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Ideals with componentwise linear powers

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Abstract. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K, and let A be a finitely generated standard graded S-algebra. We show that if the defining ideal of A has a quadratic initial ideal, then all the graded components of A are componentwise linear. Applying this result to the Rees ring $\mathcal{R}(I)$ of a graded ideal I gives a criterion on I to have componentwise linear powers. Moreover, for any given graph G, a construction on G is presented which produces graphs whose cover ideals I_G have componentwise linear powers. This, in particular, implies that for any Cohen–Macaulay Cameron–Walker graph G all powers of I_G have linear resolutions. Moreover, forming a cone on special graphs like unmixed chordal graphs, path graphs, and Cohen–Macaulay bipartite graphs produces cover ideals with componentwise linear powers.

1 Introduction

Componentwise linear ideals were first introduced by Herzog and the first author of this paper in [4]. Since their introduction, they have emerged as an intriguing class of ideals deserving special attention, due to some of their interesting characterizations in combinatorics and commutative algebra (see [1, 3, 11, 14, 15]). One research theme in this context is to find ideals whose all powers are componentwise linear or have linear resolutions. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K. A graded ideal $I \subset S$ is called *componentwise linear*, if for each integer j, the ideal generated by all homogeneous elements of degree j in I has a linear resolution. In this paper, we mainly consider the question: what hypotheses on I ensure that all powers of I are componentwise linear? In particular, we consider some graph constructions whose cover ideals have componentwise linear powers. To investigate this question, it is natural to consider the Rees algebra $\mathcal{R}(I)$. For a graded ideal $I \subset S$, let $J \subset T = S[y_1, \dots, y_m]$ be the defining ideal of $\mathcal{R}(I)$. The ideal J is said to satisfy the x-condition with respect to a monomial order $<$ on T, if any minimal monomial generator of $\mathrm{in}_<(J)$ is of the form vw with $v \in S$ of degree at most one and $w \in K[y_1, \dots, y_m]$. This property was first defined in [10]. When J has the x-condition property, then it is proved in [10, Corollary 1.2] that if I is equigenerated, then each power of I has a linear resolution. It is natural to ask when I is not equigenerated, how the x-condition affects the powers of I. This is considered in [7], where it is shown that if J satisfies the

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x-condition with respect to some special monomial order, then I^k has linear quotients with respect to some set of generators for all k, which may not be minimal. But once it is minimal, it implies the componentwise linearity of I^k (see [6, Theorem 8.2.15]). When I is a monomial ideal and $\mathrm{in}_<(J)$ is generated by quadratic monomials, it is shown in [7, Theorem 3.6] that this set of generators is minimal; and hence, all powers of I are componentwise linear. In the first section of this paper, we extend this result to any graded ideal I (see Theorem 2.1 and Corollary 2.2).

The question of having linear or componentwise linear powers has attracted special attention for the ideals arising from graphs and it has been studied in several papers. Some families of such ideals whose powers inherit componentwise linear or linear property are cover ideals of Cohen–Macaulay bipartite graphs [5], unmixed chordal graphs [8, Theorem 2.7], chordal graphs that are $(C_4, 2K_2)$-free [2, Theorem 3.7], trees [13, Corollary 3.5], path graphs, biclique graphs and Cameron–Walker graphs whose bipartite graph is a complete bipartite graph [7, Corollary 4.7], and edge ideals with linear resolutions [10, Theorem 3.2]. In Section 2 of this paper, we consider a construction on a graph G denoted by $G(H_1, \dots, H_n)$ which attaches to each vertex x_i of G a graph H_i. As the main result, we show in Theorem 3.2 that if for each i, the defining ideal J_{H_i} of $\mathcal{R}(I_{H_i})$ has a quadratic Gröbner basis, then each power of the vertex cover ideal $I_{G(H_1,\dots,H_n)}$ is componentwise linear. To this aim, we first show in Theorem 3.1 that if each J_{H_i} satisfies the x-condition, then $J_{G(H_1,\dots,H_n)}$ satisfies the x-condition with respect to some monomial order. Cohen–Macaulay Cameron–Walker graph and cone graphs are examples of such constructions.

2 Algebras with componentwise linear graded components

Let K be a field, and let $A = \bigoplus_{i,j} A_{(i,j)}$ be a bigraded K-algebra with $A_{(0,0)} = K$. Set $A_j = \bigoplus_i A_{(i,j)}$. We assume that A_0 is the polynomial ring $S = K[x_1, \dots, x_n]$ with the standard grading. Then $A = \bigoplus_j A_j$ is a graded S-algebra, and each A_j is a graded S-module with grading $(A_j)_i = A_{(i,j)}$ for all i. We assume in addition that A is a finitely generated standard graded S-algebra and $(A_1)_i = 0$ for $i < 0$.

We fix a system of homogeneous generators $f_1, \dots, f_m$ of A_1 with $\deg f_i = d_i$ for $i = 1, \dots, m$. Let $T = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the bigraded polynomial ring over K with the grading induced by $\deg x_i = (1, 0)$ for $i = 1, \dots, n$ and $\deg y_j = (d_j, 1)$ for $j = 1, \dots, m$. Define the K-algebra homomorphism $\varphi\, T \to A$ with $\varphi(x_i) = x_i$ for $i = 1, \dots, n$ and $\varphi(y_j) = f_j$ for $j = 1, \dots, m$. Then φ is a surjective K-algebra homomorphism of bigraded K-algebras, and hence $J = \mathrm{Ker}(\varphi)$ is a bigraded ideal in T. The defining ideal J of A is said to satisfy the *x-condition*, with respect to a monomial order $<$ on T if all $u \in \mathcal{G}(\mathrm{in}_<(J))$ are of the form vw with $v \in S$ of degree ≤ 1 and $w \in K[y_1, \dots, y_m]$. Here, $\mathcal{G}(I)$ denotes the minimal set of monomial generators of a monomial ideal I.

Given a monomial order $<'$ on the polynomial ring $K[y_1, \dots, y_m]$ and a monomial order $<_x$ on $K[x_1, \dots, x_n]$, let $<$ be a monomial order on T such that

(1) $$\prod_{i=1}^{n} x_i^{a_i} \prod_{i=1}^{m} y_i^{b_i} < \prod_{i=1}^{n} x_i^{a'_i} \prod_{i=1}^{m} y_i^{b'_i},$$

if

$$\prod_{i=1}^{m} y_i^{b_i} <' \prod_{i=1}^{m} y_i^{b'_i} \quad \text{or} \quad \prod_{i=1}^{m} y_i^{b_i} = \prod_{i=1}^{m} y_i^{b'_i} \quad \text{and} \quad \prod_{i=1}^{n} x_i^{a_i} <_x \prod_{i=1}^{n} x_i^{a'_i}.$$

We call $<$ the order induced by the orders $<'$ and $<_x$.

A graded S-module M has *linear quotients*, if there exists a system of homogeneous generators $f_1, \dots, f_m$ of M with the property that each of the colon ideals $(f_1, \dots, f_{j-1}) : f_j$ is generated by linear forms. With the above notation and terminology, we have the following.

Theorem 2.1 *Let J be the defining ideal of the K-algebra A. If $\mathrm{in}_<(J)$ is generated by quadratic monomials, then for all $k \geq 1$, the S-module A_k has linear quotients with respect to a minimal generating set and hence it is componentwise linear.*

Proof Let $k \geq 1$ be an integer. For any element $h = f_{i_1} \dots f_{i_k} \in A_k$, we set $h^* = y_{i_1} \dots y_{i_k}$. Let $\{h_1^*, \dots, h_d^*\}$ be the set of all standard monomials of bidegree $(*, k)$ in T which are of the form h^*. Then by [7, Lemma 3.2], $h_1, \dots, h_d$ is a system of generators of A_k. We show that this is indeed a minimal system of generators of A_k. Since $f_1, \dots, f_m$ is a minimal generating set of A_1, there is nothing to prove for $k = 1$. Now, let $k > 1$. By contradiction suppose that there exists an integer j such that $h_j = \sum_{\ell \neq j} p_\ell h_\ell$ for some homogeneous polynomials $p_\ell \in S$. Then $q = h_j^* - \sum_{\ell \neq j} p_\ell h_\ell^* \in J$. Without loss of generality, assume that q has the least initial term among all the expressions of the form $h_j^* - \sum_{\ell \neq j} p'_\ell h_\ell^* \in J$. Since h_j^* is a standard monomial, we have $\mathrm{in}_<(q) = \mathrm{in}_<(p_i)h_i^* \in \mathrm{in}_<(J)$ for some $i \neq j$ with $p_i \neq 1$. By assumption, there exists an element g in the Gröbner basis of J such that $\mathrm{in}_<(g)$ has degree two and $\mathrm{in}_<(g)$ divides $\mathrm{in}_<(p_i)h_i^*$. If $\mathrm{in}_<(g) = y_r y_t$ for some r and t, then $y_r y_t$ divides h_i^*, which implies that $h_i^* \in \mathrm{in}_<(J)$, a contradiction. So $\mathrm{in}_<(g) = x_r y_t$ for some r and t and then x_r divides $\mathrm{in}_<(p_i)$ and y_t divides h_i^*. Let $g = x_r y_t - \sum_{\ell=1}^{s} c_\ell u_\ell y_{j_\ell}$ for some $c_\ell \in K$ and monomials $u_\ell \in S$. Then

$$x_r f_t = \sum_{\ell=1}^{s} c_\ell u_\ell f_{j_\ell}. \tag{2}$$

We have $u_\ell \neq 1$ for all ℓ. Indeed, if $u_\lambda = 1$ for some λ, then

$$f_{j_\lambda} = c_\lambda^{-1} x_r f_t - \sum_{\ell \neq \lambda} c_\lambda^{-1} c_\ell u_\ell f_{j_\ell}. \tag{3}$$

This is a contradiction, since $f_1, \dots, f_m$ is a minimal generating set of A_1. Thus $u_\ell \neq 1$ for all ℓ. We have

$$\mathrm{in}_<(p_i)h_i = (\mathrm{in}_<(p_i)/x_r)x_r f_t(h_i/f_t) =$$
$$(\mathrm{in}_<(p_i)/x_r)(\sum_{\ell=1}^{s} c_\ell u_\ell f_{j_\ell})(h_i/f_t).$$

Hence $\mathrm{in}_<(p_i)h_i = \sum_{\ell=1}^{s} c_\ell w_\ell h'_\ell$, for the monomials $w_\ell = (\mathrm{in}_<(p_i)/x_r)u_\ell \neq 1$ and elements $h'_\ell = f_{j_\ell}(h_i/f_t) \in A_k$. Let $p'_i = p_i - \mathrm{in}_<(p_i)$. Then

$$(4)\qquad h_j = (\mathrm{in}_<(p_i) + p_i')h_i + \sum_{\ell\neq i,j} p_\ell h_\ell = \sum_{\ell=1}^{s} c_\ell w_\ell h_\ell' + p_i' h_i + \sum_{\ell\neq i,j} p_\ell h_\ell.$$

We show that $h_\ell' \neq h_j$ for all $1 \leq \ell \leq s$. Indeed, by the equality (2) and that $u_\ell \neq 1$, we have $\deg(f_t) \geq \deg(f_{j_\ell})$. Hence $\deg(h_\ell') \leq \deg(h_i)$. While $\deg(h_j) = \deg(p_i) + \deg(h_i) > \deg(h_i)$. Hence $\deg(h_j) > \deg(h_\ell')$ which implies that $h_\ell' \neq h_j$. Also $y_{j_\ell} <' y_t$ implies that $(h_\ell')^* <' h_i^*$. By (4), we have

$$(5)\qquad q' = h_j^* - \sum_{\ell=1}^{s} c_\ell w_\ell (h_\ell')^* - p_i' h_i^* - \sum_{\ell\neq i,j} p_\ell h_\ell^* \in J.$$

Since $(h_\ell')^* \neq h_j^*$ for all ℓ and $\mathrm{in}_<(q') < \mathrm{in}_<(q)$, by the assumption on q, we conclude that there exists at least one integer $1 \leq \ell \leq s$ such that the monomial $(h_\ell')^*$ is not standard. By [7, Lemma 2.2], for any such ℓ, there exist standard monomials $h_{t_{\ell,1}}^*, \ldots, h_{t_{\ell,b}}^*$ with $h_{t_{\ell,1}}^* < \cdots < h_{t_{\ell,b}}^* < (h_\ell')^*$ and homogeneous polynomials $v_{\ell,\lambda}$ such that

$$(6)\qquad (h_\ell')^* - \sum_{\lambda=1}^{b} v_{\ell,\lambda} h_{t_{\ell,\lambda}}^* \in J.$$

Again notice that $j \notin \{t_{\ell,1}, \ldots, t_{\ell,b}\}$, since $\deg(h_j) > \deg(h_\ell') \geq \deg(h_{t_{\ell,\lambda}})$ for any $1 \leq \lambda \leq b$. Set $I_1 = \{\ell : 1 \leq \ell \leq s,\ (h_\ell')^* \text{ is not standard}\}$ and $I_2 = [s]\backslash I_1$. By (5) and (6), we obtain an expression

$$q'' = h_j^* - \sum_{\ell\in I_1}\sum_{\lambda=1}^{b} c_\ell w_\ell v_{\ell,d}\, h_{t_{\ell,d}}^* - \sum_{\ell\in I_2} c_\ell w_\ell (h_\ell')^* - p_i' h_i^* - \sum_{\ell\neq i,j} p_\ell h_\ell^* \in J$$

in terms of standard monomials. Since $\mathrm{in}_<(q'') < \mathrm{in}_<(q)$, we get a contradiction.

Thus, $h_1, \ldots, h_d$ is a minimal system of generators of A_k. Now, using [7, Theorem 2.3], we conclude that for each k, A_k has linear quotients with respect to its minimal set of generators. Thus, it follows from [6, Theorem 8.2.15] that A_k is componentwise linear. ■

Applying Theorem 2.1 to the Rees algebra of a graded ideal, we obtain the following corollary which generalizes [6, Corollary 10.1.8] and [7, Theorem 3.6].

Corollary 2.2 *Let $I \subset S$ be a graded ideal, and let J be the defining ideal of the Rees algebra $\mathcal{R}(I)$. If $\mathrm{in}_<(J)$ is generated by quadratic monomials with respect to the monomial order defined in (1), then for any $k \geq 1$, I^k has linear quotients with respect to its minimal monomial generating set and hence it is componentwise linear.*

3 Cover ideals of graphs with componentwise linear powers

Let G be a finite simple graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $E(G)$ be the set of edges of G. A subset $C \subseteq V(G)$ is called a *vertex cover* of G, when it intersects any edge of G. Moreover, C is called a *minimal vertex cover* of G, if it is a vertex cover and no proper subset of C is a vertex cover of G. Let as before $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K. We associate each subset

$C \subset V(G)$ with the monomial $u_C = \prod_{x_i \in C} x_i$ of S. Let $C_1, \ldots, C_q$ denote the minimal vertex covers of G. The *cover ideal* of G is defined as $I_G = (u_{C_1}, \ldots, u_{C_q})$. The Rees algebra of I_G is the toric ring

$$\mathcal{R}(I_G) = K[x_1, \ldots, x_n, u_{C_1}t, \ldots, u_{C_q}t] \subset S[t].$$

Let $T = S[y_1, \ldots, y_q]$ denote the polynomial ring and define the surjective map $\pi : T \to \mathcal{R}(I_G)$ by setting $\pi(x_i) = x_i$ and $\pi(y_j) = u_{C_j}t$. The toric ideal $J_G \subset T$ of $\mathcal{R}(I_G)$ is the kernel of π. Let $<_{\text{lex}}$ denote the pure lexicographic order on S induced by the ordering $x_1 > \cdots > x_n$ and suppose that $u_{C_q} <_{\text{lex}} \cdots <_{\text{lex}} u_{C_1}$. Let $<'_{\text{lex}}$ denote the pure lexicographic order on $K[y_1, \ldots, y_q]$ induced by the ordering $y_1 > \cdots > y_q$. We let $<^\sharp$ be the monomial order on T which is induced by the orders $<'_{\text{lex}}$ and $<_{\text{lex}}$.

Given finite simple graphs $H_1, \ldots, H_n$ with $V(H_i) = \{z_1^{(i)}, \ldots, z_{r_i}^{(i)}\}$, we construct the graph $G(H_1, \ldots, H_n)$ on $V(G) \cup V(H_1) \cup \cdots \cup V(H_n)$ whose set of edges is

$$E(G) \cup E(H_1) \cup \cdots \cup E(H_n) \cup \left(\bigcup_{\substack{1 \le i \le n \\ 1 \le j \le r_i}} \{x_i, z_j^{(i)}\} \right)$$

Theorem 3.1 *Suppose that each J_{H_i} satisfies the x-condition with respect to $z_1^{(i)} > \cdots > z_{r_i}^{(i)}$. Then $J_{G(H_1,\ldots,H_n)}$ satisfies the x-condition with respect to*

$$z_1^{(1)} > z_2^{(1)} > \cdots > z_{r_1}^{(1)} > z_1^{(2)} > \cdots > z_{r_2}^{(2)} > \cdots > z_{r_n}^{(n)} > x_1 > \cdots > x_n.$$

Proof Let $S = K[x_1, \ldots, x_n, z_1^{(1)}, \ldots, z_{r_n}^{(n)}]$ denote the polynomial ring in $n + r_1 + \cdots + r_n$ variables over a field K. Let $C_1^{(i)}, \ldots, C_{s_i}^{(i)}$ denote the minimal vertex covers of H_i. Given a vertex cover C of G, we introduce a subset $C' = C \cup C^{(i)} \cup \cdots \cup C^{(n)}$ of $V(G) \cup V(H_1) \cup \cdots \cup V(H_n)$ for which

$$C^{(i)} = \begin{cases} V(H_i), & \text{if} \quad x_i \notin C, \\ C_j^{(i)}, & \text{if} \quad x_i \in C, \end{cases}$$

where $1 \le j \le s_i$ is arbitrary. It follows that C' is a minimal vertex cover of G and that every minimal vertex cover of G is of the form C'.

Let $<_{\text{lex}}$ denote the pure lexicographic order on S with respect to the above ordering $z_1^{(1)} > \cdots > x_n$. Let $C_1, \ldots, C_q$ denote the minimal vertex covers of $G(H_1, \ldots, H_n)$ and suppose that $u_{C_q} <_{\text{lex}} \cdots <_{\text{lex}} u_{C_1}$.

Let $T_i = K[z_1^{(i)}, \ldots, z_{r_i}^{(i)}, y_1^{(i)}, \ldots, y_{s_i}^{(i)}]$ denote the polynomial ring in $r_i + s_i$ variables over K and $J_{H_i} \subset T_i$ the toric ideal of $\mathcal{R}(I_{H_i})$ which is the kernel of $\pi_i : T_i \to \mathcal{R}(I_{H_i})$. Let $w = z_j^{(i)} w''$ be a monomial belonging to the minimal system of monomial generators of $\text{in}_{<^\sharp}(J_{H_i})$, where w'' is a monomial in $y_1^{(i)}, \ldots, y_{s_i}^{(i)}$. Let $\pi_i(w) = z_j^{(i)} u_{C_{\xi_1}^{(i)}} t \ldots u_{C_{\xi_a}^{(i)}} t$. Let $C_{\zeta_{i'}}$ be a minimal vertex covers of $G(H_1, \ldots, H_n)$ with $C_{\zeta_{i'}} \cap V(H_i) = C_{\xi_{i'}}^{(i)}$ for each $1 \le i' \le a$. It follows that $z_j^{(i)} y_{\zeta_1} \ldots y_{\zeta_a}$ belongs to $\text{in}_{<^\sharp}(J_{G(H_1,\ldots,H_n)}) \subset T = S[y_1, \ldots, y_q]$.

Let C be a minimal vertex cover of $G(H_1, \ldots, H_n)$ with $x_i \notin C$ and

$$C' = ((C \cup \{x_i\}) \setminus V(H_i)) \cup C_j^{(i)},$$

where $1 \leq j \leq s_i$ is arbitrary. Then C' is a minimal vertex cover of $G(H_1, \ldots, H_n)$ with $u_{C'} <_{\text{lex}} u_C$. Let $C = C_e$ and $C' = C_f$. Then $e < f$ and

$$x_i y_e - \prod_{z_{j'}^{(i)} \notin C_j^{(i)}} z_{j'}^{(i)} y_f \in J_{G(H_1, \ldots, H_n)}$$

whose initial monomial is $x_i y_e$.

Let $\mathcal{A}$ denote the set of monomials of the form either $z_j^{(i)} y_{\zeta_1} \ldots y_{\zeta_a}$ or $x_i y_e$ constructed above. Let $\mathcal{B}$ denote the set of monomials in $y_1, \ldots, y_q$ belonging to the minimal system of monomial generators of $\text{in}_{<^\sharp}(J_{G(H_1,\ldots,H_n)})$.

Let $(\mathcal{A}, \mathcal{B})$ denote the monomial ideal of T generated by $\mathcal{A} \cup \mathcal{B}$. One claims that $\text{in}_{<^\sharp}(J_{G(H_1,\ldots,H_n)}) = (\mathcal{A}, \mathcal{B})$. One must prove that, for monomials u and v of T not belonging to $(\mathcal{A}, \mathcal{B})$ with $u \neq v$, one has $\pi(u) \neq \pi(v)$, see [9, Theorem 3.11]. Let $u = u_x u_z u_y$ and $v = v_x v_z v_y$ with $u \neq v$, where u_x, v_x are monomials in $x_1, \ldots, x_n$, where u_z, v_z are monomials in $z_j^{(i)}, 1 \leq i \leq n, 1 \leq j \leq r_i$, and where u_y, v_y are monomials in $y_1, \ldots, y_s$. Suppose that u and v are relatively prime and that $\pi(u) = \pi(v)$.

Let, say, $u_x \neq 1$ and x_i divide u_x. Since x_i does not divide v_x and since $\deg u_y = \deg v_y$, it follows that there is y_a which divides u_y for which $x_i \notin C_a$. Hence, $x_i y_a \in \mathcal{A}$, a contradiction. Thus, $u_x = v_x = 1$.

Let $\pi(u) = u_z \cdot u_{C_{a_1}} t \ldots u_{C_{a_p}} t$ and $\pi(v) = v_z \cdot u_{C_{a'_1}} t \ldots u_{C_{a'_p}} t$. Let, say, $u_z \neq 1$ and $z_j^{(i)}$ divide u_z. In each of $\pi(u)$ and $\pi(v)$, replace each of $x_1, \ldots, x_n$ with 1 and replace $z_j^{(i')}$ with 1 for each $i' \neq i$ and for each $1 \leq j \leq r_i$. Then $\pi(u)$ comes to

$$u'_z \cdot u_{C_{c_1}^{(i)}} t \ldots u_{C_{c_{p'}}^{(i)}} t \left(\prod_{j=1}^{r_i} z_j^{(i)} \right)^{p-p'}$$

and $\pi(v)$ comes to

$$v'_z \cdot u_{C_{c'_1}^{(i)}} t \ldots u_{C_{c'_{p'}}^{(i)}} t \left(\prod_{j=1}^{r_i} z_j^{(i)} \right)^{p-p'},$$

where each of u'_z and v'_z is a monomial in $z_1^{(i)}, \ldots, z_{r_i}^{(i)}$. One has $p' > 0$ and

$$u'_z \cdot y_{c_1}^{(i)} \ldots y_{c_{p'}}^{(i)} - v'_z \cdot y_{c'_1}^{(i)} \ldots y_{c'_{p'}}^{(i)} \in J_{H_i}.$$

Since $z_j^{(i)}$ divides u'_z, one has $u'_z \cdot y_{c_1}^{(i)} \ldots y_{c_{p'}}^{(i)} - v'_z \cdot y_{c'_1}^{(i)} \ldots y_{c'_{p'}}^{(i)} \neq 0$ and its initial monomial belongs to $\text{in}_{<^\sharp}(J_{H_i})$. Since J_{H_i} satisfies the x-condition, it follows that either $u \in (\mathcal{A}, \mathcal{B})$ or $v \in (\mathcal{A}, \mathcal{B})$, a contradiction. Thus $u_z = v_z = 1$.

Since $u = u_y, v = v_y, u - v \neq 0$ and $\pi(u) = \pi(v)$, one has either $u \in (\mathcal{B})$ or $v \in (\mathcal{B})$, a contradiction. ∎

Theorem 3.2 *Suppose that each J_{H_i} has a quadratic Gröbner basis with respect to some monomial order. Then each power of the vertex cover ideal $I_{G(H_1,\dots,H_n)}$ possesses an order of linear quotients on its minimal monomial generating set, and hence it is componentwise linear.*

Proof We keep the notation used in the proof of Theorem 3.1. Suppose that J_{H_i} has a quadratic Gröbner basis with respect to $z_1^{(i)} > \cdots > z_{r_i}^{(i)}$ for each i. By Theorem 3.1, the ideal $J = J_{G(H_1,\dots,H_n)}$ satisfies the x-condition with respect to the order $<$ induced by the orders $z_1^{(i)} > \cdots > z_{r_i}^{(i)}$. Hence by the proof of [7, Theorem 3.3], for any positive integer k, the ideal $(I_{G(H_1,\dots,H_n)})^k$ has a system of generators $h_1, \dots, h_s$, each of them of the form $h_i = u_{C_{i_1}} \dots u_{C_{i_k}}$, which possess an order of linear quotients $h_1 < \cdots < h_s$ and such that $h_i^* = y_{i_1} \dots y_{i_k}$ is a standard monomial of T with respect to $<^\sharp$ and J. We prove that $\{h_1, \dots, h_s\}$ is the minimal generating set of monomials of $(I_{G(H_1,\dots,H_n)})^k$.

Suppose that $h_j = wh_i$ for some integers i and j and a monomial $w \in S$. We should show that $w = 1$ and $i = j$. Let $h_i = u_{C_{i_1}} \dots u_{C_{i_k}}$ and $h_j = u_{C_{j_1}} \dots u_{C_{j_k}}$. If $w = 1$ and $i \neq j$, then $h_j^* - h_i^* \in J$. So either h_i^* or h_j^* belongs to $\mathrm{in}_{<^\sharp}(J)$, a contradiction. Hence $i = j$ and we are done.

Now, assume that $w \neq 1$. Let $w = w_x w_z$, where w_x is a monomial in $x_1, \dots, x_n$ and w_z is a monomial in $z_j^{(i)}, 1 \le i \le n, 1 \le j \le r_i$. Assume that $w_x \neq 1$ and $x_t | w$ for some integer t. So we have $x_t \notin C_{i_a}$ for some $1 \le a \le k$. The set

$$C' = ((C_{i_a} \cup \{x_t\}) \setminus V(H_t)) \cup C_\ell^{(t)},$$

where $1 \le \ell \le s_t$ is arbitrary, is a minimal vertex cover of $G(H_1, \dots, H_n)$ with $u_{C'} <_{\mathrm{lex}} u_C$. Let $C' = C_b$. Then $i_a < b$ and

$$x_t y_{i_a} - \prod_{z_{j'}^{(t)} \notin C_\ell^{(t)}} z_{j'}^{(t)} y_b \in J_{G(H_1,\dots,H_n)}$$

whose initial monomial is $x_t y_{i_a}$. Since $x_t | w$ and $u_{C_{i_a}} | h_i$, we may write $wh_i = w'h'$, where $w' = (w/x_t) \prod_{z_{j'}^{(t)} \notin C_\ell^{(t)}} z_{j'}^{(t)}$ and $h' = (h_i/u_{C_{i_a}}) u_{C_b}$. Therefore $h_j = w'h'$ with $\deg(w'_x) < \deg(w_x)$. Repeating this procedure, we obtain $h_j = uf$, where u is a monomial with $u_x = 1$ and $u_z \neq 1$ and $f = u_{C_{\ell_1}} \dots u_{C_{\ell_k}}$ for some $\ell_1, \dots, \ell_k$. Let $w''(h'')^*$ be the unique standard monomial in T with respect to $<^\sharp$ and J such that $\pi(w''(h'')^*) = f$, where w'' is a monomial in S.

Then $h_j = uf = uw''h''$. Let $h'' = u_{C_{s_1}} \dots u_{C_{s_k}}$ If $x_t | w''$ for some integer t, then we have $x_t \notin C_{s_a}$ for some a and then $x_t y_{s_a} \in \mathrm{in}_{<^\sharp}(J)$. Since $x_t y_{s_a}$ divides $w''(h'')^*$, this implies that $w''(h'')^* \in \mathrm{in}_{<^\sharp}(J)$, a contradiction. Hence $w''_x = 1$. Note that $(h'')^*$ is a standard monomial as well. So the equality $h_j = uf = u_z w''_z h''$ shows that we may reduce to the case that $w_x = 1$. Hence $w = w_z$ with $w_z \neq 1$ and $h_j = w_z h_i$.

Let, say, $z_j^{(i)}$ divides w_z. In each of h_j, h_i and w, replace each of $x_1, \dots, x_n$ with 1 and replace $z_j^{(i')}$ with 1 for each $i' \neq i$ and for each $1 \le j \le r_{i'}$. Then h_j comes to

$$u_{C_{c_1}^{(i)}} t \dots u_{C_{c_{p'}}^{(i)}} t \left(\prod_{j=1}^{r_i} z_j^{(i)} \right)^{p-p'}$$

and h_i comes to

$$u_{C^{(i)}_{c'_1}} t \dots u_{C^{(i)}_{c'_{p'}}} t \left(\prod_{j=1}^{r_i} z_j^{(i)} \right)^{p-p'},$$

and w comes to v, where v is a monomial in $z_1^{(i)}, \dots, z_{r_i}^{(i)}$. One has $p' > 0$ and

$$y_{c_1}^{(i)} \dots y_{c_{p'}}^{(i)} - v \cdot y_{c'_1}^{(i)} \dots y_{c'_{p'}}^{(i)} \in J_{H_i}. \tag{7}$$

Moreover, since h_i^* and h_j^* are standard monomials in T, $y_{c_1}^{(i)} \dots y_{c_{p'}}^{(i)}$ and $y_{c'_1}^{(i)} \dots y_{c'_{p'}}^{(i)}$ are standard monomials in T_i, i.e., they do not belong to $\mathrm{in}_<(J_{H_i})$. Since J_{H_i} has a quadratic Gröbner basis with respect to $z_1^{(i)} > \dots > z_{r_i}^{(i)}$, as was shown in the proof of [7, Theorem 3.6], the monomials $u_{C^{(i)}_{c_1}} \dots u_{C^{(i)}_{c_{p'}}}$ and $u_{C^{(i)}_{c'_1}} \dots u_{C^{(i)}_{c'_{p'}}}$ belong to the minimal set of monomial generators of $(I_{H_i})^{p'}$. This contradicts to equation (7). ■

The following corollaries are derived from Theorem 3.2.

Corollary 3.3 *Let $G' = G(H_1, \dots, H_n)$, where G is any graph on n vertices and each H_i belongs to one of the following families of graphs:*

(a) *Unmixed chordal graphs.*
(b) *Cohen–Macaulay bipartite graphs.*
(c) *Path graphs.*
(d) *Cameron–Walker graphs whose bipartite graphs are complete graphs.*

Then any powers of the vertex cover ideal $I_{G'}$ possesses an order of linear quotients on its minimal monomial generating set and hence it is componentwise linear.

Proof For any graph H_i belonging to one of the families (a) to (d), the defining ideal of $\mathcal{R}(I_{H_i})$ has a quadratic Gröbner basis, see [5], [7, Corollary 4.7], and [8, Theorem 2.7]. Hence, the desired result follows from Theorem 3.2. ■

Let G be a graph with $V(G) = \{x_1, \dots, x_n\}$. A *cone* on G is a graph G' with $V(G') = V(G) \cup \{y\}$ and $E(G') = E(G) \cup \{\{x_i, y\} : 1 \le i \le n\}$, where y is a new vertex. The vertex y is called a *universal vertex* of G' and G' is called a *cone graph*.

A *friendship graph* F_n is a planar graph with $2n + 1$ vertices and $3n$ edges. It can be constructed by joining n copies of the cycle graph C_3 with a common vertex. A *fan graph* $F_{1,n}$ is a cone on a path graph with n vertices.

Corollary 3.4 *Let G be one of the following graphs:*

(a) *A Cameron–Walker graph with the bipartite partition (X, Y) for which there exists one leaf attached to each vertex in X and at least one pendant triangle attached to each vertex of Y.*

(b) *A cone graph with a universal vertex x such that $G\setminus\{x\}$ is one of the graphs (a) to (d) in Corollary 3.3. Examples of such cone graphs are the fan graphs $F_{1,n}$, friendship graphs F_n and star graphs.*

Then all powers of the vertex cover ideal I_G have linear quotients and hence are componentwise linear.

The following result is an immediate consequence of Corollary 3.4 and the characterization of Cohen–Macaulay Cameron–Walker graphs given in [12, Theorem 1.3].

Corollary 3.5 *Let G be a Cohen–Macaulay Cameron–Walker graph. Then all power of the vertex cover ideal of G have linear resolutions.*

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