J. Austral. Math. Soc. Ser. B 37(1995), 99–120

LOW FREQUENCY SCATTERING OF ELASTIC WAVES BY A CAVITY USING A MATCHED ASYMPTOTIC EXPANSION METHOD

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(Received 14 February 1994; revised 1 May 1994)

Abstract

This work deals with low-frequency asymptotic solutions using the method of matched asymptotic expansions. It is based on two papers by Buchwald [3] and Buchwald and Tran Cong [4] who studied the diffraction of elastic waves by a small circular cavity and a small elliptic cavity, respectively, in an otherwise unbounded domain. Here we clarify and systematize some aspects of their work and extend it to the diffraction of elastic waves by a small cylindrical cavity with a hypotrochoidal boundary. Results for the case of an incident P-wave are compared, in the special case of an elliptic boundary, with the results from the numerical solution of the boundary integral equation method.

1. Introduction

In two recent works [1][2], we considered the problem of scattering of time-harmonic stress waves by an infinite cylindrical cavity of arbitrary smooth cross-section, in an otherwise unbounded, homogenous, isotropic, linearly elastic solid. This problem was solved using the boundary integral equation (B.I.E.) method. However, it is known that the simplest integral equations fail to have a unique solution at the so-called irregular frequencies (I.F.). Two methods for overcoming this difficulty were then presented along with some numerical results. In this paper, we set out to solve, for low frequencies, the boundary value problem stated above, using the method of matched asymptotic expansions (M.A.E.). This method, extensively used in fluid dynamics, has only recently been adopted to study the scattering of elastic waves by small cylindrical inhomogeneities as reviewed by Datta [6]. Buchwald [3] developed a method for studying the diffraction of elastic waves by a small circular cylindrical cavity in an otherwise unbounded domain. His method is based on the establishment of a relationship between the equations of plane elastodynamics and elastostatics.

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Later Buchwald and Tran Cong [4] extended this method, using Muskhelishvili's [8] conformal mapping method, to the diffraction of elastic waves by a small elliptic cylindrical cavity. Here we shed some light on some aspects of Buchwald and Tran Cong's [4] work and extend the use of this method to the diffraction of elastic waves by a small cylindrical cavity whose smooth arbitrary cross-section can be mapped onto a circle by one of a certain class of mappings (see below). Some numerical results for the elliptic case are presented and compared with those obtained from the B.I.E. method.

2. Basic formulation

Let us recall the formulation of the boundary value problem as it is stated in [1]: Determine a function $\underset{\sim}{u}^{sc}$ for $P \in D$, satisfying

1. Elastodynamic equations of motion in D

$$k^{-2}.\nabla(\nabla.\underset{\sim}{u}^{sc}(P)) - K^{-2}.\nabla x(\nabla x\underset{\sim}{u}^{sc}(P)) + \underset{\sim}{u}^{sc}(P) = 0, \quad P \in D. \tag{2.1}$$

2. Stress-free boundary condition on ∂D

$$T\underset{\sim}{u}^{sc}(p) = \underset{\sim}{\tau}^{sc}.\hat{n}(p) = -\underset{\sim}{\tau}^{inc}(p).\hat{n}(p) = -T\underset{\sim}{u}^{inc}(p), \quad p \in \partial D. \tag{2.2}$$

3. The radiation conditions as defined in [1].

It is known that the solution of (2.1) can be expressed in terms of two potentials which satisfy the Helmoltz equation. Following Buchwald [3], this expression has the form

$$\mu.u_x^{sc} = \mu'.\partial\phi^{sc}/\partial x - \partial\psi^{sc}/\partial y, \tag{2.3a}$$

$$\mu.u_y^{sc} = \mu'.\partial\phi^{sc}/\partial y - \partial\psi^{sc}/\partial x, \tag{2.3b}$$

where u_x^{sc} and u_y^{sc} are the scattered components of the displacements in the x and y-directions, respectively, and ϕ^{sc}, ψ^{sc} are the corresponding potentials. The dimensionless constants λ' and μ' are given in terms of the Lamé constants λ and μ by

$$\lambda' = \lambda/(\lambda + 2\mu) = (1 - 2\tau^2), \tag{2.4a}$$

$$\mu' = \mu/(\lambda + 2\mu) = \tau^2, \tag{2.4b}$$

so that $\lambda' + 2\mu' = 1$.

Substituting (2.3) into (2.1) yields, after some manipulation,

$$\nabla^2\left(\frac{\partial\phi^{sc}}{\partial x}-\frac{\partial\psi^{sc}}{\partial y}\right)=K^2\left(\mu'\frac{\partial\phi^{sc}}{\partial x}-\frac{\partial\psi^{sc}}{\partial y}\right), \tag{2.5a}$$

$$\nabla^2\left(\frac{\partial\phi^{sc}}{\partial y}+\frac{\partial\psi^{sc}}{\partial x}\right)=K^2\left(\mu'\frac{\partial\phi^{sc}}{\partial y}+\frac{\partial\psi^{sc}}{\partial x}\right), \tag{2.5b}$$

where ∇^2 is the two-dimensional Laplacian operator.

If $\hat{n}$ is the unit normal at a point on the boundary and $\hat{s}$ is the unit tangent at the same point, such that the three vectors $(\hat{n};\hat{s};\hat{e}_3)$ form a right-handed local orthonormal coordinate system, the boundary condition (2.2) can then be expressed as

$$\tau^{sc}_{nn}=-\tau^{inc}_{nn};\quad \tau^{sc}_{sn}=-\tau^{inc}_{sn}, \tag{2.6}$$

where τ^{sc}_{nn}, τ^{sc}_{sn} are the scattered components of the stress tensor with respect to that local coordinate system and τ^{inc}_{nn}, τ^{inc}_{sn} correspond to the incident field.

3. The inner problem and expansion

We introduce the inner variables (x', y') and the corresponding polar coordinates (r', θ') such that

$$x'=x/L,\quad y'=y/L,\quad r'=r/L\quad\text{and}\quad \theta'=\theta, \tag{3.1}$$

where L is a characteristic constant length of the scatterer. In terms of these dimensionless variables, (2.5) becomes

$$\nabla'^2\left(\frac{\partial\phi^{sc}}{\partial x'}-\frac{\partial\psi^{sc}}{\partial y'}\right)=(KL)^2\left(\mu'\frac{\partial\phi^{sc}}{\partial x'}-\frac{\partial\psi^{sc}}{\partial y'}\right), \tag{3.2a}$$

$$\nabla'^2\left(\frac{\partial\phi^{sc}}{\partial y'}+\frac{\partial\psi^{sc}}{\partial x'}\right)=(KL)^2\left(\mu'\frac{\partial\phi^{sc}}{\partial y'}+\frac{\partial\psi^{sc}}{\partial x'}\right), \tag{3.2b}$$

where ∇'^2 denotes the two-dimensional dimensionless Laplacian operator.

As $KL \ll 1$, we assume the following asymptotic expansions for ϕ^{sc} and ψ^{sc}.

$$\phi^{sc}=\sum_{n=0}^{\infty}(iKL)^n\phi^{sc}_n\quad\text{and}\quad \psi^{sc}=\sum_{n=0}^{\infty}(iKL)^n\psi^{sc}_n. \tag{3.3}$$

Substituting (3.3) into (3.2), and comparing coefficients of powers of iKL, we get

$$\nabla'^2\left(\frac{\partial\phi^{sc}_l}{\partial x'}-\frac{\partial\psi^{sc}_l}{\partial y'}\right)=\left(\mu'\frac{\partial\phi^{sc}_{l-2}}{\partial x'}-\frac{\partial\psi^{sc}_{l-2}}{\partial y'}\right), \tag{3.4a}$$

$$\nabla'^2\left(\frac{\partial\phi^{sc}_l}{\partial y'}+\frac{\partial\psi^{sc}_l}{\partial x'}\right)=\left(\mu'\frac{\partial\phi^{sc}_{l-2}}{\partial y'}+\frac{\partial\psi^{sc}_{l-2}}{\partial x'}\right), \tag{3.4b}$$

for $l = 0, 1, 2, \dots$ and the right-hand sides are zero when $l = 0, 1$. It follows that

$$\nabla'^4 \phi_l^{sc} = \nabla'^2 \phi_{l-2}^{sc} \quad \text{and} \quad \nabla'^4 \psi_l^{sc} = \nabla'^2 \psi_{l-2}^{sc} \quad \text{for } l \geq 2 \tag{3.5}$$

and for $l = 0, 1$, the functions ϕ_l^{sc}, ψ_l^{sc} satisfy the biharmonic equation

$$\nabla'^4 \phi_l^{sc} = \nabla'^4 \psi_l^{sc} = 0 \quad \text{for } l = 0, 1. \tag{3.6}$$

In order to avoid solving the inhomogeneous equations (3.5), we shall only consider (3.6), and so we are restricting ourselves to the first two terms in the expansions (3.3). The solution of (3.6) is obtained using Muskhelishvili's [8] technique. Let

$$W_l^{sc} = \phi_l^{sc}(x', y') + i.\psi_l^{sc}(x', y') \quad \text{for } l = 0, 1. \tag{3.7}$$

The solution can then be expressed as follows (see [3]):

$$W_l^{sc} = \bar{z}.\Omega_l(z) + \int \omega_l(z) dz \quad l = 0, 1, \tag{3.8}$$

where $\Omega_l(z)$, $\omega_l(z)$ are functions of the complex variable $z = x' + i.y'$ which are analytic in the appropriate domain, and $\bar{z} = x' - i.y'$.

We now introduce the following expressions which are obtained after the substitution of (3.8) into the expressions of the Cartesian components of the stress tensor and displacements:

$$\begin{aligned} \Theta_l^{sc} &= (\tau_{xx}^{sc} + \tau_{yy}^{sc}) = \frac{2}{L^2}.(1-\mu')\nabla'^2 \phi_l^{sc} = \frac{8}{L^2}.(1-\mu')\frac{\partial^2 \phi_l^{sc}}{\partial z \partial \bar{z}} \\ &= \frac{4}{L^2}.(1-\mu')\left\{\frac{\partial^2 W_l^{sc}}{\partial z \partial \bar{z}} + \frac{\partial^2 \bar{W}_l^{sc}}{\partial z \partial \bar{z}}\right\} = \frac{4}{L^2}.(1-\mu')\left\{\Omega_l'(z) + \overline{\Omega_l'(z)}\right\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \Phi_l^{sc} &= (\tau_{xx}^{sc} - \tau_{yy}^{sc} + 2i.\tau_{xy}^{sc}) = \frac{4}{L^2}.\left\{(1+\mu')\frac{\partial^2 W_l^{sc}}{\partial \bar{z}^2} - (1-\mu')\frac{\partial^2 \bar{W}_l^{sc}}{\partial \bar{z}^2}\right\} \\ &= -\frac{4}{L^2}.(1-\mu')\left\{z.\overline{\Omega_l''(z)} + \overline{\omega_l'(z)}\right\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} D_l &= \frac{\mu}{1-\mu'}.(u_x^{sc} + i.u_y^{sc})_l = \frac{1}{L}.\left\{\sigma.\frac{\partial W_l^{sc}}{\partial \bar{z}} - \frac{\partial \bar{W}_l^{sc}}{\partial \bar{z}}\right\} \\ &= \frac{1}{L}.\left\{\sigma.\Omega_l(z) - z.\overline{\Omega_l'(z)} - \overline{\omega_l(z)}\right\}, \end{aligned} \tag{3.11}$$

where $\Omega'(z)$, $\Omega''(z)$ are the first and second derivatives of $\Omega(z)$ with respect to z and

$$\sigma = (1+\mu')/(1-\mu') = (1+\tau^2)/(1-\tau^2).$$

At this point, it should be noted that if C and E are constants,

$$\Omega_{\mathbf{R}}(z) = i.E.z, \quad \omega_{\mathbf{R}}(z) = C, \quad W_{\mathbf{R}} = i.E.z.\bar{z} + C.z = \phi^{\mathbf{R}} + i.\psi^{\mathbf{R}}, \tag{3.12}$$

correspond to a rigid body translation with zero stresses. Furthermore, given any analytic function $\chi(z)$, a substitution in (3.9)–(3.11) shows that

$$W^* = \phi^* + i.\psi^* = \sigma.\chi(z) + \overline{\chi(z)} \tag{3.13}$$

corresponds to zero displacements and stresses.

We shall now give the expression of the boundary condition in terms of the complex potentials $\Omega_l(z)$, $\omega_l(z)$. If β is the angle the unit normal, at a point on the boundary, makes with the x-axis, it can be shown that

$$(\tau_{\rm nn}^{\rm sc} + i.\tau_{\rm sn}^{\rm sc})_l = \tfrac{1}{2}.\Theta_l^{\rm sc} + \tfrac{1}{2}.\Phi_l^{\rm sc}.e^{-2i\beta}. \tag{3.14}$$

The boundary condition (2.6) is then given by

$$(\tau_{\rm nn}^{\rm sc} + i.\tau_{\rm sn}^{\rm sc})_l = \tfrac{1}{2}.\Theta_l^{\rm sc} + \tfrac{1}{2}.\Phi_l^{\rm sc}.e^{-2i\beta} = -\tau_l^{\rm I}, \tag{3.15}$$

where

$$\tau_l^{\rm I} = (\tau_{\rm nn}^{\rm I} + i.\tau_{\rm sn}^{\rm I})_l. \tag{3.16}$$

Substituting for $\Theta_l^{\rm sc}$ and $\Phi_l^{\rm sc}$ leads to

$$\left\{\Omega_l'(z) + \overline{\Omega_l'(z)}\right\} - \left\{z.\overline{\Omega_l''(z)} + \overline{\omega_l'(z)}\right\}.e^{-2i\beta} = -L^2.\tau_l^{\rm I}/2(1-\mu'). \tag{3.17}$$

It can be shown (see [7]) that (3.17) can be rewritten as

$$\frac{d}{dz}\left\{\Omega_l(z) + z.\overline{\Omega_l'(z)} + \overline{\omega_l(z)}\right\} = -L^2.\tau_l^{\rm I}/2(1-\mu'). \tag{3.18}$$

Denote by z_0 and z the complex numbers which correspond to two points on the boundary such that z_0 is fixed and z is variable. Integrating (3.18) over the boundary anti-clockwise from z_0 to z yields

$$\Omega_l(z) + z.\overline{\Omega_l'(z)} + \overline{\omega_l(z)} = R_l(z) + \text{ constant }, \tag{3.19}$$

where the constant is the value of the left-hand side expression for $z = z_0$, and $R_l(z)$ is given by

$$R_l(z) = -\frac{L^2}{2(1-\mu')}.\int_{z_0}^{z} \tau_l^{\rm I} dz. \tag{3.20}$$

4. Determination of the complex potentials $\Omega_l(z)$, $\omega_l(z)$

The complex potentials $\Omega_l(z)$ and $\omega_l(z)$ are determined through the implementation of the boundary condition (3.19). This determination requires the specification of the

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conformal transformation $z = m(\xi)$ which maps the exterior of the unit circle C_1 in the ξ-plane, on to the exterior of the boundary ∂D in the z-plane. We assume $m(\xi)$ to be single-valued and analytic in the domain exterior to C_1. Moreover, we shall assume that the derivative of $m(\xi)$ with respect to ξ is nonzero for $|\xi| \geq 1$. When ξ is on C_1, it will be denoted by t. In the ξ-plane, the boundary condition (3.19) is then expressed as

$$\Omega_l(t) + \frac{m(t)}{\overline{m'(t)}}.\overline{\Omega_l'(t)} + \overline{\omega_l(t)} = R_l(t) + \text{ constant.} \tag{4.1}$$

It can be shown [7] that the complex potentials are of the form

$$\Omega_l(\xi) = -\frac{i}{2.\pi}.\frac{(X_l + i.Y_l)}{(1+\sigma)}.\mathrm{Ln}\,\xi + \Omega_l^*(\xi) \tag{4.2}$$

$$\omega_l(\xi) = -\frac{i}{2.\pi}.\frac{(X_l - i.Y_l)}{(1+\sigma)}.\mathrm{Ln}\,\xi + \omega_l^*(\xi), \tag{4.3}$$

where $\Omega_l^*(\xi)$, $\omega_l^*(\xi)$ are analytic and single-valued for $|\xi| \geq 1$ and are bounded as $|\xi| \to \infty$. The resultant force $X_l + i.Y_l$ over the hole is given by

$$X_l + i.Y_l = -\frac{L^2}{2(1-\mu')}.\int_{\partial D} \tau_l^{\mathrm{I}} dz. \tag{4.4}$$

Note that the complex potentials given by (4.2) and (4.3) are multiple-valued and analytic functions for $|\xi| \geq 1$. However, the complex displacements and stresses they generate are single-valued for $|\xi| \geq 1$.

A substitution of (4.2) and (4.3) into (4.1) gives

$$\Omega_l^*(t) + \frac{m(t)}{\overline{m'(t)}}.\overline{\Omega_l^*(t)}' + \overline{\omega_l^*(t)} = F_l(t), \tag{4.5}$$

where

$$F_l(t) = R_l(t) + \frac{i}{2.\pi}.(X_l + i.Y_l).\mathrm{Ln}\,t - \frac{i}{2.\pi}.\frac{(X_l - i.Y_l)}{(1+\sigma)}.\frac{m(t)}{\overline{t.m'(t)}} + \text{constant}$$

and is single-valued on C_1.

In order to go further in the determination of the complex potentials, we need to specify the function $m(\xi)$. This is chosen to be

$$z = m(\xi) = \xi + m/\xi^n \quad (0 \leq m < 1/n), \tag{4.6}$$

where m is real and n an integer. The condition $0 \leq m < 1/n$ ensures that the boundary ∂D does not have loops or cusps. This conformal transformation maps the exterior of the unit circle C_1 onto the exterior of a hypotrochoid. When $n = 1, 2$ or 3, the unit circle C_1 in the ξ-plane is mapped onto an ellipse, a curvilinear triangle or a curvilinear square, respectively. From (4.6) it can be verified that $m(\xi)$ is a single-valued analytic function and its derivatives $m'(\xi) \neq 0$ for $|\xi| \geq 1$.

Having specified $m(\xi)$, it can be shown [7] that $\Omega_l^*(t)$ and $\omega_l^*(t)$ can be written as follows:

$$\omega_l^*(\xi) = -\frac{1}{2\pi i}\int_{C_1}\frac{F_l(t)}{t-\xi}dt + m.\sum_{i=1}^{n-2} i.\alpha_l^{-i}/\xi^{n-(i+1)} \quad \text{for } |\xi| \geq 1, \tag{4.7}$$

$$\Omega_l^*(\xi) = -\frac{1}{2\pi i}\int_{C_1}\frac{\overline{F_l(t)}}{t-\xi}dt - \frac{\overline{m(1/\bar{\xi})}}{m'(\xi)}.\Omega_l^{*\prime}(\xi) - m.\sum_{i=1}^{n-1} i.\alpha_l^i.\xi^{n-(i+1)} \quad \text{for } |\xi| \geq 1, \tag{4.8}$$

where

$$\alpha_l^i = \begin{cases} -\dfrac{\beta_l^{i-1} + m.[n-(i+1)].a_{n-2}.\bar{\beta}_l^{n-(i+2)}}{i.[n-(i+1)]^2 - 1} & \text{for } 1 \leq i \leq n-2 \\ \beta_l^{i-1} & \text{for } i > n-2, \end{cases} \tag{4.9}$$

with

$$\beta_l^i = \frac{1}{2\pi i}.\int_{C_1} F_l(t).t^i.dt \quad 0 \leq i < \infty \tag{4.10}$$

and

$$a_n = \begin{cases} 0 & n \leq 0, \\ 1 & n > 0. \end{cases}$$

We shall now consider the case of an incident P-wave propogating along the x-axis, that is,

$$\Phi^I = e^{ikx} \quad \text{and} \quad \Psi^I = 0. \tag{4.11}$$

Therefore τ_l^{I} is given by

$$\tau_l^{\mathrm{I}} = -\frac{(1-\mu')}{L^2}.(kL)^2.\left\{1 + \gamma_1.e^{-2i\beta}\right\}.\begin{cases} 1 & \text{for } l = 0 \\ \sqrt{\mu'}.x' & \text{for } l = 1, \end{cases} \tag{4.12}$$

where

$$\gamma_1 = \mu'/(1-\mu').$$

In terms of $m(\xi)$, $e^{-2i\beta}$ is given by

$$e^{-2i\beta} = \frac{\overline{\xi.m'(\xi)}}{\xi.m(\xi)}. \tag{4.13}$$

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From (4.4), (4.12) and (4.13), we get for $X_l + i.Y_l$,

$$X_l + i.Y_l = \begin{cases} 0 & \text{for } l = 0 \\ 2\pi i(1+\sigma).\delta_1 & \text{for } l = 1, \end{cases} \tag{4.14}$$

with

$$\delta_1 = \sqrt{\mu'}.(1-n.m)^2.(kL)^2/8. \tag{4.15}$$

The complex potentials $\Omega_l(\xi)$, $\omega_l(\xi)$, in this case, are given by

$$\Omega_0(\xi) = \frac{(kL)^2}{2}\left\{\frac{\gamma_1}{(n-2).m^2.a_{n-2}-1}\cdot\frac{1}{\xi} + \frac{m.\gamma_1.a_{n-2}}{(n-2).m^2.a_{n-2}-1}\cdot\frac{1}{\xi^{n-2}} + \frac{m}{\xi^n}\right\}, \tag{4.16}$$

$$\omega_0(\xi) = \frac{(kL)^2}{2}\left[(a_{n-2}-1)m\gamma_1\xi^{n-2} + (1+nm^2)/\xi - m\gamma_1/\xi^n + \frac{(1+nm^2)}{(\xi^{n+1}-nm)}.\left\{\frac{\gamma_1}{(n-2)m^2a_{n-2}-1}.\xi^{n-2} + \frac{a_{n-2}.m.(n-2)\gamma_1}{(n-2)m^2a_{n-2}-1}.\xi + mn/\xi\right\}\right], \tag{4.17}$$

$$\Omega_1(\xi) = \delta_1.\mathrm{Log}\,\xi + \sqrt{\mu'}.\frac{(kL)^2}{8}\left[\frac{\gamma_1}{2(n-3).m^2.a_{n-3}-1}.\frac{1}{\xi^2} + \frac{2m\gamma_1a_{n-3}}{2(n-3).m^2.a_{n-3}-1}.\frac{1}{\xi^{n-3}} + m(1+nm^2)a_{n-1}/\xi^{n-1} + \frac{2m}{n+1}(n-\gamma_1)/\xi^{n+1} + m^2/\xi^{2n}\right], \tag{4.18}$$

$$\begin{aligned}\omega_1(\xi) = -\sigma.\delta_1.\mathrm{Log}\,\xi + \sqrt{\mu'}\frac{(kL)^2}{8}\Bigg[&2(a_{n-2}-1)m\gamma_1\xi^{n-3} \\ &+\{1+2m^2(n-\gamma_1)\}/\xi^2 + \frac{2m}{n+1}\{1+n(n+1)m^2-n\gamma_1\}/\xi^{n+1} \\ &-\gamma_1\{2ma_{n-1}/\xi^{n-1}+m^2/\xi^{2n}\} \\ &+\frac{(1+nm^2)}{\xi^{n+1}-mn}\left\{\frac{2\gamma_1}{2(n-3)m^2a_{n-3}-1}.\xi^{n-3} - (1-nm^2)\xi^{n-1} + \frac{2m(n-3)\gamma_1a_{n-3}}{2(n-3)m^2a_{n-3}-1}.\xi^2 + (n-1)(1+nm^2)m + 2m(n-\gamma_1)/\xi^2 + 2nm^2/\xi^{n+1}\right\}\Bigg].\end{aligned} \tag{4.19}$$

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5. The outer problem and expansion

We introduce the outer variables (X', Y') and the corresponding polar coordinates (R, Θ) such that

$$X' = K.X; \quad Y' = K.Y; \quad R = K.r; \quad \Theta = \theta. \tag{5.1}$$

Following Sabina and Willis [9], the general solution of the outer problem, for small KL, can be expressed, in terms of the outer coordinates, as follows:

$$\phi^{sc} = \sum_{j=0}^{\infty} \varepsilon^j.\left[A_j.\cos(j\theta) + A'_j.\sin(j\theta)\right].H_j^{(1)}(\sqrt{\mu'}.R), \tag{5.2}$$

$$\psi^{sc} = \sum_{j=0}^{\infty} \varepsilon^j.\left[B_j.\cos(j\theta) + B'_j.\sin(j\theta)\right].H_j^{(1)}(R), \tag{5.3}$$

where the notation $\varepsilon = KL$ has been used and will be kept hereafter. The symbol $H_j^{(1)}(\cdot)$ denotes the Hankel function of the first kind and order j and A_j, A'_j, B_j, and B'_j are unknown constants which may depend on ε if the matching requires it. They are assumed to be of $\mathrm{O}(1)$ or smaller for $j > 0$ and $\mathrm{O}(\varepsilon)$ for $j = 0$. As in Sabina and Willis [9], we write the unknown constants as follows:

$$A_j = a_j^{(0)} + \varepsilon.a_j^{(1)} + \varepsilon^2.a_j^{(2)} + \cdots, \tag{5.4a}$$

$$A'_j = a_j'^{(0)} + \varepsilon.a_j'^{(1)} + \varepsilon^2.a_j'^{(2)} + \cdots, \tag{5.4b}$$

$$B_j = b_j^{(0)} + \varepsilon.b_j^{(1)} + \varepsilon^2.b_j^{(2)} + \cdots, \tag{5.4c}$$

$$B'_j = b_j'^{(0)} + \varepsilon.b_j'^{(1)} + \varepsilon^2.b_j'^{(2)} + \cdots, \tag{5.4d}$$

where we take $a_0^{(0)} = b_0^{(0)} = 0$.

It can be verified that (5.2)-(5.3) do satisfy the equations of motion and the radiation conditions at infinity. The boundary condition on ∂D is redundant.

6. The matching of the inner and outer expansions

We shall now proceed to relate the inner and outer expansions in order to determine the unknown constants of the outer expansions. This is done using the asymptotic matching principle described in Crighton and Leppington [5] and which is briefly outlined below.

We write ϕ_{inner}, ψ_{inner} for the potentials of the inner solution and ϕ_{outer}, ψ_{outer} for those corresponding to the outer solution. Note that each potential is expressed in terms of its corresponding coordinates, that is, inner coordinates for ϕ_{inner}, ψ_{inner} and

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outer coordinates for ϕ_{outer}, ψ_{outer}. We introduce the notation $\phi_{\text{inner}}^{(p)}$ ($\psi_{\text{inner}}^{(p)}$) for the asymptotic expansion of ϕ_{inner} (ψ_{inner}) up to and including all terms $O(\varepsilon^p)$ for fixed inner coordinates; and we write $\phi_{\text{inner}}^{(p,q)}$ ($\psi_{\text{inner}}^{(p,q)}$) for the result of rewriting $\phi_{\text{inner}}^{(p)}$ ($\psi_{\text{inner}}^{(p)}$) in terms of outer coordinates and expanding up to and including all terms $O(\varepsilon^q)$ for fixed outer coordinates. Similar notation is also used for the corresponding potentials of the outer solution. Note that terms which are $O(\varepsilon.\text{Log}\,\varepsilon)$ and $O(\varepsilon)$ are here both regarded as $O(\varepsilon)$. With this notation, the matching principle is expressed as follows:

$$\phi_{\text{inner}}^{(p,q)} = \phi_{\text{outer}}^{(q,p)}, \tag{6.1a}$$

$$\psi_{\text{inner}}^{(p,q)} = \psi_{\text{outer}}^{(q,p)}. \tag{6.1b}$$

Note that a transformation of $\phi_{\text{inner}}^{(p,q)}$, $\psi_{\text{inner}}^{(p,q)}$ back into inner coordinates or $\phi_{\text{outer}}^{(q,p)}$, $\psi_{\text{outer}}^{(q,p)}$ back into outer oordinates must be made before the identification is performed.

The potentials ϕ_{inner}, ψ_{inner}, ϕ_{outer}, ψ_{outer} in our problem are given by

$$\phi_{\text{inner}} = \phi_0 + i\varepsilon.\phi_1 + \cdots + \phi^* + \phi^{\mathbf{R}}, \tag{6.2a}$$

$$\psi_{\text{inner}} = \psi_0 + i\varepsilon.\psi_1 + \cdots + \psi^* + \psi^{\mathbf{R}}, \tag{6.2b}$$

$$\phi_{\text{outer}} = \sum_{j=0}^{\infty} \varepsilon^j . \left[A_j . \cos(j\theta) + A'_j . \sin(j\theta)\right] . H_j^{(1)}(\sqrt{\mu'}.R), \tag{6.3a}$$

$$\psi_{\text{outer}} = \sum_{j=0}^{\infty} \varepsilon^j . \left[B'_j . \cos(j\theta) + B_j . \sin(j\theta)\right] . H_j^{(1)}(R), \tag{6.3b}$$

where the potentials ϕ^*, ψ^*, $\phi^{\mathbf{R}}$, $\psi^{\mathbf{R}}$, introduced in Section 3, are included to enable us to eliminate or add terms if the matching requires it. However, these potentials must satisfy the conditions mentioned in Section 3, that is, ϕ^*, ψ^* generate no displacements and no stresses and $\phi^{\mathbf{R}}$, $\psi^{\mathbf{R}}$ represent a rigid body translation. The potentials ϕ^*, ψ^* will be, hereafter, referred to as the null-potentials.

As only ϕ_0, ψ_0, ϕ_1, ψ_1 are known, which incidentally are all of order ε^2 (see Section 4), the inner potentials ϕ_{inner}, ψ_{inner} can only be expanded up to and including $O(\varepsilon^3)$, that is,

$$\phi_{\text{inner}} = \phi_0 + i\varepsilon.\phi_1 + \phi^* + \phi^{\mathbf{R}}, \tag{6.4a}$$

$$\psi_{\text{inner}} = \psi_0 + i\varepsilon.\psi_1 + \psi^* + \psi^{\mathbf{R}}. \tag{6.4b}$$

This, as it will be seen later, puts a restriction on the approximation to which the coefficients of the outer expansions can be determined.

As an illustration, we shall now apply the matching principle for the special case $n = 1$. The expressions for the complex potentials $\Omega_0(\xi)$, $\omega_0(\xi)$, $\Omega_1(\xi)$, $\omega_1(\xi)$ are

$$\Omega_0(\xi) = \beta_1/\xi; \quad \omega_0(\xi) = (1+m^2)\beta_1/\xi(\xi^2 - m) + \beta_2/\xi, \tag{6.5}$$

where

$$\beta_1 = (m-\gamma_1)(kL)^2/2; \quad \beta_2 = (1+m^2-2m\gamma_1)(kL)^2,$$

and

$$\Omega_1(\xi) = \delta_1 \mathrm{Log}\,\xi + \delta_2/\xi^2, \tag{6.6a}$$

$$\omega_1(\xi) = \delta_3 \mathrm{Log}\,\xi + \delta_5/\xi^5 + \delta_6/(\xi^2 - m) + \delta_7/\xi^2(\xi^2 - m), \tag{6.6b}$$

where

$$\delta_1 = \sqrt{\mu'}(1-m^2)(kL)^2/8; \quad \delta_2 = \sqrt{\mu'}(1+m)\beta_1/4; \quad \delta_3 = -(1+2\gamma_1)\delta_1;$$

$$\delta_5 = \sqrt{\mu'}(1+m)(1+2m^2-3m\gamma_1)(kL)^2/8; \quad \delta_6 = -(1+m^2)\delta_1; \quad \delta_7 = 2(1+m^2)\delta_2.$$

Note that Buchwald and Tran Cong [4] have an additional term δ_4 in their expression for $\omega_1(\xi)$; we discuss this at the end of the present section.

Noting that the inverse of (4.8), for $n = 1$, is

$$\xi = [z + (z-4m)^{\frac{1}{2}}]/2, \tag{6.7}$$

it is found when expressing in terms of the outer coordinates and expanding in powers of ε that

$$\varepsilon^{-1} = \varepsilon/Z + m\varepsilon^3/Z^3 + 2m^2\varepsilon^5/Z^5 + 5m^3\varepsilon^7/Z^7 + \mathrm{O}(\varepsilon^9),$$

$$(\varepsilon^2 - m)^{-1} = \varepsilon^2/Z^2 + 3m\varepsilon^4/Z^4 + 10m^2\varepsilon^6/Z^6 + 35m^3\varepsilon^8/Z^8 + \mathrm{O}(\varepsilon^{10}),$$

$$\mathrm{Log}\,\xi = \mathrm{Log}\,(Z\varepsilon) - m\varepsilon^2/Z^2 - 3m^2\varepsilon^4/2Z^4 - 10m^3\varepsilon^6/3Z^6 + \mathrm{O}(\varepsilon^8),$$

where $Z = Re^{i\theta}$. Substituting the above expressions into the complex potentials, the following asymptotic expansions, in terms of the outer coordinates, for the potentials ϕ_0, ψ_0, ϕ_1, ψ_1, are eventually obtained:

$$\phi_0 = C_1 \mathrm{Log}\,R + \left(C_2 + C_3\frac{\varepsilon^2}{R^2}\right)\cos(2\theta) + m\left(C_2 + C_4\frac{\varepsilon^2}{R^2}\right)\frac{\varepsilon^2}{R^2}\cos(4\theta)$$
$$+m^2\left(2C_2 + C_5\frac{\varepsilon^2}{R^2}\right)\frac{\varepsilon^4}{R^4}\cos(6\theta) + C_6\frac{\varepsilon^6}{R^6}\cos(8\theta) + \mathrm{O}(\varepsilon^{10}), \tag{6.8a}$$

$$\psi_0 = C_1.\theta - \left(C_2 + C_3\frac{\varepsilon^2}{R^2}\right)\sin(2\theta) - m\left(C_2 + C_4\frac{\varepsilon^2}{R^2}\right)\frac{\varepsilon^2}{R^2}\sin(4\theta)$$
$$-m^2\left(2C_2 + C_5\frac{\varepsilon^2}{R^2}\right)\frac{\varepsilon^4}{R^4}\sin(6\theta) - C_6\frac{\varepsilon^6}{R^6}\sin(8\theta) + \mathrm{O}(\varepsilon^{10}), \tag{6.8b}$$

where

$$C_1 = \beta_2; \quad C_2 = \beta_1; \quad C_3 = -\{m\beta_2 + (1+m^2)\beta_1\}/2;$$
$$C_4 = -\{m\beta_2/2 + (1+m^2)\beta_1\}; \quad C_5 = 5\{m\beta_2/3 + (1+m^2)\beta_1\}/2;$$
$$C_6 = 5m^3\beta_1,$$

and

$$\begin{aligned}\phi_1 = {} & D_1\varepsilon^{-1}R\theta\sin(\theta) + \{\mu' D_2 R\varepsilon^{-1}\mathrm{Log}\,(R\varepsilon^{-1}) + D_3 R\varepsilon^{-1} + D_4\varepsilon/R\}\cos(\theta) \\ & + \{D_5 + D_6\varepsilon^2/R^2\}\frac{\varepsilon}{R}\cos(3\theta) + \{D_7 + D_8\varepsilon^2/R^2\}\frac{\varepsilon^3}{R^3}\cos(5\theta) \qquad (6.9\mathrm{a}) \\ & + D_9\frac{\varepsilon^3}{R^5}\cos(7\theta) + \mathrm{O}(\varepsilon^9),\end{aligned}$$

$$\begin{aligned}\psi_1 = {} & -\mu' D_1\varepsilon^{-1}R\theta\cos(\theta) + \{D_2 R\varepsilon^{-1}\mathrm{Log}\,(R\varepsilon^{-1}) + D_3 R\varepsilon^{-1} - D_4\varepsilon/R\}\sin(\theta) \\ & - \{D_5 + D_6\varepsilon^2/R^2\}\frac{\varepsilon}{R}\sin(3\theta) - \{D_7 + D_8\varepsilon^2/R^2\}\frac{\varepsilon^3}{R^3}\sin(5\theta) \qquad (6.9\mathrm{b}) \\ & - D_9\frac{\varepsilon^5}{R^5}\sin(7\theta) + \mathrm{O}(\varepsilon^9),\end{aligned}$$

where

$$D_1 = 2(1+\gamma_1)\delta_1; \quad D_2 = -D_1; \quad D_3 = \sigma.\delta_1; \quad D_4 = m\delta_3 - \delta_5 - \delta_6;$$
$$D_5 = \delta_2 - m\delta_1; \quad D_6 = (3m^2\delta_3 - 4m\delta_5 - 6m\delta_6 - 2\delta_7)/6; \quad D_7 = m(4\delta_2 - 3m\delta_1)/2;$$
$$D_8 = m(2m^2\delta_3 - 3m\delta_5 - 6m\delta_6 - \delta_7)/3; \quad D_9 = 5m^2(\delta_2 - 2m\delta_1/3).$$

A comparison of (6.8)-(6.9) with (6.3) shows that, to this approximation, we may express the outer solution as

$$\phi_{\text{outer}} = \sum_{j=0}^{8}\varepsilon^j.A_j.\cos(j\theta).H_j^{(1)}(\sqrt{\mu'}.R), \qquad (6.10\mathrm{a})$$

$$\psi_{\text{outer}} = \sum_{j=0}^{8}\varepsilon^j.B_j.\sin(j\theta).H_j^{(1)}(R). \qquad (6.10\mathrm{b})$$

Let us now apply (6.1) with $p = 3$ and $q = 4$. Using (6.4), (6.8) and (6.9), we get for $\phi_{\text{inner}}^{(3,4)}$ and $\psi_{\text{inner}}^{(3,4)}$, written in inner coordinates, the following:

$$\begin{aligned}\phi_{\text{inner}}^{(3,4)} = {} & C_1.\mathrm{Log}\,\varepsilon + iD_1\varepsilon r'\theta\sin(\theta) + C_1\mathrm{Log}\,r' \\ & + i\varepsilon(\mu' D_2 r'\mathrm{Log}\,r' + D_3 r' + D_4/r')\cos(\theta) \\ & + (C_2 + C_3/r'^2)\cos(2\theta) + iD_5\varepsilon.\cos(3\theta)/r' \\ & + mC_2.\cos(4\theta)/r'^2 + \phi^* + \phi^{\mathrm{R}}, \qquad (6.11\mathrm{a})\end{aligned}$$

$$\begin{aligned}\psi^{(3,4)}_{\text{inner}} = {} & C_1.\theta - i\mu' D_1 \varepsilon r' \theta \sin(\theta)\\ & + i\varepsilon (D_2 r' \operatorname{Log} r' + D_3 r' - D_4/r')\sin(\theta)\\ & -(C_2 + C_3/r'^2)\sin(2\theta) - iD_5\varepsilon.\sin(3\theta)/r'\\ & - mC_2.\sin(4\theta)/r'^2 + \psi^* + \psi^{\mathrm{R}}.\end{aligned} \tag{6.11b}$$

For $\phi^{(4,3)}_{\text{outer}}$ and $\psi^{(4,3)}_{\text{outer}}$, written in inner coordinates, we get

$$\begin{aligned}\phi^{(4,3)}_{\text{outer}} = {} & \left(\varepsilon.a_0^{(1)} + \varepsilon^2 a_0^{(2)} + \varepsilon^3 a_0^{(3)}\right)\left[1 + \frac{2i}{\pi}\left\{\gamma + \operatorname{Log}(\varepsilon\sqrt{\mu'}\, r'/2)\right\}\right]\\ & - \varepsilon.a_0^{(1)}\left[1 + \frac{2i}{\pi}\left\{\gamma - 1 + \operatorname{Log}(\varepsilon\sqrt{\mu'}\, r'/2)\right\}\right]\mu'\varepsilon^2 r'^2/4\\ & + \left[-\frac{2i}{\pi}\left\{a_1^{(0)} + \cdots + \varepsilon^3 a_1^{(3)}\right\}\frac{1}{\sqrt{\mu'}\, r'} + \frac{2i}{\pi}\left\{a_1^{(0)} + \varepsilon.a_1^{(1)}\right\}\right.\\ & \qquad \left. .\left[2\gamma - \pi i - 1 + 2\operatorname{Log}(\sqrt{\mu'}\,\varepsilon r'/2)\right].\sqrt{\mu'}\,\varepsilon^2 r'\right]\cos(\theta)\\ & - \frac{i}{\pi}\left[\left\{a_2^{(0)} + \cdots + \varepsilon^2 a_2^{(2)}\right\}\frac{4}{\mu' r'^2} + \varepsilon^2\left\{a_2^{(0)} + \varepsilon.a_2^{(1)}\right\}\right]\cos(2\theta)\\ & - \frac{i}{\pi}\left[\left\{a_3^{(0)} + \varepsilon.a_3^{(1)}\right\}\frac{16}{\mu'^{3/2}.r'^3} + \varepsilon^2\left\{a_3^{(0)} + \varepsilon.a_3^{(1)}\right\}\frac{2}{\sqrt{\mu'}\, r'}\right]\cos(3\theta)\\ & - \frac{2i}{\pi}\left[a_4^{(0)}\frac{48}{\mu'^2 r'^4} + \varepsilon^2 a_4^{(0)}\frac{4}{\mu' r'^2}\right]\cos(4\theta),\end{aligned} \tag{6.12a}$$

$$\begin{aligned}\psi^{(4,3)}_{\text{outer}} = {} & \left[-\frac{2i}{\pi}\left\{b_1^{(0)} + \cdots + \varepsilon^3 b_1^{(3)}\right\}\frac{1}{r'}\right.\\ & \left. + \frac{2i}{\pi}\left\{b_1^{(0)} + \varepsilon.b_1^{(1)}\right\}.\left[2\gamma - \pi i - 1 + 2\operatorname{Log}(\varepsilon r'/2)\right]\varepsilon^2 r'\right]\sin(\theta)\\ & - \frac{1}{\pi}\left[\left\{b_2^{(0)} + \cdots + \varepsilon^2 b_2^{(2)}\right\}\frac{4}{r'^2} + \varepsilon^2\left\{b_2^{(0)} + \varepsilon.b_2^{(1)}\right\}\right]\sin(2\theta)\\ & - \frac{i}{\pi}\left[\left\{b_3^{(0)} + \varepsilon.b_3^{(1)}\right\}\frac{16}{r'^3} + \varepsilon^2\left\{b_3^{(0)} + \varepsilon.b_3^{(1)}\right\}\frac{2}{r'}\right]\sin(3\theta)\\ & - \frac{2i}{\pi}\left[b_4^{(0)}\frac{48}{r'^4} + \varepsilon^2 b_4^{(0)}\frac{4}{r'^2}\right]\sin(4\theta),\end{aligned} \tag{6.12b}$$

where we have used the following asymptotic expansions of the Hankel functions for small arguments x:

$$H_0^{(1)}(x) = 1 - x^2/4 + \frac{2i}{\pi}\left[(1 - x^2/4)(\gamma + \operatorname{Log}(x/2)) + x^2/4\right] + \mathrm{O}(x^4 \operatorname{Log} x),$$

$$H_1^{(1)}(x) = -\frac{2i}{\pi}\frac{1}{x} + \left[2\gamma - \pi i - 1 + 2\operatorname{Log}(x/2)\right]\frac{i}{2\pi}x + \mathrm{O}(x^3 \operatorname{Log} x),$$

$$H_j^{(1)}(x) = -\frac{i}{\pi}(j-2)!\left[(j-1)\left(\frac{2}{x}\right)^j + \left(\frac{2}{x}\right)^{j-2}\right] + \mathrm{O}(x^{4-j}) \qquad \text{for } j > 1;$$

γ, here, denotes Euler's constant.

An examination of (6.10) and (6.12) seems to suggest the following form for the null-potentials ϕ^*, ψ^*:

$$\begin{aligned} \phi^* = {} & X_0 + X_1 \mathrm{Log}\, r' \\ & + X_2 \left(r'\theta\sin(\theta) - r'\mathrm{Log}\,(r')\cos(\theta)\right) + (X_3 + X_4/r')\cos(\theta) \\ & + X_5\cos(2\theta)/r'^2 + X_6\cos(3\theta)/r'^3 + X_7\cos(4\theta)/r'^4, \end{aligned} \tag{6.13a}$$

$$\begin{aligned} \psi^* = \mu'\Big[& X_1\theta - X_2\left(r'\theta\cos(\theta) + r'\mathrm{Log}\,(r')\sin(\theta)\right) + \left(X_3 - X_4/r'\right)\sin(\theta) \\ & - X_5\sin(2\theta)/r'^2 - X_6\sin(3\theta)/r'^3 - X_7\sin(4\theta)/r'^4\Big], \end{aligned} \tag{6.13b}$$

while for $\phi^{\mathbf{R}}$ and $\psi^{\mathbf{R}}$ we have

$$\phi^{\mathbf{R}} = C_{\mathbf{R}} r'\cos(\theta) \qquad \text{and} \qquad \psi^{\mathbf{R}} = C_{\mathbf{R}} r'\sin(\theta). \tag{6.14}$$

The constants $X_0, X_1, \ldots, X_7$ and $C_{\mathbf{R}}$ are to be determined by the matching process. Note that the expressions given by (6.13) satisfy the required conditions, that is, they generate no displacements or stresses, while the expressions given by (6.14) give rise to a rigid body translation but no stresses. Using (6.1), (6.10), (6.12), (6.13) and (6.14), and identifying coefficients of the same type leads to a set of equations, which when solved yield the following:

$$A_0 = i\pi/2 + \mathrm{O}(\varepsilon^4); \quad \varepsilon A_1 = 2\pi\delta_1/\sqrt{\mu'} - \pi\sqrt{\mu'}D_4\varepsilon^2/2 + \mathrm{O}(\varepsilon^5);$$

$$\varepsilon^2 A_2 = i\pi C_2 + i\pi\mu' C_3\varepsilon^2/4 + \mathrm{O}(\varepsilon^5); \quad \varepsilon^3 A_3 = -\pi\sqrt{\mu'}D_5\varepsilon^2 + \mathrm{O}(\varepsilon^5);$$

$$\varepsilon^4 A_4 = i\pi\mu' m C_2\varepsilon^2/8 + \mathrm{O}(\varepsilon^5);$$

$$\varepsilon B_1 = -2\pi\delta_1 + \pi D_4\varepsilon^2/2 + \mathrm{O}(\varepsilon^5); \quad \varepsilon^2 B_2 = -i\pi C_2\varepsilon^2 - i\pi C_3\varepsilon^2/4 + \mathrm{O}(\varepsilon^5);$$

$$\varepsilon^3 B_3 = \pi D_5\varepsilon^2/2 + \mathrm{O}(\varepsilon^5); \quad \varepsilon^4 B_4 = -i\pi m C_2\varepsilon^2/8 + \mathrm{O}(\varepsilon^5),$$

with

$$X_0 = \frac{i\pi}{2}\frac{C_1}{\gamma_1}\left[1 + \frac{2i}{\pi}\left\{\gamma + \mathrm{Log}\,(\sqrt{\mu'}/2) + (1+\gamma_1)\mathrm{Log}\,\varepsilon\right\}\right];$$

$$X_1 = -C_1/\mu'; \quad X_2 = -iD_1\varepsilon; \quad X_3 = \frac{2i\delta_1\varepsilon}{(1-\mu')}\left[\mathrm{Log}\,(\sqrt{\mu'}\,\varepsilon^2) - b'\right];$$

$$X_4 = -4i\delta_1/\mu'\varepsilon; \quad X_5 = 4C_2/\mu'\varepsilon^2; \quad X_6 = 8iD_5/\mu'\varepsilon; \quad X_7 = 12mC_2/\mu'\varepsilon^2;$$

$$C_{\mathbf{R}} = i\delta_1\varepsilon\left[(1+\gamma_1)\left\{b'(1+\mu') - 2\mathrm{Log}\,\varepsilon - 2\mu'\mathrm{Log}\,(\sqrt{\mu'}\varepsilon)\right\} - \sigma\right],$$

where $b' = \pi i + 1 + 2\text{Log}\, 2 - 2\gamma$.

From (6.12a), it can be seen that no matter how far we expand $\phi^{(q)}_{\text{outer}}$, the coefficient A_0 of the outer expansion can only be determined up to the order ε^3. This is due to the fact that in the expansion of $\phi^{(q,p)}_{\text{outer}}$, the value of p cannot go beyond 3 for reasons which were mentioned earlier in this section. It follows, therefore, that for consistency all terms of order ε^4 and higher in the remaining coefficients should be dropped. As a result of this, the expressions (6.10) for the scattered potentials become

$$\phi_{\text{outer}} = \sum_{j=0}^{2} \varepsilon^j A_j H_j^{(1)}(\sqrt{\mu'}R)\cos(j\theta), \tag{6.15a}$$

$$\psi_{\text{outer}} = \sum_{j=0}^{2} \varepsilon^j B_j H_j^{(1)}(R)\sin(j\theta), \tag{6.15b}$$

with the coefficients $\varepsilon^j A_j$, $\varepsilon^j B_j$ given by

$$A_0 = \frac{i\pi}{2}\frac{C_1}{\gamma_1} + \mathrm{O}(\varepsilon^4);$$

$$\varepsilon A_1 = 2\pi\delta_1/\sqrt{\mu'} + \mathrm{O}(\varepsilon^4); \quad \varepsilon B_1 = -2\pi\delta_1 + \mathrm{O}(\varepsilon^4);$$
$$\varepsilon^2 A_2 = i\pi C_2 + \mathrm{O}(\varepsilon^4); \quad \varepsilon^2 B_2 = -i\pi C_2 + \mathrm{O}(\varepsilon^4).$$

The above results are the same as those derived by Buchwald and Tran Cong [4]. However, in the expression of the complex potential $\omega_1(\xi)$, given by (6.6b), we do not have a constant term which Buchwald and Tran Cong [4] have included in their expression. The value assigned to that constant is $-m\delta_1$. We have not been able to explain how that constant was obtained. However, it can be seen from our systematic calculation that its absence does not affect in any way the result obtained.

7. Results of the matching for $n > 1$

In this section, we present the results of the matching for $n > 1$. The details of these calculations are not included as they are similar to those for the case treated in the previous section ($n = 1$). In this case ($n > 1$), however, the explicit expression of the inverse transformation of (4.8) cannot be obtained. Nevertheless, it can be seen from the previous section that only the asymptotic expansion, in terms of the outer coordinates, is required. This can be shown (see Appendix) to be of the following form:

$$\xi = Z\varepsilon^{-1} + \frac{a'_n}{Z^n}\varepsilon^n + \frac{a'_{2n+1}}{Z^{2n+1}}\varepsilon^{2n+1} + \frac{a'_{3n+2}}{Z^{3n+2}}\varepsilon^{3n+2} + \frac{a'_{4n+3}}{Z^{4n+3}}\varepsilon^{4n+3} + \mathrm{O}(\varepsilon^{5n+4}), \tag{7.1}$$

where $Z = Re^{i\theta}$ and

$$a'_n = -m; \quad a'_{2n+1} = -nm^2; \quad a'_{3n+2} = -\frac{n(n+1)}{2}m^3;$$

$$a'_{4n+3} = -\frac{n(2n+1)(4n+1)}{3}m^4.$$

Note that (7.1) is also valid for $n = 1$.

To our order of approximation, the scattered potentials are given by

$$\phi_{\text{outer}} = \sum_{j=0}^{3} \varepsilon^j A_j H_j^{(1)}(\sqrt{\mu'}R)\cos(j\theta), \tag{7.2a}$$

$$\psi_{\text{outer}} = \sum_{j=0}^{3} \varepsilon^j B_j H_j^{(1)}(R)\sin(j\theta), \tag{7.2b}$$

where for $n = 2$ we have

$$A_0 = i\pi(C_1 + iD_4\varepsilon)/2\gamma_1; \quad \varepsilon A_1 = 2\pi\delta_0/\sqrt{\mu'} + i\pi\sqrt{\mu'}C_2\varepsilon/2;$$

$$\varepsilon B_1 = -(2\pi\delta_0 + i\pi C_2\varepsilon/2); \quad \varepsilon^2 A_2 = i\pi(C_3 + iD_6\varepsilon); \quad \varepsilon^2 B_2 = -i\pi(C_3 + iD_6\varepsilon);$$

$$\varepsilon^3 A_3 = i\pi\sqrt{\mu'}C_5\varepsilon/2; \quad \varepsilon^3 B_3 = -i\pi C_5\varepsilon/2,$$

all with errors of $O(\varepsilon^4)$, with

$$C_1 = (1+2m^2)\beta_3; \quad C_2 = -m\beta_1; \quad C_3 = \beta_1; \quad C_5 = \beta_2; \quad D_4 = \delta_6; \quad D_6 = \delta_1$$

and

$$\beta_1 = -\gamma_1(kL)^2/2; \quad \beta_2 = m(kL)^2/2; \quad \beta_3 = (kL)^2/2;$$

$$\delta_0 = \sqrt{\mu'}(1-2m^2)(kL)^2/8; \quad \delta_1 = \sqrt{\mu'}m(1+2m^2)(kL)^2/8; \quad \delta_6 = -\sqrt{\mu'}m\gamma_1(kL)^2/2.$$

For $n > 2$, the coefficients $\varepsilon^j A_j$, $\varepsilon^j B_j$ are given in the table below, again with errors of $O(\varepsilon^4)$.

In Table 1

$$C_1 = (1+nm^2)\beta_3; \quad C_2 = \beta_1; \quad C_4 = m\beta_1; \quad D_7 = \delta_2;$$

with

$$\beta_1 = \frac{\gamma_1}{(n-2)m^2 - 1}(kL)^2/2; \quad \beta_3 = (kL)^2/2;$$

$$\delta_0 = \sqrt{\mu'}(1-nm^2)(kL)^2/8; \quad \delta_2 = \sqrt{\mu'}\frac{2m\gamma_1 a_{n-3}}{2(n-3)m^2 a_{n-3} - 1}(kL)^2/8$$

TABLE 1. Values of the coefficients for $n > 2$

$n =$	3	4	≥ 5
A_0	$i\pi C_1/2\gamma_1$	$i\pi C_1/2\gamma_1$	$i\pi C_1/2\gamma_1$
εA_1	$2\pi\delta_0/\sqrt{\mu'}$	$2\pi\delta_0/\sqrt{\mu'}$	$2\pi\delta_0/\sqrt{\mu'}$
εB_1	$-2\pi\delta_0$	$-2\pi\delta_0$	$-2\pi\delta_0$
$\varepsilon^2 A_2$	$i\pi(C_2 + C_4)$	$i\pi C_2 - \pi D_7\varepsilon$	$i\pi C_2$
$\varepsilon^2 B_2$	$-i\pi(C_2 + C_4)$	$-i\pi C_2 + \pi D_7\varepsilon$	$-i\pi C_2$
$\varepsilon^3 A_3$	0	$i\pi\sqrt{\mu'}C_4\varepsilon/2$	0
$\varepsilon^3 B_3$	0	$-i\pi C_4\varepsilon/2$	0

and

$$a_n = \begin{cases} 0 & \text{for } n \leq 0, \\ 1 & \text{for } n > 0. \end{cases}$$

The constant $C_{\mathbf{R}}$ in the rigid body translation potentials is given by

$$C_{\mathbf{R}} = i\delta_0\varepsilon\left[(1+\gamma_1)\left\{b'(1+\mu') - 2\mathrm{Log}\,\varepsilon - 2\mu'\mathrm{Log}\,(\sqrt{\mu'}\varepsilon)\right\} - \sigma\right],$$

where

$$\delta_0 = \sqrt{\mu'}(1 - nm^2)(kL)^2/8 \qquad \text{and} \qquad b' = i\pi + 1 + 2\mathrm{Log}\,2 - 2\gamma.$$

We do not record the lengthy expressions for the null-potentials here; they do not contribute to the displacement field itself.

8. Comparison of the numerical results of the M.A.E. method with those of the B.I.E. method for the special case of the ellipse (n = 1)

We conclude this work by presenting some numerical results obtained from the M.A.E. method for the special case of an elliptic cavity ($n = 1$). The ellipse is characterized by the ratio $H = a/b$ of its semi-major axis, a, to its semi-minor axis, b. The characteristic length L is chosen to be a. The incident field is as defined in Section 4, that is, a P-wave at zero incidence.

The quantities computed are the total dimensionless complex components in the x and y directions of the surface displacements, which are given by

$$u_x^* = \frac{L.\mu.u_x}{i\mu' kL} = e^{ikL(1+m)\cos\eta} + \mathrm{Real}\,(D_0)/i\gamma_1 kL$$

$$+\,\mathrm{Real}\,(D_1)/\gamma_1\sqrt{\mu'} - C_{\mathbf{R}}/i\gamma_1 kL, \tag{8.1a}$$

$$u_y^* = \frac{L.\mu.u_y}{i\mu' kL} = \mathrm{Im}\,(D_0)/i\gamma_1 kL + \mathrm{Im}\,(D_1)/\gamma_1\sqrt{\mu'}, \tag{8.1b}$$

where Real and Im denote the real and imaginary parts,

$$\begin{aligned} D_0 &= \sigma\beta_1\bar{\xi} + (m\beta_1 - \beta_2)\xi, \\ D_1 &= -m\delta_1 + \sigma\delta_2\bar{\xi}^2 - (\delta_1 + \delta_5)\xi^2 \\ &\quad - \left[(m\delta_1 - 2\delta_2 + \delta_7)\xi^2 + m(m\delta_1 - 2\delta_2) + \delta_6\right]/(\xi^2 - m). \end{aligned}$$

The constants β_1, β_2, δ_1, δ_2, $\ldots$, δ_7 and $C_{\mathbf{R}}$ are given in section 6, $\xi = e^{i\eta}$ with $0 \le \eta \le 2\pi$ and $\bar{\xi}$ denotes the conjugate of ξ. In terms of H, m is expressed as

$$m = (H - 1)/2H.$$

Numerical results from the B.I.E. method are also presented and a comparison of these results with those obtained from the M.A.E. method is made. Results are presented for $Ka = 0.1$ and $H = 1.5$ and 3.0, while Poisson's ratio ν is fixed at 0.25. As D_0 and D_1 are of order ε^2 and $C_{\mathbf{R}}$ of order ε^3, it can be seen from (8.1) that we expect the results from the M.A.E. method to agree with those of the B.I.E. method to the order of ε^2.

In Table 2 (3), we have the results for $\varepsilon = 0.1$ and $H = 1.5$ (3.0). In the first column we have the angle η which determines the position of the station on the boundary. In the second and third columns, we have the real and imaginary parts of the complex component of the displacement in the x-direction and in the fourth and fifth columns we have those corresponding to the y-direction. At every station, the upper (lower) entry corresponds to the results from the B.I.E. (M.A.E.) method. Note that since there is a symmetry about the x-axis, the results are only given for the upper half of the boundary. It can clearly be seen that the results agree to the order expected, that is, two decimal places.

While the B.I.E. method provides the solution for any type of boundary, no significant simplifications can be made when ε becomes small. Furthermore, there is a limit on how small ε can be before the solution from the B.I.E. method becomes numerically unstable. Although this difficulty may be overcome, it may lead to an increase in the time and hence cost in computing the solution. The M.A.E. method, as we have seen, gives results with a reasonable error (of order ε^3) and ε can be taken as small as we wish. However, as ε gets bigger the error incurred is no longer negligible. In this case, one can always resort to the B.I.E. method.

TABLE 2. Numerical values of the displacement for $\varepsilon = 0.1$ and $H = 1.5$. The upper (lower) entry corresponds to the B.I.E. (M.A.E.) method.

$\eta(°)$	Real (u_x^*)	Im (u_x^*)	Real (u_y^*)	Im (u_y^*)
0	0.9868 0.9878	0.1553 0.1536	0.0000 0.	0.0000 0.
18	0.9873 0.9883	0.1475 0.1459	0.0000 −0.0001	0.0044 0.0045
36	0.9888 0.9896	0.1247 0.1233	0.0000 −0.0002	0.0082 0.0085
54	0.9907 0.9913	0.0893 0.0882	0.0000 −0.0002	0.0115 0.0117
72	0.9923 0.9926	0.0446 0.0440	0.0000 −0.0001	0.0133 0.0137
90	0.9932 0.9931	−0.0050 −0.0051	0.0003 0.0000	0.0142 0.0144
108	0.9930 0.9926	−0.0545 −0.0541	0.0004 0.0001	0.0133 0.0137
126	0.9920 0.9913	−0.0993 −0.0984	0.0005 0.0002	0.0115 0.0116
144	0.9906 0.9896	−0.1345 −0.1335	0.0004 0.0002	0.0082 0.0085
162	0.9894 0.9883	−0.1575 −0.1560	0.0002 0.0001	0.0044 0.0045
180	0.9890 0.9878	−0.1653 −0.1638	0.0000 0.0000	0.0000 0.0000

Acknowledgement

This work was carried out at the Mathematics Department of the University of Manchester. The author wishes to thank Dr P. A. Martin for his useful help throughout the preparation of this work.

TABLE 3. Numerical values of the displacement for $\varepsilon = 0.1$ and $H = 3.0$. The upper (lower) entry corresponds to the B.I.E. (M.A.E.) method.

$\eta(°)$	Real (u_x^*)	Im (u_x^*)	Real (u_y^*)	Im (u_y^*)
0	0.9874	0.1404	0.0000	0.0000
	0.9879	0.1396	0.	0.
18	0.9879	0.1334	−0.0004	0.0090
	0.9884	0.1325	−0.0003	0.0089
36	0.9894	0.1128	−0.0006	0.0171
	0.9898	0.1121	−0.0005	0.0170
54	0.9913	0.0808	−0.0006	0.0237
	0.9915	0.0802	−0.0005	0.0233
72	0.9928	0.0403	−0.0004	0.0277
	0.9929	0.0399	−0.0003	0.0274
90	0.9935	−0.0046	−0.0001	0.0293
	0.9935	−0.0046	0.0000	0.0289
108	0.9932	−0.0494	0.0002	0.0277
	0.9929	−0.0492	0.0003	0.0274
126	0.9920	−0.0899	0.0005	0.0237
	0.9915	−0.0895	0.0005	0.0233
144	0.9904	−0.1219	0.0005	0.0171
	0.9898	−0.1214	0.0005	0.0170
162	0.9891	−0.1425	0.0003	0.0090
	0.9884	−0.1419	0.0003	0.0089
180	0.9886	−0.1496	0.0000	0.0000
	0.9879	−0.1489	0.0000	0.0000

Appendix

Consider the conformal transformation defined by (4.6), which we rewrite here

$$z = \xi + m/\xi^n, \tag{A1}$$

where n is a positive integer and m is real with $0 \le m < 1/n$. We wish to find the asymptotic expansion of the inverse transformation in terms of the outer coordinates. The equation which determines the inverse of (A1) has n solutions. Special care must,

https://doi.org/10.1017/S0334270000007591 Published online by Cambridge University Press

therefore, be taken in choosing the right one. This can be done by looking at the behaviour of z for large $|\xi|$. Taking this into account, we assume the following form:

$$\xi = z + \sum_{j=1}^{\infty} a'_j / z^j, \tag{A2}$$

where a'_j are coefficients to be determined. Re-expressing (A2) in terms of outer coordinates gives

$$\xi = Z.\varepsilon^{-1} + \sum_{j=1}^{\infty} a'_j \varepsilon^j / Z^j. \tag{A3}$$

Substituting for ξ, given by (A3), into (A1) leads to

$$0 = \sum_{j=1}^{\infty} a'_j \varepsilon^j / Z^j + m \left(\sum_{j=1}^{\infty} a'_j \varepsilon^j / Z^j \right)^{-n}. \tag{A4}$$

In order to determine the coefficients a'_j, we expand the expression multiplied by m in (A4) for small ε and then equate to zero all coefficients of the powers of $1/Z$. This leads to an infinite system of equations which gives for the first few coefficients the following expressions:

$$a'_1 = a'_2 = \cdots = a'_{n-1} = 0,$$

$$a'_n = -m,$$

$$a'_{n+1} = a'_{n+2} = \cdots = a'_{2n} = 0,$$

$$a'_{2n+1} = -nm^2,$$

$$a'_{2n+2} = a'_{2n+3} = \cdots = a'_{3n+1} = 0,$$

$$a'_{3n+2} = -n(3n+1)m^3/2,$$

$$a'_{3n+3} = a'_{3n+4} = \cdots = a'_{4n+2} = 0,$$

$$a'_{4n+3} = -n(2n+1)(4n+1)m^4/3.$$

Substituting the above expressions into (A3) gives

$$\xi = Z\varepsilon^{-1} + a'_n \varepsilon^n / Z^n + a'_{2n+1} \varepsilon^{2n+1} / Z^{2n+1} + a'_{3n+2} \varepsilon^{3n+2} / Z^{3n+2} + a'_{4n+3} \varepsilon^{4n+3} / Z^{4n+3} + O(\varepsilon^{5n+4}). \tag{A5}$$

References

[1] L. Bencheikh, "Modified fundamental solutions for the scattering of elastic waves by a cavity", *Q. J. Mech. Appl. Math. Part 1* **43** (1990) 57–73.

[2] L. Bencheikh, "Modified fundamental solutions for the scattering of elastic waves by a cavity: numerical results", *Int. J. Num. Meth. Eng.* **36** (1993) 3283–3302.

[3] V. T. Buchwald, "The diffraction of elastic waves by small cylindrical cavities", *J. Austral. Math. Soc. Ser. B* **20** (1978) 495–507.

[4] V. T. Buchwald and T. Tran Cong, "The diffraction of long elastic waves by elliptical cylindrical cavities", *J. Austral. Math. Soc. Ser. B* **26** (1984) 108–118.

[5] D. G. Crighton and F. G. Leppington, "Singular perturbation methods in acoustics: diffraction by a plate of finite thickness", *Proc. Roy. Soc. Lond. Ser. A* **335** (1973) 313–339.

[6] S. K. Datta, "Scattering of elastic waves", in *Mechanics Today* (ed. S. Nemat-Nasser), Volume 4, (Pergamon Press, New York, 1978) 149–208.

[7] A. H. England, *Complex variable methods in elasticity* (Wiley-Interscience, 1971).

[8] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity* (Noordhoff, Groningen, The Netherlands, 1963).

[9] F. J. Sabina and J. R. Willis, "Scattering of SH waves by a rough half-space of arbitrary slope", *Geophy. J. Roy. Astr. Soc.* **42** (1975) 685–703.