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BLOCKING SETS IN PROJECTIVE SPACES

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1. Introduction. Blocking sets in projective spaces have been of interest for quite some time, having applications to game theory (see [**6**; **7**]) as well as finite nets and partial spreads (see [**5**]). In [**4**] Bruen showed that if B is a blocking set in a projective plane of order n, then

$$n + \sqrt{n} + 1 \leqq |B| \leqq n^2 - \sqrt{n}.$$

In this paper it is shown that if B is a blocking set in $PG(k, q)$, where $k \geqq 3$, then

$$|B| \geqq q^h + q^{h-1} + \ldots + q + 1 \quad \text{if } k = 2h + 1, \text{ and}$$
$$|B| \geqq q^h + 2q^{h-1} + q^{h-2} + \ldots + q + 1 \quad \text{if } k = 2h.$$

2. Background. The concept of game theory was first introduced in [**8**] by von Neumann and Morgenstern, and was later expounded by several other people. The definitions of the terms used in this paper can be found in [**7**]. Let $\Sigma = PG(k, q)$ denote k-dimensional projective space over the finite field $GF(q)$, where q is any prime-power. To define a *projective game* Σ, we take the $k/2$-dimensional subspaces of $PG(k, q)$ as minimal winning coalitions if k is even and the $(k + 1)/2$-dimensional subspaces as minimal winning coalitions if k is odd. A *blocking set* in $\Sigma = PG(k, q)$ is a collection of points in Σ that meets every minimal winning coalition but contains no minimal winning coalition.

Since the complement of a blocking set B is again a blocking set, to find bounds for $|B|$ it is sufficient to find a lower bound. The difficulty of this problem depends heavily upon whether k is even or odd.

3. The odd dimensional case. It is easy to find sharp bounds for $|B|$ when B is a blocking set in $PG(2h + 1, q)$, as the following theorems show. The obvious proofs will be omitted.

THEOREM (1). *Let B denote a blocking set in $PG(2h + 1, q)$, where h is any positive integer. Then $|B| \geqq q^h + q^{h-1} + \ldots + q + 1$.*

Clearly an h-space of $\Sigma = PG(2h + 1, q)$ meets every $(h + 1)$-space of Σ but contains no $(h + 1)$-space of Σ. Hence an h-space of Σ is a blocking set of Σ, and the bound of Theorem (1) is sharp. The next result states that every blocking set in Σ of minimum cardinality is an h-space.

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THEOREM (2). *Let B denote a blocking set in $\Sigma = PG(2h + 1, q)$ such that $|B| = q^h + q^{h-1} + \ldots + q + 1$. Then B is an h-space of Σ.*

COROLLARY. *Let B denote a blocking set of $PG(2h + 1, q)$. Then $|B| \geqq q^h + \ldots + q + 1$. Moreover, equality holds if and only if B is an h-space of $PG(2h + 1, q)$.*

4. The even dimensional case. As mentioned earlier, finding good bounds for $|B|$ when B is a blocking set in an even dimensional projective space is a much more difficult problem. The following results are due to A. Bruen and proofs can be found in [**3**; **4**].

THEOREM (3). *Let π denote a projective plane of order n. If B is a blocking set in π, then $|B| \geqq n + \sqrt{n} + 1$. Moreover, if $n = m^2$, then $|B| = m^2 + m + 1$ if and only if B is a Baer subplane of order m.*

In Desarguesian projective planes of order p^t, where p is a prime, T. G. Ostrom pointed out (see [**4**, Theorem 5.3]) that blocking sets of cardinality $p^t + p^{t-1} + 1$ always exist. This is the best method so far known for constructing small blocking sets in projective planes of non-square order. It is not known whether or not Bruen's lower bound of $n + \sqrt{n} + 1$ can be improved when n is a non-square. Moreover, Bruen's methods do not extend to projective spaces of dimension higher than two.

Our main thrust for the remainder of this paper will be to study blocking sets in $PG(2h, q)$, where $h \geqq 2$ is an integer. A blocking set B in $\Sigma = PG(2h, q)$ meets every h-space of Σ but contains no h-space of Σ. Since there are $q^h + \ldots + q + 1$ h-spaces containing a given $(h - 1)$-space, $|B| \geqq q^h + \ldots + q + 1$. An h-space of Σ contains precisely $q^h + \ldots + q + 1$ points, but an h-space is not a blocking set of Σ since it contains a minimal winning coalition, namely itself. The following results show that the lower bound given above can be significantly increased in any even dimensional projective space. In fact, an additional term of q^{h-1} can be added to this bound.

LEMMA (1). *Suppose B is a set of points in $\Sigma = PG(n, q)$ that meets every k-space of Σ, where n is a positive integer and $0 \leqq k \leqq n$. Then $|B| \geqq q^{n-k} + \ldots + q + 1$.*

Proof. Follows immediately.

Before looking at blocking sets in arbitrary even dimensional projective spaces, we first study the situation in $PG(4, q)$. The proof of the next lemma, although quite simple, will serve as a prototype to keep in mind when reading the fairly complicated induction presented in Theorem (4).

LEMMA (2). *Let B denote a blocking set in $\Sigma = PG(4, q)$. Then $|B| \geqq q^2 + 2q + 1$.*

Proof. Lemma (1) implies that we can assume there exists a line l of Σ that misses B. Since there are $q^2 + q + 1$ planes of Σ containing l, we can assume that one of these planes, say π, meets B in at most one point and hence in exactly one point P. Let $\Gamma_1, \ldots, \Gamma_{q+1}$ denote the $q + 1$ 3-spaces of Σ that contain π. Clearly $\Gamma_i \cap B$ is a blocking set of Γ_i for $i = 1, 2, \ldots, q + 1$. According to the corollary of Theorem (2), $|\Gamma_i \cap B| \geqq q + 1$ for all i and $|\Gamma_i \cap B| = q + 1$ if and only if $\Gamma_i \cap B$ is a line through the point P.

Suppose, first of all, that at most one Γ_i meets B in a line containing P. The rest of the 3-spaces containing π meet B in at least $q + 2$ points, only one of which lies on π. Thus

$$|B| \geqq q(q + 1) + 1(q) + 1 = q^2 + 2q + 1$$

and we are done.

Hence we can assume that at least two Γ_i's each meet B in a line containing P. Let π_0 denote the plane determined by these two intersecting lines of B. Since B is a blocking set of Σ, π_0 contains a point R such that $R \notin B$. Choose a collection of planes in Σ, one being π_0, with the property that together the planes cover all the points of Σ but any two planes meet precisely in the point R. It is easy to see from the theory of spreads that such a collection of planes can always be chosen, and, in fact, a generalization of this construction will be proved in Theorem (4). Such a collection has $q^2 + 1$ members, each meeting B in at least one point. Moreover, the member π_0 meets B in at least $2q + 1$ points. The distinctness of all these intersection points implies that

$$|B| \geqq q^2(1) + 1(2q + 1) = q^2 + 2q + 1,$$

and the lemma is proved.

THEOREM (4). *Let B denote a blocking set in $\Sigma = PG(2h, q)$, where $h \geqq 2$ is an integer. Then $|B| \geqq q^h + 2q^{h-1} + q^{h-2} + \ldots + q + 1$.*

Proof. By Lemma (1) we can assume that B misses some $(h - 1)$-space Γ of Σ. Since there are $q^h + \ldots + q + 1$ h-spaces containing Γ, we can assume that some h-space γ^0 meets B in at most one point and hence in exactly one point P. If χ is an $(h + k)$-space of Σ, then $B \cap \chi$ is a set of points in χ that meets every h-space of χ. Thus $|B \cap \chi| \geqq q^k + \ldots + q + 1$ by Lemma (1). In particular, every $(h + 1)$-space of Σ meets B in at least $q + 1$ points.

There are $q^{h-1} + \ldots + q + 1$ $(h + 1)$-spaces of Σ containing γ^0. If each of these meets B in at least $q + 2$ points, then each meets B in at least $q + 1$ points other than P and

$$|B| \geqq (q^{h-1} + \ldots + q + 1)(q + 1) + 1 = q^h + 2q^{h-1} + 2q^{h-2} + \ldots + 2q + 2$$

and we are done. Thus we can assume that some $(h + 1)$-space γ^1 containing γ^0 meets B in eaxctly $q + 1$ points. Clearly $B \cap \gamma^1$ is a line l of γ^1.

A simple induction shows that for $k = 2, 3, \ldots, h - 1$, we can assume there exists an $(h + k)$-space γ^k of Σ containing γ^{k-1} that meets B in at most $q^k + 2q^{k-1} + q^{k-2} + \ldots + q$ points. Let $\gamma^0 \subseteq \gamma^1 \subseteq \ldots \subseteq \gamma^{h-1}$ be chosen as above. Moreover, at the ith stage, we can assume that γ^i has been chosen so that $|B \cap \gamma^i|$ is as small as possible. We now claim that for $k = 0, 1, \ldots, h - 1$, if χ is any $(h + k + 1)$-space containing γ^k, then χ meets B in at least $q^{k+1} + 2q^k + q^{k-1} + \ldots + q + 1$ points or else $B \cap \chi$ contains a $(k + 1)$-space. As shown earlier, every $(h + 1)$-space containing γ^0 either meets B in at least $q + 2$ points or else meets B in a line. Thus the claim is true for $k = 0$. Assume by induction that the claim is true for all $j \leqq k$, where $k \leqq h - 2$, and we will now prove that the claim is true for $k + 1$. Thus, according to the preceding paragraph, we can assume that $B \cap \gamma^j$ contains a j-space ϵ^j for $j = 0, 1, \ldots, k + 1$. Letting χ be any $(h + k + 2)$-space containing γ^{k+1}, we want to show that $|B \cap \chi| \geqq q^{k+1} + 2q^{k+1} + q^k + \ldots + q + 1$ or else else $B \cap \chi$ contains a $(k + 2)$-space.

There are $q + 1$ $(h + k + 1)$-spaces of χ containing γ^k, one of which is γ^{k+1}. The proof of the claim will be broken up into two cases.

Case (1). Suppose, first of all, that two $(h + k + 1)$-spaces Σ_1 and Σ_2 of χ containing γ^k meet B in at most $q^{k+1} + 2q^k + q^{k-1} + \ldots + q$ points. Then $\Sigma_1 \cap B$ and $\Sigma_2 \cap B$ both contain $(k + 1)$-spaces by induction. Say that Σ_i^* is a $(k + 1)$-space contained in $\Sigma_i \cap B$ for $i = 1, 2$. Now $(\Sigma_1 \cap B) \cap (\Sigma_2 \cap B) = \gamma^k \cap B$ and $|\gamma^k \cap B| \leqq q^k + 2q^{k-1} + q^{k-2} + \ldots + q$. Since a $(k + 1)$-space contains $q^{k+1} + \ldots + q + 1$ points, Σ_1^* and Σ_2^* are *distinct* $(k + 1)$-spaces contained in B. Moreover, since $|\Sigma_i \cap B| \leqq q^{k+1} + 2q^k + q^{k-1} + \ldots + q$ and $|\Sigma_i^*| = q^{k+1} + \ldots + q + 1$, at most $q^k - 1$ points of $\Sigma_i \cap B$ lie outside Σ_i^* for $i = 1, 2$. Thus, since $\epsilon^k \subseteq \gamma^k \cap B \subseteq \Sigma_i \cap B$ and $|\epsilon^k| = q^k + \ldots + q + 1$, we see that $|\epsilon^k \cap \Sigma_i^*| \geqq q^{k-1} + \ldots + q + 2$ for $i = 1, 2$ and $\epsilon^k \cap \Sigma_i^*$ is a k-space by an order argument. Hence ϵ^k is a projective subspace of Σ_i^* for $i = 1, 2$ and thus $\epsilon^k = \Sigma_1^* \cap \Sigma_2^*$. Therefore B contains two $(k + 1)$-spaces Σ_1^* and Σ_2^* intersecting in a k-space, namely ϵ^k. Let $\alpha = \langle \Sigma_1^*, \Sigma_2^* \rangle$ be the $(k + 2)$-space of χ generated by Σ_1^* and Σ_2^*.

If α is contained in B, then $B \cap \chi$ contains the $(k + 2)$-space α and we have proved the claim. Thus we may as well assume that α is not contained in B. Let Q be a point of α such that $Q \notin B$. Since we are trying to prove that $|B \cap \chi| \geqq q^{k+2} + 2q^{k+1} + q^k + \ldots + q + 1$, Lemma (1) applied to the point set $B \cap \chi$ in χ allows us to assume that there exists an $(h - k - 2)$-space δ of χ containing the point Q that misses B. If $\delta \cap \alpha$ contains a line m, then the line m must meet the $(k + 1)$-space Σ_1^* at some point of the $(k + 2)$-space α, yielding the contradiction that $\delta \cap B \neq \emptyset$. Thus $\delta \cap \alpha = Q$ and $\langle \delta, \alpha \rangle$ is an h-space of χ.

Next take a collection of h-spaces in χ, one member being $\langle \delta, \alpha \rangle$, with the property that together they cover all the points of χ and any two members meet precisely in the $(h - k - 2)$-space δ. Such a construction is attained by taking a spread of pairwise disjoint $(k + 1)$-spaces covering all the points in the quotient space $\chi/\delta \cong PG(2k + 3, q)$, one member being $\langle\delta, \alpha\rangle/\delta \cong PG(k + 1, q)$, and then "pulling back" to χ (see [**2**, Section 5]). Such a collection will have $q^{k+2} + 1$ members. Each member meets $B \cap \chi$ in at least one point since B is a blocking set, and the member $\langle\delta, \alpha\rangle$ meets $B \cap \chi$ in at least the $2q^{k+1} + q^k + \ldots + q + 1$ distinct points of $\Sigma_1^* \cup \Sigma_2^*$. Moreover, all these points are distinct since $\delta \cap B = \emptyset$. Therefore $|B \cap \chi| \geqq q^{k+2} + 2q^{k+1} + q^k + \ldots + q + 1$, and the claim is proved in this case.

Case (2): The only remaining case to be considered is when only one $(h + k + 1)$-space of χ containing γ^k, namely γ^{k+1}, meets B in at most $q^{k+1} + 2q^k + q^{k-1} + \ldots + q$ points. Let $N_i = |\gamma^i \cap B|$ for $i = 0, 1, \ldots, h - 1$. As shown earlier, $N_0 = 1$ and $N_1 = q + 1$. Since there are $q + 1$ $(h + k + 1)$-spaces of χ containing γ^k and each of these, except γ^{k+1}, meets B in at least $q^{k+1} + 2q^k + q^{k-1} + \ldots + q + 1$ points,

$$\begin{aligned}|B \cap \chi| &\geqq q(q^{k+1}\, 2q^k + q^{k-1} + \ldots + q + 1 - N_k) + (N_{k+1} - N_k) + N_k \\ &= q^{k+2} + 2q^{k+1} + q^k + \ldots + q - qN_k + N_{k+1}.\end{aligned}$$

Thus, to prove the claim, it suffices to show that $N_{k+1} - qN_k \geqq 1$.

There are $q + 1$ $(h + k)$-spaces of γ^{k+1} containing γ^{k-1}, one of which is γ^k. Also γ^k was chosen so that $|\gamma^k \cap B| \leqq |\gamma^* \cap B|$ for all $(h + k)$-spaces γ^* containing γ^{k-1}. Thus

$$N_{k+1} = |\gamma^{k+1} \cap B| \geqq (q + 1)(N_k - N_{k-1}) + N_{k-1} = (q + 1)N_k - qN_{k-1}.$$

Since we want to show that $N_{k+1} \geqq qN_k + 1$, it suffices to show that $(q + 1)N_k - qN_{k-1} \geqq qN_k + 1$. But this is equivalent to showing that $N_k \geqq qN_{k-1} + 1$. Repeating the same procedure one step at a time, the problem eventually reduces to showing that $N_1 \geqq qN_0 + 1$. But this inequality is apparent since $N_1 = q + 1$ and $N_0 = 1$. Hence, all cases considered, the claim has been proved by induction.

Taking $k = h - 1$ in the above claim, since $\Sigma = PG(2h, q)$ is a $(2h)$-space containing γ^{h-1}, either Σ meets B in at least $q^h + 2q^{h-1} + q^{h-2} + \ldots + q + 1$ points or $\Sigma \cap B = B$ contains an h-space. Since B contains no h-spaces of Σ by definition,

$$|\Sigma \cap B| = |B| \geqq q^h + 2q^{h-1} + q^{h-2} + \ldots + q + 1$$

and the theorem is proved.

The following theorems give some examples of relatively small blocking sets in even dimensional projective spaces, and show that the bound given in Theorem (4) is reasonably good.

THEOREM (5). *Let $\Sigma = PG(2h, r^2)$, where r is a prime-power and $h \geqq 2$ is an integer. Let Σ_0 be a Baer sub-$2h$-space of Σ of order r. Then Σ_0 is a blocking set of Σ and*

$$|\Sigma_0| = r^{2h} + r^{2h-1} + \ldots + r + 1.$$

Proof. Follows immediately from the definition of a Baer subspace.

This theorem shows that when q is a square, $PG(2h, q)$ always contains a blocking set of cardinality

$$q^h + q^{(2h-1)/2} + q^{h-1} + \ldots + q + q^{1/2} + 1.$$

THEOREM (6). *Let $\Sigma = PG(2h, q)$, where $h \geqq 2$ is an integer and q is any prime-power. Then there exists a blocking set B in Σ such that $|B| = 2q^h + q^{h-1} + q^{h-2} + \ldots + q$.*

Proof. Let Γ be an h-space of Σ, let R be a point of Γ, and let Σ_0 be any $(h + 1)$-space containing Γ. Suppose Γ^* is an h-space of Σ that meets Γ precisely in the point R. A dimension argument shows that $\langle \Gamma^*, \Sigma_0 \rangle$ is the entire space Σ since Γ and Γ^* are h-spaces of $\langle \Gamma^*, \Sigma_0 \rangle$ that meet in a single point. Another dimension argument shows that $\Gamma^* \cap \Sigma_0$ is a line l, and this line passes through R since the line l and the h-space Γ must meet in Σ_0. Thus every h-space that meets Γ in precisely the point R intersects Σ_0 in a line that meets Γ in R.

There are q^h lines of Σ_0 that meet Γ in precisely the point R, and we pick a point on each of these lines so that all the chosen points are distinct from R and they do not all lie on the same h-space of Σ. If $\{P_i\}$ denotes the q^h chosen points, let B denote $\{P_i\}$ together with the points of Γ other than R. Then $|B| = 2q^h + q^{h-1} + \ldots + q$.

Since every h-space of Σ meets Γ, the argument given in the first paragraph shows that B meets every h-space. To show that B is a blocking set of Σ, it suffices to show that B contains no h-space. Suppose Γ' is an h-space of Σ such that $\Gamma' \subseteq B$. Since $B \subseteq \Sigma_0$, a dimension argument shows that $\Gamma \cap \Gamma'$ is an $(h - 1)$-space of Σ_0. Thus there are q^h points of Γ' lying outside Γ, and these must be all the points of $\{P_i\}$. This contradicts the choice of $\{P_i\}$, and the theorem is proved.

5. An application to msp spreads. If W denotes a maximal strictly partial spread (msp spread) of $\Sigma = PG(2h + 1, q)$, a well-known problem is to find bounds on $|W|$. For arbitrary $h \geqq 2$ the only available bounds are $q + 1 \leqq |W| \leqq q^{h+1} - \sqrt{q}$. Although $|W| \geqq q + 1$ can be shown in a straightforward manner, this lower bound can also be obtained using techniques similar to those of Bruen [**5**] and Beutelspacher [**1**]. That is, choose a hyperplane H of Σ containing no member of W and let S denote the points of H lying in any member of W. Then S is a blocking set of H and $|S| = |W|(q^{h-1} + \ldots + q + 1)$. The result now follows by taking the trivial lower bound of $q^h + q^{h-1} + \ldots + q + 1$ for $|S|$.

The lower bound given in Theorem (4) for $|S|$ does not improve the above result. However, a significantly improved lower bound for $|S|$, such as that given in the conjecture of the following section, will also give a new and improved lower bound for $|W|$ as above.

6. Concluding remarks. The author firmly believes that the lower bound given in Theorem (4) is not sharp. As mentioned above, a significantly better bound would not only be of interest in its own right but would also yield a new result in the msp spread problem stated in Section 5. It is also of great interest to find examples of minimal (or at least very small) blocking sets in $\Sigma = PG(2h, q)$. It should be pointed out that the blocking set given in Theorem (6) has all its points contained in an $(h + 1)$-space of Σ. To find smaller blocking sets, one presumable should scatter the points more than this. The author has been unsuccessful so far in constructing such blocking sets.

We close this paper with the following conjecture, stemming from Theorems (3) and (5).

CONJECTURE. *If S is a blocking set in $PG(2h, q)$, then*

$$|S| \geqq q^h + q^{(2h-1)/2} + q^{h-1} + \ldots + q + q^{1/2} + 1.$$

REFERENCES

1. A. Beutelspacher, *Partial spreads in finite projective spaces and partial designs*, Math. Z. *145* (1975), 211–229.
2. R. H. Bruck and R. C. Bose, *The construction of translation planes from projective spaces*, J. Algebra *1* (1964), 85–102.
3. A. Bruen, *Baer subplanes and blocking sets*, Bull. Amer. Math. Soc. *76* (1970), 342–344.
4. ——— *Blocking sets in finite projective planes*, SIAM J. Appl. Math. *21* (1971), 380–392.
5. ——— *Partial spreads and replaceable nets*, Can. J. Math. *23* (1971), 381–391.
6. A. J. Hoffman and M. Richardson, *Block design games*, Can. J. Math. *13* (1961), 110–128.
7. M. Richardson, *On finite projective games*, Proc. Amer. Math. Soc. *7* (1956), 453–465.
8. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior* (2nd ed., Princeton Univ. Press, Princeton, 1947).

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