Bull. Aust. Math. Soc. **99** (2019), 242–249
doi:10.1017/S0004972718001181

EXISTENCE AND BLOW-UP OF SOLUTIONS TO A PARABOLIC EQUATION WITH NONSTANDARD GROWTH CONDITIONS

YANG LIU

(Received 30 May 2018; accepted 29 September 2018; first published online 11 December 2018)

Abstract

We study the initial boundary value problem for a fourth-order parabolic equation with nonstandard growth conditions. We establish the local existence of weak solutions and derive the finite time blow-up of solutions with nonpositive initial energy.

2010 *Mathematics subject classification*: primary 35K35; secondary 35A01, 35B44.

Keywords and phrases: fourth-order parabolic equation, existence, blow-up.

1. Introduction

In this paper, we study the following initial boundary value problem for a parabolic equation of $p(x)$-biharmonic type

$$\begin{cases} u_t + \Delta^2_{p(x)} u = |u|^{q(x)-2}u & \text{for all } (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{for all } x \in \Omega, \\ u(x,t) = \Delta u(x,t) = 0 & \text{for all } (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, and

$$\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u).$$

The functions $p : \overline{\Omega} \to [p_-, p_+] \subset (1,\infty)$ and $q : \overline{\Omega} \to [q_-, q_+] \subset (1,\infty)$ are measurable, and

$$p_- = \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\overline{\Omega}} p(x), \quad q_- = \operatorname*{ess\,inf}_{x\in\overline{\Omega}} q(x), \quad q_+ = \operatorname*{ess\,sup}_{x\in\overline{\Omega}} q(x).$$

This work was supported by the Science and Technology Plan Project of Gansu Province in China (Grant No. 17JR5RA279) and the Talent Introduction Research Project of Northwest Minzu University in China (Grant No. xbmzdxyjrc201701).

In addition, $p(x)$ is log-Hölder continuous on $\overline{\Omega}$ (see [5]), that is, there exists a constant $C > 0$ such that, for any $x_1, x_2 \in \overline{\Omega}$,

$$|p(x_1) - p(x_2)| \le \frac{C}{\ln(e + |x_1 - x_2|^{-1})}. \tag{1.2}$$

Recently, differential equations and variational problems with nonstandard growth conditions have attracted more and more attention (see, for example, [2–4, 8, 9]). They arise from various physical problems involving anisotropic phenomena such as nonlinear electrorheological fluids [1, 10] and elastic mechanics [14]. These applications have been facilitated by the development of Lebesgue and Sobolev spaces with variable exponents.

In the case $p(x) \equiv p$, a constant, the differential equation $(1.1)_1$ without source term becomes

$$u_t + \Delta_p^2 u = 0. \tag{1.3}$$

Equation (1.3) describes the image intensity, u, in a You–Kaveh model [12], which has been demonstrated to be effective for the trade-off between noise removal and edge preservation in image processing. The differential equation $(1.1)_1$ is a generalised nonlinear version of (1.3). Of course, the nonlinearity comes not only from the source term but also from the diffusion term.

The stationary equation corresponding to the differential equation $(1.1)_1$ is treated in [3, 4, 7]. Drábek and Ôtani [7] studied the nonlinear eigenvalue problem for the p-biharmonic operator with $p > 1$, that is,

$$\Delta_p^2 u = \lambda |u|^{p-2} u,$$

with the Navier boundary condition $(1.1)_3$. They obtained a principal positive eigenvalue that is simple and isolated and proved that the corresponding eigenfunction $u_1 = u_1(p) > 0$ satisfies $\Delta u_1 < 0$ in Ω and $\partial u_1/\partial n < 0$ on $\partial\Omega$.

When $p(x)$ is not a constant, the $p(x)$-biharmonic operator possesses more complicated nonlinearity than the p-biharmonic operator. The existence of weak solutions to the equation

$$\Delta_{p(x)}^2 u = \lambda |u|^{q(x)-2} u$$

with boundary condition $(1.1)_3$ was investigated by Ayoujil and El Amrouss when $p(x) = q(x)$ [3] and when $p(x) \neq q(x)$ [4].

Our purpose is to seek sufficient conditions for existence and blow-up of weak solutions to the problem (1.1). The paper is organised as follows. In Section 2, we display some notation, definitions and known facts on the Lebesgue and Sobolev spaces with variable exponents and present our main results on problem (1.1). In Section 3, we prove the local existence of solutions to problem (1.1). Section 4 is devoted to the proof of finite time blow-up of solutions to problem (1.1).

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press

2. Notation and main results

We first recall some definitions and basic properties of the Lebesgue and Sobolev spaces with variable exponents (see [5, 6, 13] for further details).

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable real-valued functions with

$$\int_\Omega |u|^{p(x)}\,dx < \infty.$$

Equipped with the norm

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\mu > 0 \;\Big|\; \int_\Omega \Big|\frac{u}{\mu}\Big|^{p(x)}\,dx \le 1\Big\},$$

$L^{p(x)}(\Omega)$ is a separable and reflexive Banach space, and

$$\min\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\} \le \int_\Omega |u|^{p(x)}\,dx \le \max\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\}.$$

We denote the dual space of $L^{p(x)}(\Omega)$ by $L^{p'(x)}(\Omega)$, where the conjugate exponent is $p'(x) = p(x)/(p(x)-1)$.

The Sobolev space with variable exponents is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where $k \ge 1$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$, and $D^\alpha u$ is the αth-weak partial derivative of u. Equipped with the norm

$$\|u\|_{k,p(x)} := \|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{p(x)},$$

$W^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space. If $q(x) \le p_*(x)$, the variable exponent Sobolev embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ holds for any $u \in W^{k,p(x)}(\Omega)$, where

$$p_*(x) = \begin{cases} \dfrac{Np(x)}{N - kp(x)} & \text{if } kp(x) < N,\\ \infty & \text{if } kp(x) \ge N.\end{cases}$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. From (1.2), $C_0^\infty(\Omega)$ is dense in $W_0^{k,p(x)}(\Omega)$. We write $\mathcal{W} := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ equipped with the norm $\|u\|_{\mathcal{W}} = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$. Then $\|u\|_{\mathcal{W}}$, $\|u\|_{2,p(x)}$ and $\|\Delta u\|_{p(x)}$ are equivalent.

Throughout the paper, for the sake of simplicity, we denote $\Omega \times [0, T]$ by Q_T, the norm $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$, the norm $\|\cdot\|_2$ by $\|\cdot\|$ and $\int_\Omega uv\,dx$ by (u, v).

Definition 2.1. A function $u \in L^\infty(0, T; \mathcal{W})$ with $u_t \in L^2(0, T; L^2(\Omega))$ is called a weak solution to problem (1.1) if $u(0) = u_0$ and

$$(u_t, v) + (|\Delta u|^{p(x)-2}\Delta u, \Delta v) = (|u|^{q(x)-2}u, v)$$

for any $v \in C_0^\infty(\Omega)$ and almost all $t \in (0, T]$.

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press

We define the energy functional associated with problem (1.1) by

$$E(u) = \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx. \tag{2.1}$$

Our main results can now be stated as follows.

THEOREM 2.2. *Assume that* $u_0(x) \in \mathcal{W}$ *and* p_-, p_+, q_-, q_+ *satisfy*

$$\text{(A)} \qquad \frac{2N}{N+2} < p_- < N, \quad p_+ < q_-, \quad 2 < q_+ < \frac{Np_-}{N-p_-}.$$

Then there exists a constant $T > 0$ *such that problem (1.1) admits a solution* u *with* $u \in L^\infty(0,T;\mathcal{W})$ *and* $u_t \in L^2(0,T;L^2(\Omega))$. *Moreover, we have the energy inequality*

$$E(u(t)) + \int_0^t \|u_\tau(\tau)\|^2\,d\tau \le E(u_0). \tag{2.2}$$

THEOREM 2.3. *Suppose that* (A) *holds and* $p_- \ge 2$. *Assume that* $u_0(x) \in \mathcal{W}$ *and* $E(u_0) \le 0$. *Then the solutions to problem (1.1) blow up in finite time.*

3. Proof of Theorem 2.2

Choose an orthonormal basis $\{w_j(x)\}_{j=1}^\infty$ of $L^2(\Omega)$ and consider approximate solutions

$$u_n(x,t) = \sum_{j=1}^n \xi_{jn}(t) w_j(x) \quad \text{for } n = 1,2,\ldots$$

satisfying

$$(u_{nt}, w_j) + (|\Delta u_n|^{p(x)-2}\Delta u_n, \Delta w_j) = (|u_n|^{q(x)-2}u_n, w_j) \quad \text{for } j = 1,2,\ldots,n, \tag{3.1}$$

$$u_n(x,0) = \sum_{j=1}^n \xi_{jn}(0) w_j(x) \to u_0(x) \quad \text{in } \mathcal{W}. \tag{3.2}$$

The approximate problem (3.1), (3.2) can be reduced to an ordinary differential system in the variables $\xi_{jn}(t)$. In terms of the standard theory for ordinary differential equations, there exists a classical solution u_n.

Multiplying (3.1) by $\xi_{jn}(t)$ and summing over j gives

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \int_\Omega |\Delta u_n|^{p(x)}\,dx = \int_\Omega |u_n|^{q(x)}\,dx. \tag{3.3}$$

Since

$$\int_\Omega |u_n|^{q(x)}\,dx \le \|u_n\|_{q_+}^{q_+} + |\Omega|,$$

we deduce from the Gagliardo–Nirenberg interpolation inequality that there exists a constant $C_1 > 0$, independent of u_n, such that

$$\int_\Omega |u_n|^{q(x)}\,dx \le C_1 \|\nabla u_n\|_{p_-}^{\theta q_+} \|u_n\|^{(1-\theta)q_+} + |\Omega|,$$

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press

where

$$\theta = \frac{Np_-(q_+ - 2)}{q_+(Np_- + 2p_- - 2N)}.$$

It is easy to see that

$$\int_\Omega |u_n|^{q(x)}\,dx \le C_2 \|\Delta u_n\|_{p_-}^{\theta q_+} \|u_n\|^{(1-\theta)q_+} + |\Omega|,$$

which together with Young's inequality with ϵ gives

$$\int_\Omega |u_n|^{q(x)}\,dx \le C_2(\epsilon \|\Delta u_n\|_{p_-}^{p_-} + C(\epsilon)\|u_n\|^\beta) + |\Omega|,$$

where

$$\beta = \frac{2Np_- + 2p_-q_+ - 2Nq_+}{Np_- + 2p_- - Nq_+}.$$

Consequently, from

$$\|\Delta u_n\|_{p_-}^{p_-} \le \int_\Omega |\Delta u_n|^{p(x)}\,dx + |\Omega|,$$

we have

$$\int_\Omega |u_n|^{q(x)}\,dx \le C_2\Big(\epsilon \int_\Omega |\Delta u_n|^{p(x)}\,dx + C(\epsilon)\|u_n\|^\beta\Big) + (\epsilon C_2 + 1)|\Omega|. \tag{3.4}$$

Taking $\epsilon = 1/C_2$ and substituting (3.4) into (3.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 \le C_3(\|u_n\|^\beta + 1),$$

for some sufficiently large C_3. Hence there exists T_0 such that

$$\|u_n\| \le \mathfrak{C}_1(T) \tag{3.5}$$

for all $t \in [0, T]$ with $T < T_0$, where $\mathfrak{C}_1(T)$ stands for a constant that is independent of u_n but depends on T.

Multiplying (3.1) by $\xi'_{jn}(t)$ and summing over j, we obtain

$$\|u_{nt}\|^2 + \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx = \frac{d}{dt}\int_\Omega \frac{|u_n|^{q(x)}}{q(x)}\,dx.$$

Integrating both sides of this equality with respect to t,

$$\int_0^t \|u_{n\tau}\|^2\,d\tau + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx = \int_\Omega \frac{|u_n|^{q(x)}}{q(x)}\,dx - \int_\Omega \frac{|u_n(0)|^{q(x)}}{q(x)}\,dx + \int_\Omega \frac{|\Delta u_n(0)|^{p(x)}}{p(x)}\,dx. \tag{3.6}$$

Note that

$$\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx \ge \frac{1}{p_+}\int_\Omega |\Delta u_n|^{p(x)}\,dx,$$

and

$$\int_\Omega \frac{|u_n|^{q(x)}}{q(x)}\,dx \le \frac{1}{q_-}\int_\Omega |u_n|^{q(x)}\,dx.$$

It follows from (3.4)–(3.6) that

$$\int_0^t \|u_{n\tau}\|^2\,d\tau \le \mathfrak{C}_2(T), \qquad \int_\Omega |\Delta u_n|^{p(x)}\,dx \le \mathfrak{C}_3(T). \tag{3.7}$$

From (3.7), there exist u, χ and a subsequence of $\{u_n\}$ (we shall always relabel subsequences as the same sequence), such that, as $n \to \infty$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty(0,T;\mathcal{W}) \quad\text{and}\quad u_n \to u \text{ almost everywhere in } Q_T, \tag{3.8}$$

$$\begin{aligned} u_{nt} &\rightharpoonup u_t \text{ weakly in } L^2(0,T;L^2(\Omega)),\\ |\Delta u_n|^{p(x)-2}\Delta u_n &\rightharpoonup \chi \text{ weakly in } L^{p'(x)}(Q_T). \end{aligned} \tag{3.9}$$

From the monotonicity of the operator $s \mapsto |s|^{p-2}s$ and the mean value inequality in [11], $\chi = |\Delta u|^{p(x)-2}\Delta u$ by arguments similar to the proof of [2, Lemma 3.5]. Therefore, we can pass to the limit in the approximate problem (3.1) and (3.2). Thus $u(t)$ is a solution to problem (1.1) in the sense of Definition 2.1.

Next we prove (2.2). Indeed, from (3.8), (3.9), (3.6) and (3.2), it follows that

$$\begin{aligned} \int_0^t \|u_\tau\|^2\,d\tau + \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\,dx &\le \liminf_{n\to\infty}\Big(\int_0^t \|u_{n\tau}\|^2\,d\tau + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx\Big)\\ &= \liminf_{n\to\infty}\Big(\int_\Omega \frac{|u_n|^{q(x)}}{q(x)}\,dx - \int_\Omega \frac{|u_n(0)|^{q(x)}}{q(x)}\,dx + \int_\Omega \frac{|\Delta u_n(0)|^{p(x)}}{p(x)}\,dx\Big)\\ &= \int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx - \int_\Omega \frac{|u(0)|^{q(x)}}{q(x)}\,dx + \int_\Omega \frac{|\Delta u(0)|^{p(x)}}{p(x)}\,dx. \end{aligned}$$

Thus the proof of Theorem 2.2 is complete.

4. Proof of Theorem 2.3

Let u be a solution to problem (1.1) and let $T_{\max}$ be the maximum existence time of $u(t)$. Next we prove that $T_{\max} < \infty$. If it is false, then $T_{\max} = \infty$. We consider the auxiliary function

$$M(t) = \frac{1}{2}\int_0^t \|u\|^2\,d\tau.$$

A direct calculation gives

$$M'(t) = \tfrac{1}{2}\|u\|^2,$$

and

$$M''(t) = (u, u_t) = -\Big(\int_\Omega |\Delta u|^{p(x)}\,dx - \int_\Omega |u|^{q(x)}\,dx\Big). \tag{4.1}$$

Recalling (2.1), we see that

$$E(u) \ge \frac{q_- - p_+}{p_+ q_-}\int_\Omega |\Delta u|^{p(x)}\,dx + \frac{1}{q_+}\Big(\int_\Omega |\Delta u|^{p(x)}\,dx - \int_\Omega |u|^{q(x)}\,dx\Big).$$

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press

Hence

$$\int_\Omega |\Delta u|^{p(x)}\,dx - \int_\Omega |u|^{q(x)}\,dx \le -\frac{(q_- - p_+)q_+}{p_+q_-}\int_\Omega |\Delta u|^{p(x)}\,dx + q_+E(u),$$

which together with (2.2) gives

$$\int_\Omega |\Delta u|^{p(x)}\,dx - \int_\Omega |u|^{q(x)}\,dx \le -\frac{(q_- - p_+)q_+}{p_+q_-}\int_\Omega |\Delta u|^{p(x)}\,dx + q_+E(u_0) - q_+\int_0^t \|u_\tau\|^2\,d\tau.$$

Combining this with (4.1),

$$M''(t) \ge \frac{(q_- - p_+)q_+}{p_+q_-}\int_\Omega |\Delta u|^{p(x)}\,dx - q_+E(u_0) + q_+\int_0^t \|u_\tau\|^2\,d\tau.$$

Thus

$$M''(t) \ge q_+\int_0^t \|u_\tau\|^2\,d\tau. \tag{4.2}$$

Now, it is easy to see that there exists a $t_0 > 0$ such that $M'(t) \ge M'(t_0) > 0$ and $M(t) \ge M'(t_0)(t - t_0) + M(t_0)$ for all $t \in [t_0, \infty)$. Consequently,

$$\lim_{t\to\infty} M(t) = \infty. \tag{4.3}$$

On the other hand, we deduce from (4.2) and the Cauchy–Schwarz inequality that

$$\begin{aligned} M(t)M''(t) &\ge \frac{q_+}{2}\int_0^t \|u\|^2\,d\tau \int_0^t \|u_\tau\|^2\,d\tau \\ &\ge \frac{q_+}{2}\Big(\int_0^t (u, u_\tau)\,d\tau\Big)^2 \\ &= \frac{q_+}{2}(M'(t) - M'(0))^2. \end{aligned}$$

Hence there exists a $\gamma > 0$ such that

$$M(t)M''(t) \ge (1+\gamma)M'(t)^2.$$

It is easy to verify that $M^{-\gamma}(t) > 0$ is decreasing and concave on $[t_0, \infty)$, which contradicts (4.3). Hence $T_{\max} < \infty$. Thus the proof of Theorem 2.3 is complete.

Acknowledgement

The author would like to thank the referees for the valuable suggestions.

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press

References

[1] E. Acerbi and G. Mingione, 'Regularity results for stationary electro-rheological fluids', *Arch. Ration. Mech. Anal.* **164**(3) (2002), 213–259.

[2] S. Antontsev and S. Shmarev, 'Anisotropic parabolic equations with variable nonlinearity', *Publ. Mat.* **53**(2) (2009), 355–399.

[3] A. Ayoujil and A. R. El Amrouss, 'On the spectrum of a fourth order elliptic equation with variable exponent', *Nonlinear Anal.* **71**(10) (2009), 4916–4926.

[4] A. Ayoujil and A. R. El Amrouss, 'Continuous spectrum of a fourth order nonhomogeneous elliptic equation with variable exponent', *Electron. J. Differential Equations* **2011**(24) (2011), 12 pp.

[5] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents* (Springer, Heidelberg, 2011).

[6] L. Diening, P. Nägele and M. Růžička, 'Monotone operator theory for unsteady problems in variable exponent spaces', *Complex Var. Elliptic Equ.* **57**(11) (2012), 1209–1231.

[7] P. Drábek and M. Ôtani, 'Global bifurcation result for the p-biharmonic operator', *Electron. J. Differential Equations* **2001**(48) (2001), 19 pp.

[8] J. Giacomoni, S. Tiwari and G. Warnault, 'Quasilinear parabolic problem with $p(x)$-Laplacian: existence, uniqueness of weak solutions and stabilization', *Nonlinear Differ. Equ. Appl.* **23** (2016), Article ID 24.

[9] P. Harjulehto, P. Hästö, Út V. Lê and M. Nuortio, 'Overview of differential equations with non-standard growth', *Nonlinear Anal.* **72**(12) (2010), 4551–4574.

[10] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory* (Springer, Berlin, 2000).

[11] J. Simon, 'Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$', in: *Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977)*, Lecture Notes in Mathematics, 665 (Springer, Berlin, 1978), 205–227.

[12] Y. You and M. Kaveh, 'Fourth-order partial differential equations for noise removal', *IEEE Trans. Image Process.* **9**(10) (2000), 1723–1730.

[13] A. Zang and Y. Fu, 'Interpolation inequalities for derivatives in variable exponent Lebesgue–Sobolev spaces', *Nonlinear Anal.* **69**(10) (2008), 3629–3636.

[14] V. V. Zhikov, 'Averaging of functionals of the calculus of variations and elasticity theory', *Izv. Akad. Nauk SSSR Ser. Mat.* **50**(4) (1986), 675–710.

YANG LIU, College of Mathematics and Computer Science,
Northwest Minzu University, Lanzhou 730124, PR China
and
College of Mathematics, Sichuan University, Chengdu 610065, PR China
e-mail: liuyangnufn@163.com

https://doi.org/10.1017/S0004972718001181 Published online by Cambridge University Press