for $k \in \{0,1\}$. Take an EZ decomposition (h, y) of X, say through $u \in \mathbf{R}$. By uniqueness

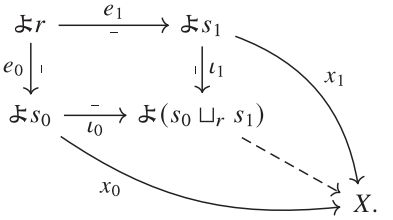
of EZ decompositions, we have isomorphisms ψ_k as shown:



Because isos act freely on lowering maps, we have $\psi_1^{-1}\psi_0 = \theta$. It follows from the universal property of the pushout in $\mathbf{R}$ that $\psi_1 h = h_1$, thus that $yh = y_1 h_1$ as desired. ■

Theorem 5.32 *If* $\mathbf{R}$ *is a pre-elegant Reedy category, then* $X \in \mathrm{PSh}(\mathbf{R})$ *is Reedy monic if and only if it sends pushouts of lowering spans to pullbacks.*

Proof One direction is Lemma 5.31. For the other, suppose X sends pushouts of lowering spans to pullbacks. By Lemma 5.24, it suffices to show X has unique EZ decompositions. Let (e_0, x_0) and (e_1, x_1) be EZ decompositions of the same element. We have an induced element as shown:



By non-degeneracy of x_0 and x_1, the maps ι_0 and ι_1 must be isomorphisms, so (e_0, x_0) and (e_1, x_1) are isomorphic. ■

Remark 5.33 A corollary of the previous theorem is that a pre-elegant Reedy category $\mathbf{R}$ is elegant if and only if all presheaves on $\mathbf{R}$ are Reedy monic. Bergner and Rezk [BR13, Proposition 3.8] show that this bi-implication actually holds for *any* Reedy category. That is, if all presheaves on $\mathbf{R}$ are Reedy monic, then $\mathbf{R}$ is necessarily pre-elegant (and thus elegant).

5.3 Relative elegance

Now, we come to our central definition, elegance of a category *relative to* a full subcategory.

Definition 5.34 We say that a pre-elegant Reedy category $\mathbf{R}$ is *elegant relative to* a fully faithful functor $i\colon \mathbf{C} \to \mathbf{R}$ if the nerve $N_i \coloneqq i^*よ\colon \mathbf{R} \to \mathrm{PSh}(\mathbf{C})$ preserves pushouts of lowering spans. We also say that *i is relatively elegant* with the same meaning.

Remark 5.35 As pushouts in $\mathrm{PSh}(\mathbf{C})$ are computed pointwise, i is relatively elegant if and only if $\mathbf{R}(ia, -)\colon \mathbf{R} \to \mathbf{Set}$ preserves lowering pushouts for all $a \in \mathbf{C}$.

Remark 5.36 A Reedy category is elegant if and only if it is elegant relative to the identity functor, in which case the nerve is simply the Yoneda embedding. At the

other extreme, any pre-elegant Reedy category is elegant relative to the unique functor $\mathbf{0} \to \mathbf{R}$.

Lemma 5.37 *If* $\mathbf{R}$ *is elegant relative to* $i\colon \mathbf{C} \to \mathbf{R}$*, then* $N_i\colon \mathbf{R} \to \mathrm{PSh}(\mathbf{C})$ *sends lowering maps to epimorphisms.*

Proof By Lemma 5.29, any $e \in \mathbf{R}^-$ fits in the pushout square

$$\begin{array}{ccc} r & \xrightarrow{e} & s \\ {\scriptstyle e}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{id}} \\ s & \xrightarrow[\mathrm{id}]{} & s, \end{array}$$

which is then preserved by N_i. ∎

Corollary 5.38 *If* $\mathbf{R}$ *is elegant relative to* $i\colon \mathbf{C} \to \mathbf{R}$*, then objects in the image of* i *are* $\mathbf{R}^-$*-projective: given a lowering map* $e\colon r \to s$ *and* $f\colon ia \to s$*, there exists a lift as below.*

$$\begin{array}{ccc} & & r \\ & \nearrow & \downarrow{\scriptstyle e} \\ ia & \xrightarrow[f]{} & s. \end{array}$$

Proof By Lemma 5.37, $N_i e\colon N_i r \to N_i s$ is epic; this means exactly post-composition with e is a surjective map $\mathbf{R}(ia, r) \to \mathbf{R}(ia, s)$. ∎

Remark 5.39 As a special case of the corollary above, we recover the fact that lowering maps in elegant Reedy categories are split epimorphisms [BR13, Proposition 3.8]. Split epis are lowering maps in any Reedy category (Corollary 2.15), so, in the elegant case, they coincide. It is not generally the case that the lowering maps in a Reedy category $\mathbf{R}$ elegant relative to some i are exactly those sent to epimorphisms by N_i: consider that $\mathbf{R}$ is always elegant relative to $\mathbf{0} \to \mathbf{R}$.

On the basis of Remark 5.35, we can identify the maximal subcategory relative to which a pre-elegant Reedy category $\mathbf{R}$ is elegant.

Definition 5.40 Let $\mathbf{R}$ be a pre-elegant Reedy category. We define its *elegant core* $\mathbf{R}^{\mathrm{ec}}$ to be the full subcategory of $\mathbf{R}$ consisting of objects r such that $\mathbf{R}(r, -)$ preserves lowering pushouts.

Proposition 5.41 *A fully faithful functor* $i\colon \mathbf{C} \to \mathbf{R}$ *into a pre-elegant Reedy category is relatively elegant exactly if it factors through the inclusion* $\mathbf{R}^{\mathrm{ec}} \to \mathbf{R}$.

We can give another characterization of relative elegance in terms of the right Kan extension $i_*\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{R})$.

Lemma 5.42 *Let* $\mathbf{R}$ *be a pre-elegant Reedy category. Then,* $i\colon \mathbf{C} \to \mathbf{R}$ *is relatively elegant if and only if* $i_* X \in \mathrm{PSh}(\mathbf{R})$ *is Reedy monic for every* $X \in \mathrm{PSh}(\mathbf{C})$.

Proof By definition, $i\colon \mathbf{C} \to \mathbf{R}$ is relatively elegant exactly if $N_i = i^* \text{よ}$ preserves lowering pushouts. Testing pushouts by mapping out of them, this holds exactly if

$\mathrm{PSh}(\mathbf{C})(i^*よ-, X)$ sends lowering pushouts to pullbacks for every $X \in \mathrm{PSh}(\mathbf{C})$. Using the natural isomorphism

$$\mathrm{PSh}(\mathbf{C})(i^*よ-, X) \cong \mathrm{PSh}(\mathbf{R})(よ-, i_*X) \cong i_*X,$$

this rewrites to i_*X sending lowering pushouts to pullbacks. ∎

This property of presheaves extends to morphisms as follows.

Definition 5.43 A map $m\colon X \to Y$ in $\mathrm{PSh}(\mathbf{R})$ *reflects degeneracy* if has the right lifting property against lowering maps $e\colon よr \bar{\to} よs$.

This means that for any $x \in X_r$, if $m_r(x)$ factors through some $e\colon よr \bar{\to} よs$, then x also factors through e.

Lemma 5.44 *Let* $\mathbf{R}$ *be a Reedy category, let* $Y \in \mathrm{PSh}(\mathbf{R})$ *be Reedy monic, and let* $m\colon X \rightarrowtail Y$ *be a degeneracy-reflecting monomorphism. Then,* m *is Reedy monic.*

Proof By Proposition 5.23, it suffices to show, for any $r \in \mathbf{R}$, that the pushout gap map in the naturality square

$$\begin{array}{ccc} L_rX_- & \longrightarrow & X_r \\ \downarrow & & \downarrow \\ L_rY_- & \longrightarrow & Y_r \end{array}$$

is monic. The bottom and right maps are monic by assumption. Because m reflects degeneracy, the square is a weak pullback, i.e., the pullback gap map is surjective. This means that the pushout gap map, seen as an object over Y_r, is the union of the subobjects given by the bottom and right maps. ∎

Corollary 5.45 *If* $i\colon \mathbf{C} \to \mathbf{R}$ *is relatively elegant, then* i_*m *is Reedy monic for every* $m\colon X \rightarrowtail Y$ *in* $\mathrm{PSh}(\mathbf{C})$.

Proof By Lemma 5.44, it suffices to show that i_*m reflects degeneracy. For any $e\colon r \bar{\to} s$, $N_i e$ is epic by Lemma 5.37, so has left lifting against monos. By transposition, e has left lifting against i_*m. ∎

In any presheaf category, all monomorphisms can be presented as cell complexes (transfinite composites of cobase changes of coproducts) of monomorphisms whose codomains are quotients of representables [Cis06, Proposition 1.2.27]. With Corollary 5.45, we can give an alternative—not necessarily comparable—set of generators in terms of the boundary inclusions in $\mathbf{R}$.

Theorem 5.46 *If* $i\colon \mathbf{C} \to \mathbf{R}$ *is relatively elegant, then every monomorphism in* $\mathrm{PSh}(\mathbf{C})$ *is a cell complex of maps of the form* $i^*(\partial^n\mathbf{R} \circledast_{\mathbf{R}[n]^{\mathrm{op}}} (よr/H))$, *where* $r \in \mathbf{R}$ *and* $H \leq \mathrm{Aut}_{\mathbf{R}}(r)$.

Proof Let $m\colon X \rightarrowtail Y$ in $\mathrm{PSh}(\mathbf{C})$. By Corollary 5.16, i_*m has a cellular presentation by maps of the form $\partial^n\mathbf{R} \mathbin{\widehat{\circledast}_{\mathbf{R}[n]^{\mathrm{op}}}} \widehat{\ell}_n(i_*m)$; by Corollary 5.45, each $\widehat{\ell}_n(i_*m)$ is monic in $\mathrm{PSh}(\mathbf{R}[n])$. In $\mathrm{PSh}(\mathbf{R}[n])$, any monomorphism is a cell complex of maps of the form $0 \rightarrowtail よr/H$ for some $r \in \mathbf{R}[n]$ and $H \leq \mathrm{Aut}_{\mathbf{R}}(r)$, because $\mathrm{PSh}(\mathbf{R}[n])$ is Boolean and any

$\mathbf{R}[n]$-set decomposes as a coproduct of orbits. By [RV14, Lemma 5.7], it follows that $i_* m$ is a cell complex of maps $\partial^n \mathbf{R} \mathbin{\widehat{\otimes}}_{\mathbf{R}[n]^{\mathrm{op}}} (0 \rightarrowtail \text{ɹ} r/H)$. Finally, i^* preserves colimits and thus cell complexes. ■

Finally, we exploit the fact that i^* preserves the operations of saturation by monomorphisms to transfer the induction principle on the Reedy monic presheaves of $\mathrm{PSh}(\mathbf{R})$ given by Theorem 5.27 to $\mathrm{PSh}(\mathbf{C})$.

Theorem 5.47 *Let* $\mathbf{R}$ *be elegant relative to* $i\colon \mathbf{C} \to \mathbf{R}$. *Let* $\mathcal{P} \subseteq \mathrm{PSh}(\mathbf{C})$ *be a class of objects such that*

- *for any* $r \in \mathbf{R}$ *and* $H \leq \mathrm{Aut}_{\mathbf{R}}(r)$, *we have* $N_i r/N_i H \in \mathcal{P}$;
- $\mathcal{P}$ *is saturated by monomorphisms.*

Then, $\mathcal{P}$ *contains every presheaf in* $\mathrm{PSh}(\mathbf{C})$.

Proof As a left and right adjoint, i^* preserves colimits and monomorphisms. The class $(i^*)^{-1}\mathcal{P}$ of $X \in \mathrm{PSh}(\mathbf{R})$ such that $i^* X \in \mathcal{P}$ is thus saturated by monomorphisms. By our first assumption and the fact that i^* preserves colimits, we have $\text{ɹ} r/H \in (i^*)^{-1}\mathcal{P}$ for every $r \in \mathbf{R}$ and $H \leq \mathrm{Aut}_{\mathbf{R}}(r)$. By Theorem 5.27 and Lemma 5.42, we thus have $i_* X \in (i^*)^{-1}\mathcal{P}$ for all $X \in \mathrm{PSh}(\mathbf{C})$. Hence, $X \cong i^* i_* X \in \mathcal{P}$ for all $X \in \mathrm{PSh}(\mathbf{C})$. ■

6 Reedy structures on categories of finite algebras

6.1 Finite algebras

Per Section 4, $\square_\vee$ and its idempotent completion can be regarded as full subcategories of the category $\mathbf{SLat}_{\mathrm{fin}}$ of finite semilattices. Any category of finite algebras of a Lawvere theory carries a natural Reedy structure: the degree of an object is its cardinality, and the lowering and raising maps are given by the (surjective, mono) factorization system. Here, we observe that this Reedy structure is pre-elegant and characterizes its elegant core in the case where free finitely-generated algebras are finite. As a corollary, the embedding $\square_\vee \to \mathbf{SLat}_{\mathrm{fin}}$ and its restriction $\square_\vee \to \mathbf{SLat}_{\mathrm{fin}}^{\mathrm{inh}}$ to inhabited algebras are relatively elegant.

For this section, we fix a Lawvere theory $\mathbf{T}$. We recall a few basic properties of its category of algebras.

Proposition 6.1 [ARV10, Corollary 3.5] *A morphism* f *in* $\mathrm{Alg}(\mathbf{T})$ *is regular epic if and only* Uf *is surjective.*

Proposition 6.2 [ARV10, Corollary 3.7] *Any morphism in* $\mathrm{Alg}(\mathbf{T})$ *factors as a regular epi followed by a mono.*

Write $\mathrm{Alg}(\mathbf{T})_{\mathrm{fin}} \to$ and $\mathrm{Alg}(\mathbf{T})_{\mathrm{fin}}^{\mathrm{inh}}$ for the full subcategories of $\mathrm{Alg}(\mathbf{T})$ consisting of algebras with finite and finite inhabited underlying sets, respectively. When we write $\mathrm{Alg}(\mathbf{T})_{\mathrm{fin}}^{(\mathrm{inh})}$ below, the relevant statement or proof applies to both of these.

Corollary 6.3 *The (surjective, mono) factorization system restricts to a Reedy structure on* $\mathrm{Alg}(\mathbf{T})_{\mathrm{fin}}^{(\mathrm{inh})}$ *with degree map given by cardinality.*

As any category of algebras has limits and colimits [ARV10, Proposition 1.21, Theorem 4.5], $\mathrm{Alg}(\mathbf{T})$ has in particular pushouts of spans of surjections.

Corollary 6.4 *The Reedy structure on* $\mathrm{Alg}(\mathbf{T})^{(\mathrm{inh})}_{\mathrm{fin}}$ *is pre-elegant.*

Proof The pushout of a span of surjections has cardinality bounded by those of the objects in the span, as surjections are left maps and thus closed under cobase change. ∎

Recall that the forgetful functor U preserves limits. While U does not generally preserve colimits, we can show that it preserves pushouts of surjective spans using the technology of sifted colimits.

Definition 6.5 A small category $\mathbf{D}$ is

- *filtered* if $\mathrm{colim}_{\mathbf{D}} : [\mathbf{D}, \mathbf{Set}] \to \mathbf{Set}$ commutes with finite limits;
- *sifted* if $\mathrm{colim}_{\mathbf{D}} : [\mathbf{D}, \mathbf{Set}] \to \mathbf{Set}$ commutes with finite products.

A *filtered (sifted) colimit* is a colimit over a filtered (sifted) category.

Recall that a *reflexive coequalizer* is a coequalizer of maps $f_0, f_1 : A \to B$ with a mutual section, that is, some $d : B \to A$ such that $f_0 d = f_1 d = \mathrm{id}$. Reflexive coequalizers are sifted (but not filtered) colimits [ARV10, Remark 3.2].

Lemma 6.6 *Let* $F : \mathbf{C} \to \mathbf{D}$ *be a functor between regular categories preserving finite limits and sifted colimits. Then, F preserves pushouts of regular epi spans.*

Proof Let a span $B_0 \overset{e_0}{\twoheadleftarrow} A \overset{e_1}{\twoheadrightarrow} B_1$ in $\mathbf{C}$ be given. We compute the following reflexive coequalizer:

$$A \times_{B_0} A \times_{B_1} A \underset{\pi_2}{\overset{\pi_0}{\rightrightarrows}} A \overset{e}{\dashrightarrow\!\!\!\!\twoheadrightarrow} B. \qquad \langle \mathrm{id}, \mathrm{id}, \mathrm{id} \rangle$$

It is straightforward to check, using the characterizations of e_0, e_1 as the coequalizers of their kernel pairs, that we have induced maps $B_0 \twoheadrightarrow B \twoheadleftarrow B_1$ exhibiting B as the pushout of our span. As F preserves the diagram above, it preserves this pushout. ∎

Corollary 6.7 $U : \mathrm{Alg}(\mathbf{T}) \to \mathbf{Set}$ *preserves pushouts of surjective spans.*

Proof U preserves limits and sifted colimits [ARV10, Proposition 2.5]. ∎

We now assume that any $\mathbf{T}$-algebra free on a finite set has a finite underlying set. In this case, the elegant core coincides with the class of *perfectly presentable* (also called *strongly finitely presentable*) algebras.

Definition 6.8 [ARV10, Definition 5.3] An object A of a category $\mathbf{C}$ is

- *finitely presentable* if $\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Set}$ preserves filtered colimits;
- *perfectly presentable* if $\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Set}$ preserves sifted colimits.

Proposition 6.9 [ARV10, Corollary 5.16 and Proposition 11.28] *Let* $A \in \mathrm{Alg}(\mathbf{T})$. *The following are equivalent:*

- *A is perfectly presentable;*
- *A is finitely presentable and regular projective;*
- *A is a retract of a finitely-generated free algebra.*

Theorem 6.10 *Suppose that every finitely-generated free algebra in* $\mathrm{Alg}(\mathbf{T})$ *has a finite underlying set. Then, the elegant core of* $\mathrm{Alg}(\mathbf{T})^{(\mathrm{inh})}_{\mathrm{fin}}$ *is the subcategory of objects perfectly presentable in* $\mathrm{Alg}(\mathbf{T})$.

Proof Suppose $A \in \mathrm{Alg}(\mathbf{T})^{(\mathrm{inh})}_{\mathrm{fin}}$ is in the elegant core of the Reedy structure. By assumption, the free algebra FUA belongs to $\mathrm{Alg}(\mathbf{T})^{(\mathrm{inh})}_{\mathrm{fin}}$, and the counit $\varepsilon_A : FUA \to A$ is clearly surjective. Then, by Corollary 5.38, we have a lift

$$\begin{array}{ccc} & & FUA \\ & \overset{\nearrow}{} & \downarrow{\scriptstyle \varepsilon_A} \\ A & \xrightarrow[\mathrm{id}]{} & A \end{array}$$

exhibiting A a retract of a free algebra. Thus, A is perfectly presentable. Conversely, if A is perfectly presentable, then $\mathrm{Alg}(\mathbf{T})(A, -) : \mathrm{Alg}(\mathbf{T}) \to \mathbf{Set}$ preserves finite limits and sifted colimits, so it preserves pushouts of lowering spans by Lemma 6.6. ∎

6.2 Semilattice cubes

Applying the preceding results, we have a (surjective, mono) Reedy structure on $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$. We can give a concrete description of its elegant core.

Lemma 6.11 *A semilattice* $A \in \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ *is in the elegant core of the (surjective, mono) Reedy structure if and only if* $1 \star A$ *is a distributive lattice.*

Proof By Theorem 6.10, the elegant core consists of the perfectly presentable objects in **SLat**. By Proposition 6.9, these are the finite regular projectives in **SLat**. These are characterized as above by Propositions 4.41 and 4.42. ∎

Theorem 6.12 *The inclusion* $i : \overline{\square}_\vee \to \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ *is relatively elegant.*

Proof If $A \in \mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ is a distributive lattice, then $1 \star A$ is a distributive lattice as well, so A is in the elegant core of $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$. ∎

Remark 6.13 The subcategory $\mathbf{SLat}^{\perp}_{\mathrm{fin}}$ of $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ consisting of finite semilattices with a minimum element is closed under Reedy factorizations and lowering pushouts, so $\overline{\square}_\vee \to \mathbf{SLat}^{\perp}_{\mathrm{fin}}$ is also relatively elegant. This embedding gives a more parsimonious set of generators, but $\mathbf{SLat}^{\mathrm{inh}}_{\mathrm{fin}}$ suffices for our purposes.

7 Equivalences and equalities

7.1 Equivalence with the Kan–Quillen model structure

Returning to the candidate Quillen equivalence $▲_! \dashv ▲^*$, it remains to show that its counit is valued in weak equivalences. We first note that the collection of those $X \in \mathrm{PSh}(\overline{\square}_\vee)$ for which $\varepsilon_X : ▲_! ▲^* X \to X$ is a weak equivalence is saturated by monomorphisms.

Proposition 7.1 [Cis06, Remarque 1.1.13] *Let* $F : \mathbf{E} \to \mathbf{F}$ *be a mono- and colimit-preserving functor between cocomplete categories. If* $\mathcal{P} \subseteq \mathbf{F}$ *is saturated by monos, then the class* $F^{-1}(\mathcal{P})$ *of objects whose image by* F *is in* $\mathcal{P}$ *is saturated by monos.*

Proposition 7.2 *If* $\mathbf{M}$ *has monos as cofibrations, then its class of weak equivalences is saturated by monos as a class of objects of* $\mathbf{M}^{\to}$.

Proof This is proven by Cisinski [Cis06, Remarque 1.4.16] for *localizers* [Cis06, Définition 1.4.1]; the class of weak equivalences in a model category with monos as cofibrations is always a localizer. ∎

Corollary 7.3 *Let* $\mathbf{E}$ *be a cocomplete category,* $\mathbf{N}$ *be a model category with monos as cofibrations, and* $F, G\colon \mathbf{E} \to \mathbf{N}$ *be mono- and colimit-preserving functors. For any natural transformation* $h\colon F \to G$*, the class of objects* $X \in \mathbf{E}$ *such that* $h_X\colon FX \to GX$ *is a weak equivalence is saturated by monos.*

Proof By Propositions 7.1 and 7.2, regarding h as a functor $\mathbf{E} \to \mathbf{N}^{\to}$. ∎

In particular, any natural transformation $h\colon F \to G$ of left Quillen adjoints $F, G\colon \mathbf{M} \to \mathbf{N}$ between model categories with monos as cofibrations satisfies the hypotheses of Corollary 7.3. In light of this, we only need to check that ε is a weak equivalence at generating presheaves.

Lemma 7.4 *Let* $A \in \mathbf{SLat}_{\mathrm{fin}}^{\mathrm{inh}}$ *and* $H \leq \mathrm{Aut}_{\mathbf{SLat}_{\mathrm{fin}}^{\mathrm{inh}}}(A)$ *be given. Then,* $N_i A/N_i H$ *is weakly contractible.*

Proof Per Corollary 4.51, it suffices to show that this object is a homotopy retract of 1. We have a semilattice morphism $\uparrow\colon [1] \times A \to A$ sending $(0, a) \mapsto a$ and $(1, a) \mapsto \top$. Any automorphism $g \in H$ preserves maximum elements, so we have a diagram like so:

$$\begin{array}{ccc} [1] \times A & \xrightarrow{\uparrow} & A \\ {\scriptstyle [1]\times g}\downarrow & & \downarrow{\scriptstyle g} \\ [1] \times A & \xrightarrow[\uparrow]{} & A. \end{array}$$

We thus obtain a contracting homotopy $\mathbb{I} \times (N_i A/N_i H) \to (N_i A/N_i H)$, using that $N_i([1] \times A) \cong \mathbb{I} \times N_i A$ and that $\mathbb{I} \times (-)$ commutes with colimits. ∎

Lemma 7.5 *The counit map* $\varepsilon_X\colon \blacktriangle_! \blacktriangle^* X \to X$ *is a weak equivalence for every* $X \in \mathrm{PSh}(\square_\vee)$.

Proof Recall that both $\blacktriangle_!$ and $\blacktriangle^*$ are left Quillen (Corollary 4.53 and Lemma 4.54). By Theorem 5.47 and Corollary 7.3, it suffices to show that $\varepsilon_X\colon \blacktriangle_! \blacktriangle^* X \to X$ is a weak equivalence whenever X is an automorphism quotient of an object in the image of N_i. In this case, X is weakly contractible by Lemma 7.4. As $\blacktriangle_! \blacktriangle^*$ preserves the terminal object, it preserves weak contractibility by Ken Brown's lemma; thus $\blacktriangle_! \blacktriangle^* X$ is weakly contractible and so ε_X is a weak equivalence by 2-out-of-3. ∎

Theorem 7.6 $\blacktriangle_!\colon \widehat{\Delta}^{\mathrm{kq}} \rightleftarrows \widehat{\square}_\vee^{\mathrm{ty}} : \blacktriangle^*$ *is a Quillen equivalence.*

Proof By Corollary 4.53 and Lemmas 7.5 and 4.55. ∎

Corollary 7.7 $\blacktriangle^*\colon \widehat{\square}_\vee^{\mathrm{ty}} \rightleftarrows \widehat{\Delta}^{\mathrm{kq}} : \blacktriangle_*$ *is a Quillen equivalence.*

Proof Write η' and ε' for the unit and counit of this adjunction. The counit is an isomorphism, so trivially valued in weak equivalences. To check the derived unit, let $X \in \mathrm{PSh}(\square_\vee)$ and let $m\colon \blacktriangle^* X \rightarrowtail (\blacktriangle^* X)^{\mathrm{fib}}$ be a fibrant replacement. We have the following naturality square:

$$\begin{array}{ccccc}
\blacktriangle_! \blacktriangle^* X & \xrightarrow[\cong]{\blacktriangle_! \blacktriangle^* \eta'_X} & \blacktriangle_! \blacktriangle^* \blacktriangle_* \blacktriangle^* X & \xrightarrow[\sim]{\blacktriangle_! \blacktriangle^* \blacktriangle_* m} & \blacktriangle_! \blacktriangle^* \blacktriangle_* (\blacktriangle^* X)^{\mathrm{fib}} \\
{\scriptstyle \varepsilon_X}\downarrow{\scriptstyle \wr} & & & & {\scriptstyle \wr}\downarrow{\scriptstyle \varepsilon_{\blacktriangle_* (\blacktriangle^* X)^{\mathrm{fib}}}} \\
X & \xrightarrow[\eta'_X]{} & \blacktriangle_* \blacktriangle^* X & \xrightarrow[\blacktriangle_* m]{} & \blacktriangle_* (\blacktriangle^* X)^{\mathrm{fib}}.
\end{array}$$

It follows by 2-out-of-3 that the bottom composite is a weak equivalence. ∎

Theorem 7.8 $\mathrm{T}\colon \widehat{\square}^{\mathrm{ty}}_\vee \rightleftarrows \widehat{\Delta}^{\mathrm{kq}} : N_{\boxtimes}$ *is a Quillen equivalence.*

Proof By the decomposition $\mathrm{T} \cong \blacktriangle^* \blacksquare_!$ (Lemma 4.48). ∎

In particular, both $\widehat{\square}^{\mathrm{ty}}_\vee$ and $\widehat{\square}^{\mathrm{ty}}_\vee$ present ∞-**Gpd**.

7.2 Equality with the test model structure

It is worth remarking that there is a model structure on $\mathrm{PSh}(\square_\vee)$ already known to present ∞-**Gpd**, namely, its *test model structure*. Constructed by Cisinski [Cis06] based on Grothendieck's theory of test categories [Gro83], a test model structure exists on the category of presheaves $\mathrm{PSh}(\mathbf{C})$ over any *local test category* $\mathbf{C}$. If $\mathbf{C}$ is moreover a *test category*, then this model structure is Quillen equivalent to $\widehat{\Delta}^{\mathrm{kq}}$.

Buchholtz and Morehouse observe that $\square_\vee$, among various other cube categories, is a test category [BM17, Corollary 3].[7] Thus, it supports a model structure presenting ∞-**Gpd**. However, it has not been established whether this model structure is constructive or compatible with a model of HoTT. Cisinski [Cis14] has shown that the test model structure on an elegant strict Reedy local test category is type-theoretic in the sense of Shulman [Shu19, Definition 6.1], but the strictness requirement prevents application of this result to any cube category with permutations (or any non-Reedy category).

By virtue of the Quillen equivalences to $\widehat{\Delta}^{\mathrm{kq}}$ already established, we know that $\widehat{\square}^{\mathrm{ty}}_\vee$ and $\widehat{\square}^{\mathrm{ty}}_\vee$ are Quillen equivalent to the test model structures on their respective base categories. Here, we check that they are in fact identical, adapting an argument of Streicher and Weinberger [SW21, Section 5].

We must begin by recalling the main definitions of test category theory. For more detail, we refer the reader to Maltsiniotis [Mal05], Cisinski [Cis06], or Jardine [Jar06]. The foundation of test category theory that we can relate presheaves on an arbitrary base category $\mathbf{C}$ with simplicial sets by way of the category of small categories, **Cat**. We write $N_\Delta\colon \mathbf{Cat} \to \mathrm{PSh}(\Delta)$ for the nerve of the inclusion $\Delta \to \mathbf{Cat}$.

[7] Maltsiniotis [Mal09] also observed that a cube category with one connection is a strict test category, but a different one: the subcategory of $\square_\vee$ generated by faces, degeneracies, and connections, i.e., not including diagonals and permutations.

Definition 7.9 Given a small category $\mathbf{C}$, write $i_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{Cat}$ for the slice category functor $a \mapsto \mathbf{C}/a$. We have an induced nerve functor $i_{\mathbf{C}}^*\colon \mathbf{Cat} \to \mathrm{PSh}(\mathbf{C})$. As $\mathbf{Cat}$ is cocomplete, this functor has a left adjoint $\mathrm{PSh}(\mathbf{C}) \to \mathbf{Cat}$, for which we also write $i_{\mathbf{C}}$.

The composite $N_\Delta i_{\mathbf{C}}\colon \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\Delta)$ is the means by which we can inherit a model structure on $\mathrm{PSh}(\mathbf{C})$ from $\widehat{\Delta}^{\mathrm{kq}}$ under appropriate conditions.

Remark 7.10 The definitions and results of Cisinski that we cite below are typically parameterized by an arbitrary *basic localizer* [Cis06, Définition 3.3.2], a class of functors to be regarded as the weak equivalences in $\mathbf{Cat}$. We always instantiate with the *minimal* basic localizer $\mathcal{W}_\infty$: the class of functors $f\colon \mathbf{C} \to \mathbf{D}$ such that $N_\Delta f\colon N_\Delta \mathbf{C} \to N_\Delta \mathbf{D}$ is a weak equivalence of $\widehat{\Delta}^{\mathrm{kq}}$ [Cis06, Corollaire 4.2.19].

Definition 7.11 [Cis06, Section 3.3.3 and Définition 4.1.3] We say $X \in \mathrm{PSh}(\mathbf{C})$ is *aspheric* if $N_\Delta i_{\mathbf{C}} X \in \mathrm{PSh}(\Delta)$ is weakly contractible in $\widehat{\Delta}^{\mathrm{kq}}$.

Definition 7.12 [Cis06, Définitions 4.1.8 and 4.1.12] A small category $\mathbf{C}$ is

- a *weak test category* if $i_{\mathbf{C}}^*\mathbf{D}$ is aspheric for every $\mathbf{D}$ with a terminal object;
- a *local test category* if $\mathbf{C}/a$ is a weak test category for all $a \in \mathbf{C}$;
- a *test category* if it is both a weak and local test category.

Proposition 7.13 [Cis06, Corollaire 4.2.18] *Let $\mathbf{C}$ be a local test category. There is a model structure on $\mathrm{PSh}(\mathbf{C})$ in which*

- *the cofibrations are the monomorphisms;*
- *the weak equivalences are the maps sent by $N_\Delta i_{\mathbf{C}}$ to a weak equivalence of $\widehat{\Delta}^{\mathrm{kq}}$.*

We write $\widehat{\mathbf{C}}^{\mathrm{test}}$ for this model category.

Remark 7.14 The test model structure $\widehat{\Delta}^{\mathrm{test}}$ coincides with $\widehat{\Delta}^{\mathrm{kq}}$. A proof is contained in the proof of [Cis06, Corollaire 4.2.19]: the class of weak equivalences of $\widehat{\Delta}^{\mathrm{test}}$ is by definition the preimage $N_\Delta^{-1}\mathcal{W}_\infty$, which is the minimal test Δ-localizer by Théorème 4.2.15, and said localizer is the class of weak equivalences of $\widehat{\Delta}^{\mathrm{kq}}$ by Corollaire 2.1.21 and Proposition 3.4.25.

Note that whereas cubical-type model structures come with explicit characterizations of their cofibrations and fibrations (or rather generating trivial cofibrations), the test model structure comes with explicit descriptions of its cofibrations and weak equivalences. In general, $\widehat{\mathbf{C}}^{\mathrm{test}}$ is Quillen equivalent to a slice of $\widehat{\Delta}^{\mathrm{kq}}$, namely, $\widehat{\Delta}^{\mathrm{kq}}/N_\Delta \mathbf{C}$ [Cis06, Corollaire 4.4.20]. When $\mathbf{C}$ is a test category, $N_\Delta \mathbf{C}$ is weakly contractible, and so we have an equivalence to $\widehat{\Delta}^{\mathrm{kq}}$ itself.

We recall the argument used by Buchholtz and Morehouse [BM17, Theorem 1] to show that $\square_\vee$ is a test category—actually a *strict* test category.

Definition 7.15 [Cis06, Sections 4.3.1 and 4.3.3, Proposition 4.3.2] We say a category $\mathbf{C}$ is *totally aspheric* if it is non-empty and $\yoneda a \times \yoneda b$ is aspheric for every $a, b \in \mathbf{C}$. A test category that is totally aspheric is called a *strict test category*.

Any representable is aspheric: the category $i_{\mathbf{C}}(\yoneda a)$ has a terminal object, thus a natural transformation from its identity functor to a constant functor, and this induces a contracting homotopy on $N_\Delta i_{\mathbf{C}}(\yoneda a)$. Thus, any category with binary products is totally aspheric.

The following result originates in [Gro83, Section 44(c)] and is invoked in [BM17] for a broad class of cube categories.

Proposition 7.16 [Cis06, Proposition 4.3.4] *Let* $\mathbf{C}$ *be a totally aspheric category. If* $\mathrm{PSh}(\mathbf{C})$ *contains an aspheric presheaf* I *with disjoint maps* $e_0, e_1 : 1 \to I$*, then* $\mathbf{C}$ *is a strict test category.*

In particular, both $\square_\vee$ and $\overline{\square}_\vee$ are strict test categories. To relate their test model structures to $\widehat{\Delta}^{\mathrm{kq}}$, we recall the notion of aspheric functor.

Definition 7.17 [Cis06, Section 3.3.3, Proposition 4.2.23(a $\Longleftrightarrow$ b″)] A functor $u : \mathbf{C} \to \mathbf{D}$ is *aspheric* if for every $d \in \mathbf{D}$, the presheaf $u^*(\text{よ}\, d)$ is aspheric.

An aspheric functor $u : \mathbf{C} \to \mathbf{D}$ between test categories induces a Quillen equivalence $u^* \dashv u_*$ between their test model structures [Cis06, Proposition 4.2.24]. For our purposes, the more relevant property is the following immediate consequence.

Proposition 7.18 [Cis06, Proposition 4.2.23(d)] *Let* $u : \mathbf{C} \to \mathbf{D}$ *be an aspheric functor between two test categories. Then, a map* f *in* $\mathrm{PSh}(\mathbf{D})$ *is a weak equivalence in* $\widehat{\mathbf{D}}^{\mathrm{test}}$ *if and only if* $u^* f$ *is a weak equivalence in* $\widehat{\mathbf{C}}^{\mathrm{test}}$.

Lemma 7.19 *Any idempotent completion* $i : \mathbf{C} \to \overline{\mathbf{C}}$ *is aspheric.*

Proof Any $A \in \overline{\mathbf{C}}$ is a retract of ia for some $a \in \mathbf{C}$. Then, $i^* \text{よ}\, A$ is likewise a retract of $i^* \text{よ}\, (ia) \cong \text{よ}\, a$, thus aspheric by Corollary 4.51. ∎

Lemma 7.20 $\blacktriangle : \Delta \to \overline{\square}_\vee$ *is aspheric.*

Proof For any $[1]^n \in \overline{\square}_\vee$, we have $\blacktriangle^* \text{よ}\, [1]^n \cong (\Delta^1)^n$. As Δ is a strict test category [Mal05, Proposition 1.6.14], any finite product of representables in $\mathrm{PSh}(\Delta)$ is aspheric [Cis06, Proposition 4.3.2(b)]. ∎

Lemma 7.21 *A map* f *in* $\mathrm{PSh}(\overline{\square}_\vee)$ *is a weak equivalence in* $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$ *if and only if* $\blacktriangle^* f$ *is a weak equivalence in* $\widehat{\Delta}^{\mathrm{kq}}$.

Proof Any left Quillen equivalence both preserves (Ken Brown's lemma) and reflects [Hov99, Corollary 1.3.16] weak equivalences between cofibrant objects, so this follows from Corollary 7.7. ∎

Theorem 7.22 *The model structures* $\widehat{\overline{\square}}^{\mathrm{test}}_\vee$ *and* $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$ *are identical.*

Proof As they have the same cofibrations, it suffices to show they have the same weak equivalences. This follows from Proposition 7.18 and Lemma 7.20 (together with Remark 7.14) and Lemma 7.21. ∎

Corollary 7.23 *The model structures* $\widehat{\square}^{\mathrm{test}}_\vee$ *and* $\widehat{\square}^{\mathrm{ty}}_\vee$ *are identical.*

Proof Again, it suffices to show they have the same weak equivalences. By Proposition 7.18 and Lemma 7.19, a map f is a weak equivalence in $\widehat{\square}^{\mathrm{test}}_\vee$ if and only if $\blacksquare_! f$ is a weak equivalence in $\widehat{\overline{\square}}^{\mathrm{test}}_\vee$. Likewise, f is a weak equivalence in $\widehat{\square}^{\mathrm{ty}}_\vee$ if and only if $\blacksquare_! f$ is a weak equivalence in $\widehat{\overline{\square}}^{\mathrm{ty}}_\vee$. ∎

These results can also be read as characterizations of the fibrations in the test model structures:

Corollary 7.24 *The fibrations in* $\widehat{\overline{\square}}_\vee^{\mathrm{test}}$ *and* $\widehat{\square}_\vee^{\mathrm{test}}$ *are those maps lifting against* $\delta_k \mathbin{\widehat{\times}} m$ *for all* $k \in \{0,1\}$ *and* $m\colon A \rightarrowtail B$.

Appendix. Negative results

Here, we collect a pair of negative results concerning the existence of (relative) Reedy structures on (idempotent completions of) cube categories. In Appendix A.1, we check that $\square_\vee$ and $\overline{\square}_\vee$ are not Reedy categories, motivating this article's approach. Appendix A.2 concerns the limits of relative elegance: we show that the Dedekind cube category does not embed elegantly in any Reedy category.

A.1 Semilattice cubes

The non-existence of a Reedy structure on $\square_\vee$ is easily verified: every Reedy category is idempotent complete [Bor94, Proposition 6.5.9], but we have seen in Section 4.3 that $\square_\vee$ is not. The map $(x, y) \mapsto (x, x \vee y)\colon [1]^2 \to [1]^2$ is a simple example of an idempotent with no splitting in $\square_\vee$.

It is therefore more appropriate to ask if the cube category's idempotent completion $\overline{\square}_\vee$, which we have characterized as the full subcategory of **SLat** consisting of finite inhabited distributive lattices (Definition 4.39), is Reedy. If this were so, we could simply study $\mathrm{PSh}(\square_\vee)$ by way of the equivalent $\mathrm{PSh}(\overline{\square}_\vee)$. However, this is not the case.

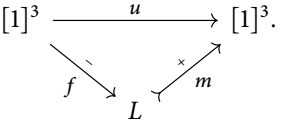
Proposition A.1 *There is no Reedy structure on* $\overline{\square}_\vee$.

Proof We consider the following morphism $u\colon [1]^3 \to [1]^3$:

$$u(x, y, z) \coloneqq (x \vee y, y \vee z, z \vee x).$$

For intuition, note that the image of u computed in **SLat** is the non-distributive *diamond lattice* $\mathfrak{M}_3$.

Suppose that we do have a Reedy structure on $\overline{\square}_\vee$. The unique map $[1]^2 \to 1$ is split epic and thus a lowering map (Corollary 2.15). Every raising map must have the right lifting property against this map, so every raising map is monic.[8] Take a Reedy factorization of u:



L is a sub-semilattice of $[1]^3$ that forms a distributive lattice and contains the image of u. Note that $\vee$, $\bot$, and $\top$ are computed in L as in $[1]^3$, but $\wedge$ may not be; we write $\wedge_L$ for the meet in L. We show that in fact $L = [1]^3$.

[8] If we only want to show $\overline{\square}_\vee$ is not *elegant* Reedy, we are already done, as observed in [Cam23, Theorem 8.12(2)]: if $\overline{\square}_\vee$ were elegant we would have a (split epi, mono) factorization of u, which would necessarily be preserved by the inclusion $\overline{\square}_\vee \to$ **SLat**, but u's (split epi, mono) factorization in **SLat** is $\mathfrak{M}_3$.

Consider the set $S \coloneqq \{011, 101, 110\} \subseteq L \subseteq [1]^3$. Let v, v', v'' be any pairwise distinct elements of S and note that we have

$$(v \wedge_L v') \vee (v \wedge_L v'') = v \wedge_L (v' \vee v'') = v \wedge_L \top = v.$$

This implies the following.

(a) $v \wedge_L v' \neq v \wedge_L v''$: otherwise, we have $(v \wedge_L v') \vee (v \wedge_L v'') = v \wedge_L v''$ and thus $v = v \wedge_L v''$, but v and v'' are incomparable.
(b) $v \wedge_L v' \neq \bot$: otherwise, we again have $(v \wedge_L v') \vee (v \wedge_L v'') = v \wedge_L v''$.

Thus, the meets $011 \wedge_L 101$, $011 \wedge_L 110$, and $011 \wedge_L 110$ are pairwise distinct and lie outside the image of u, which by a cardinality argument implies that L is the whole of $[1]^3$.

The lowering map f of our supposed factorization must then be u itself; it remains to show that u cannot be a lowering map. Consider the semilattice morphism $t\colon [1]^3 \to [2]$ defined by $t(x, y, z) \coloneqq x \vee 2y \vee 2z$. We have the following commutative diagram in $\overline{\square}_\vee$, where d_1 and s_1 are the simplex face and degeneracy maps from Definition 2.22:

$$\begin{array}{ccccc} [1]^3 & \xrightarrow{t} & [2] & \xrightarrow{s_1} & [1] \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle d_1} \\ [1]^3 & & \xrightarrow[t]{} & & [2]. \end{array}$$

The face map d_1 is split monic and therefore a raising map. If u were a lowering map, this square would have a diagonal lift. But as t is surjective, there can be no diagonal $[1]^3 \to [1]$ making the lower triangle commute. ∎

A.2 Dedekind cubes

As mentioned in the introduction, it is an open question whether the cubical-type model structure for presheaves on the Dedekind cube category $\square_{\wedge\vee}$ is equivalent to the Kan–Quillen model structure $\widehat{\Delta}^{\mathrm{kq}}$ (see Streicher and Weinberger [SW21] for further discussion). In this appendix, we show that $\square_{\wedge\vee}$ supports no relatively elegant embedding in a Reedy category, thus our argument for $\square_\vee$ admits no naive adaptation to the two-connection case.

Definition A.2 The *Dedekind cube category* $\square_{\wedge\vee}$ is the Lawvere theory of bounded distributive lattices.

$\square_{\wedge\vee}$ admits an alternative description arising from the duality between finite bounded distributive lattices and finite posets [Wra93], analogous to the description of $\square_\vee$ as a full subcategory of **SLat**.

Proposition A.3 *$\square_{\wedge\vee}$ is equivalent to the full subcategory of* **Pos** *consisting of posets of the form $[1]^n$ for $n \in \mathbb{N}$.*

We will only need this latter description.

The Dedekind cube category attracted attention [Spi16, Sat19, KV20, SW21, HR22] in the HoTT community following Cohen et al.'s interpretation of HoTT in De Morgan cubical sets [CCHM15]. As Orton and Pitts note [OP18, Remark 3.2], this

interpretation does not require all the structure of De Morgan cubes; in particular, it can be repeated with $\square_{\wedge\vee}$. The name "Dedekind" was coined by Awodey in reference to the fact that the cardinality of $\square_{\wedge\vee}([1]^n, [1])$ is the nth Dedekind number.

A.2.1 A no-go theorem

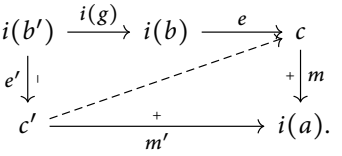
We begin by identifying a property shared by all categories $\mathbf{C}$ with a relatively elegant functor $i\colon \mathbf{C} \to \mathbf{R}$; the contrapositive will show that no such functor exists out of $\square_{\wedge\vee}$.

Definition A.4 A *sieve* on an object a of a small category $\mathbf{C}$ is a set of morphisms $\mathcal{S} \subseteq \mathbf{C}/a$ such that $g \in \mathcal{S}$ implies $gf \in \mathcal{S}$ for any composable $f \in \mathbf{C}^{\to}$. We regard the collection $\mathrm{Sv}_{\mathbf{C}}(a)$ of sieves on $a \in \mathbf{C}$ as a poset ordered by inclusion. A sieve is *principal* if it is of the form $\langle f \rangle \coloneqq \{gf \mid g \in \mathbf{C}/b\}$ for some $f\colon b \to a$; we write $\mathrm{PrSv}_{\mathbf{C}}(a) \subseteq \mathrm{Sv}_{\mathbf{C}}(a)$ for the subposet of principal sieves on a.

Recall that $\mathrm{Sv}_{\mathbf{C}}(a)$ is isomorphic to the poset of subobjects of $\yoneda a \in \mathrm{PSh}(\mathbf{C})$. The principal sieve $\langle f \rangle$ on a map $f\colon b \to a$ corresponds to the subobject $\mathrm{Im}\, f \rightarrowtail \yoneda a$. Given a relatively elegant $i\colon \mathbf{C} \to \mathbf{R}$, the following lemma deduces a well-foundedness property of these subobjects in $\mathrm{PSh}(\mathbf{C})$ from the well-foundedness of the Reedy category $\mathbf{R}$.

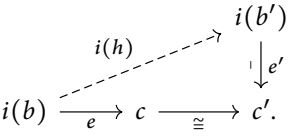
Lemma A.5 *Let $\mathbf{C}$ be a category, and let $\mathbf{R}$ be a Reedy category elegant relative to some $i\colon \mathbf{C} \to \mathbf{R}$. Then, for any $a \in \mathbf{C}$, there exists a strictly monotone map $d\colon \mathrm{PrSv}_{\mathbf{C}}(a) \to \mathbb{N}$. In particular, $\mathrm{PrSv}_{\mathbf{C}}(a)$ is well-founded.*

Proof Given a principal sieve $\langle f \rangle \in \mathrm{PrSv}_{\mathbf{C}}(a)$ generated by $f\colon b \to a$, we define $d(\langle f \rangle)$ to be the degree of $i(f)$, i.e., the degree of the intermediate object in its Reedy factorization. To see that this definition is independent of the choice of representative f and that d is order-preserving, it suffices to check that for any $f\colon b \to a$ and $f'\colon b' \to a$, if $\langle f' \rangle \subseteq \langle f \rangle$ then $d(\langle f' \rangle) \leq d(\langle f \rangle)$. If $\langle f' \rangle \subseteq \langle f \rangle$, then there exists some $g\colon b' \to b$ such that $f' = fg$. Upon Reedy factorizing $i(f') = m'e'$ and $i(f) = me$, orthogonality gives us a map as shown:



By Lemma 2.14, the lift is a raising map, so $d(\langle f \rangle) = |c'| \leq |c| = d(\langle f' \rangle)$.

To see that d is strictly monotone, suppose that additionally $|c'| = |c|$. Then, the diagonal above is an isomorphism. By $\mathbf{R}^-$-projectivity of $i(b)$ (Corollary 5.38) and fullness of i, we obtain a lift as below:



Then, $f = f'h$, so $\langle f \rangle \subseteq \langle f' \rangle$. ∎

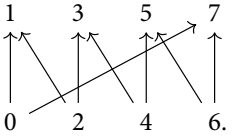
A.2.2 Principal sieves in Dedekind cubes

Now, we show that the poset of principal sieves on $[1]^3 \in \square_{\wedge\vee}$ is not well-founded. We embed a poset model of the circle $\mathfrak{C}_n \rightarrowtail [1]^n$ in each cube, then exhibit a chain of subobjects of $[1]^n$ (for any $n \geq 3$) induced by maps $\cdots \to \mathfrak{C}_{n_2} \to \mathfrak{C}_{n_1} \to \mathfrak{C}_n$ that cannot stabilize.

Definition A.6 The *fence* $\mathfrak{F} \in \mathbf{Pos}$ is the poset whose elements are integers and whose order is generated by the inequalities $i \leq i-1$ and $i \leq i+1$ for all even $i \in \mathbb{Z}$.

Definition A.7 The nth *crown poset* $\mathfrak{C}_n \in \mathbf{Pos}$ is the quotient of $\mathfrak{F}$ identifying $i, j \in \mathfrak{F}$ whenever $i = j \pmod{2n}$. We write $p_n : \mathfrak{F} \to \mathfrak{C}_n$ for the quotient map.

For example, $\mathfrak{C}_4$ is the following poset:



Remark A.8 Each crown poset is freely generated by a graph (though not the graphs usually known as *crown graphs*, which have more edges).

The simplicial nerve N_Δ sends each crown poset to a simplicial set weakly equivalent to the circle. As such, any map between crown posets can be associated with a winding number. Concretely, we can define the winding number on the level of posets as follows.

Definition A.9 Any poset map $f : \mathfrak{C}_m \to \mathfrak{C}_n$ lifts to an endomap

$$\begin{array}{ccc} \mathfrak{F} & \overset{\hat{f}}{\dashrightarrow} & \mathfrak{F} \\ {\scriptstyle p_m}\downarrow & & \downarrow{\scriptstyle p_n} \\ \mathfrak{C}_m & \xrightarrow[f]{} & \mathfrak{C}_n, \end{array}$$

which is unique modulo $2n$. The *winding number* of f is

$$\deg(f) \coloneqq \frac{\hat{f}(2m) - \hat{f}(0)}{2n}.$$

It is straightforward to check that $\deg(gf) = \deg(g)\deg(f)$ for $\mathfrak{C}_m \overset{f}{\to} \mathfrak{C}_n \overset{g}{\to} \mathfrak{C}_p$, as we expect from a winding number. Because $\mathfrak{C}_m$ is "too short" to wrap around $\mathfrak{C}_n$ when $m < n$, we have the following.

Lemma A.10 *If $m < n$, then $\deg(f) = 0$ for any $f : \mathfrak{C}_m \to \mathfrak{C}_n$.*

Proof By induction, $|\hat{f}(i) - \hat{f}(0)| \leq i$ for every $i \in \mathbb{N}$, so $|\hat{f}(2m) - \hat{f}(0)| < 2n$. ∎

Definition A.11 For $n \geq 3$, define a poset embedding $c_n : \mathfrak{C}_n \rightarrowtail [1]^n$ by

$$c_n(i)_j = \begin{cases} 1 & \text{if} \lfloor \frac{i}{2} \rfloor \leq j \leq \lceil \frac{i}{2} \rceil \\ 0 & \text{otherwise} \end{cases}.$$

Definition A.12 Given $m, n \geq 3$ and a monotone map $f : \mathfrak{C}_m \to \mathfrak{C}_n$, define an extension

$$\begin{array}{ccc} \mathfrak{C}_m & \xrightarrow{f} & \mathfrak{C}_n \\ {\scriptstyle c_m}\downarrow & & \downarrow{\scriptstyle c_n} \\ [1]^m & \xrightarrow[\overline{f}]{} & [1]^n \end{array}$$

by setting

$$\overline{f}(v) \coloneqq \begin{cases} c_n(f(i)) & \text{if } v = c_m(i), \\ \bot & \text{if } v = \bot, \\ \top & \text{otherwise.} \end{cases}$$

The mapping $f \mapsto \overline{f}$ is the functorial action of a *semifunctor* from the category of crown posets to $\square_{\wedge\vee}$: compositions are preserved, but not identities.

Lemma A.13 *The diagram in Definition A.12 is a pullback.*

Proof The three cases in the definition of $\overline{f}$ have disjoint values. ∎

Theorem A.14 *There exists no Reedy category* $\mathbf{R}$ *with a fully faithful functor* $i : \square_{\wedge\vee} \to \mathbf{R}$ *such that* $\mathbf{R}$ *is elegant relative to* i.

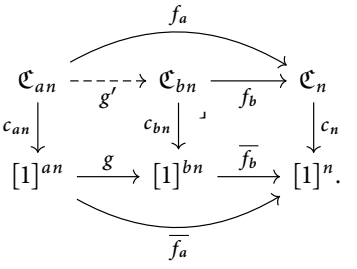
Proof Suppose for the sake of contradiction that we have some $i : \square_{\wedge\vee} \to \mathbf{R}$ such that $\mathbf{R}$ is elegant relative to i. Choose any $n \geq 3$. For every $m \geq 2$ and $a \geq 1$, the identity function on $\mathfrak{F}$ induces a map $f_a : \mathfrak{C}_{am} \to \mathfrak{C}_m$ with winding number a. We then have the following diagram in **Pos**:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{f_2} & \mathfrak{C}_{8n} & \xrightarrow{f_2} & \mathfrak{C}_{4n} & \xrightarrow{f_2} & \mathfrak{C}_{2n} \\ & & \downarrow{\scriptstyle f_8} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_2} \\ \cdots & \xrightarrow[\mathrm{id}]{} & \mathfrak{C}_n & \xrightarrow[\mathrm{id}]{} & \mathfrak{C}_n & \xrightarrow[\mathrm{id}]{} & \mathfrak{C}_n. \end{array}$$

Applying $\overline{(-)}$, we have a chain of principal sieves $\langle \overline{f_2} \rangle \supseteq \langle \overline{f_4} \rangle \supseteq \langle \overline{f_8} \rangle \supseteq \cdots$ on $[1]^n$. By Lemma A.5, this chain must stabilize; in particular, there must be some pair $a < b$ (both powers of 2) such that $\langle \overline{f_a} \rangle = \langle \overline{f_b} \rangle$. Then, there exists a map

$$\begin{array}{ccc} [1]^{an} & \xrightarrow{\quad g \quad} & [1]^{bn}. \\ & {\scriptstyle \overline{f_a}}\searrow \quad \swarrow{\scriptstyle \overline{f_b}} & \\ & [1]^n & \end{array}$$

By Lemma A.13, we have an induced map of crown posets:



But because $an < bn$, we must have $\deg(g') = 0$ by Lemma A.10, which contradicts that $\deg(f_b)\deg(g') = \deg(f_a) = a$. ∎

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