References

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108.43 An alternating recursion: proof of a conjecture by Erik Vigren

The following construction was considered by Erik Vigren in [1]. With positive numbers a_0, a_1 given, a_n is defined for $n \geqslant 2$ by an alternating recursion:

$$a_{2n} = \sqrt{a_{2n-2}a_{2n-1}},$$

the geometric mean of the previous two terms, while

$$a_{2n+1} = \frac{2a_{2n-1}a_{2n}}{a_{2n-1} + a_{2n}},$$

the harmonic mean of the previous two terms (which we denote by $H\left(a_{2n-1}, a_{2n}\right)$).

It was conjectured in [1], with support from numerical calculations, that a_n converges to $\gamma(a_0, a_1)$, where for $x < y$,

$$\gamma(x, y) = \frac{y}{\sqrt{\frac{y}{x} - 1}} \tan^{-1}\sqrt{\frac{y}{x} - 1} \tag{1}$$

while for $x > y$,

$$\gamma(x, y) = \frac{y}{\sqrt{1 - \frac{y}{x}}} \tanh^{-1}\sqrt{1 - \frac{y}{x}}. \tag{2}$$

Here we give a proof for the case where $a_0 < a_1$, so that (1) applies. The case $a_0 > a_1$ can then be proved similarly, or derived from (1) using $\tanh^{-1} z = \frac{1}{2} \ln\frac{1+z}{1-z}$ and $\ln\left(1 + ic\right) - \ln(1 - ic) = 2i \tan^{-1} c$.

Lemma 1: (a_n) tends to a limit.

Proof: First, since either type of mean of x and y lies between x and y, an easy induction shows that $a_0 < a_n < a_1$ for all $n \geqslant 2$. Now $a_2 = \sqrt{a_0a_1}$, so

$a_0 < a_2 < a_1$. Also, $a_3 = H(a_1, a_2)$, so $a_2 < a_3 < a_1$. Further, $a_3 < \frac{1}{2}(a_1 + a_2)$, since $H(x, y) < \frac{1}{2}(x + y)$. Hence $a_3 - a_2 < \frac{1}{2}(a_1 - a_2) < \frac{1}{2}(a_1 - a_0)$.

Repeating this, we see that $a_0 < a_2 < a_4 < \ldots$ and $a_1 > a_3 > a_5 > \ldots$, also $0 < a_{2n+1} - a_{2n} < \frac{1}{2^n}(a_1 - a_0)$. It follows that (a_{2n}) and (a_{2n+1}) converge to a common limit L.

Lemma 2: We have $\gamma(a_2, a_3) = \gamma(a_0, a_1)$.

Once Lemma 2 is known, the deduction that a_n tends to $\gamma(a_0, a_1)$ is easy, as follows. By repetition of Lemma 2, $\gamma(a_{2n}, a_{2n+1}) = \gamma(a_0, a_1)$ for all n. Now $\frac{a_{2n+1}}{a_{2n}} - 1 \to 0$ as $n \to \infty$ and $\frac{\tan^{-1} t}{t} \to 1$ as $t \to 0$, so we see from (1) that $\gamma(a_{2n}, a_{2n+1}) \to L$ as $n \to \infty$. But $\gamma(a_{2n}, a_{2n+1})$ has the constant value $\gamma(a_0, a_1)$, so $L = \gamma(a_0, a_1)$.

We will prove Lemma 2 by establishing two further lemmas.

Lemma 3: We have

$$\frac{a_3}{\sqrt{\frac{a_3}{a_2} - 1}} = \frac{2a_1}{\sqrt{\frac{a_1}{a_0} - 1}}.$$

Proof: Note that

$$\frac{a_3}{a_2} = \frac{2a_1}{a_1 + a_2} = \frac{2a_1}{a_1 + \sqrt{a_0 a_1}} = \frac{2\sqrt{a_1}}{\sqrt{a_1} + \sqrt{a_0}}, \tag{3}$$

so

$$\frac{a_3}{a_2} - 1 = \frac{\sqrt{a_1} - \sqrt{a_0}}{\sqrt{a_1} + \sqrt{a_0}} = \frac{\left(\sqrt{a_1} - \sqrt{a_0}\right)^2}{a_1 - a_0},$$

hence

$$\sqrt{\frac{a_3}{a_2} - 1} = \frac{\sqrt{a_1} - \sqrt{a_0}}{\sqrt{a_1 - a_0}}. \tag{4}$$

By (3) and (4), together with $a_2 = \sqrt{a_0 a_1}$, we have

$$\frac{a_3}{\sqrt{\frac{a_3}{a_2} - 1}} = \frac{2\sqrt{a_0}\, a_1}{\sqrt{a_1} + \sqrt{a_0}} \frac{\sqrt{a_1 - a_0}}{\sqrt{a_1} - \sqrt{a_0}}$$

$$= \frac{2\sqrt{a_0}\, a_1 \sqrt{a_1 - a_0}}{a_1 - a_0}$$

$$= \frac{2\sqrt{a_0}\, a_1}{\sqrt{a_1 - a_0}} = \frac{2a_1}{\sqrt{\frac{a_1}{a_0} - 1}}.$$

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Lemma 4: If $\tan\theta = \sqrt{\frac{a_3}{a_2} - 1}$, then $\tan 2\theta = \sqrt{\frac{a_1}{a_0} - 1}$.

Proof: By (4),

$$\tan\theta = \frac{\sqrt{a_1} - \sqrt{a_0}}{\sqrt{a_1 - a_0}},$$

so

$$1 - \tan^2\theta = 1 - \frac{a_1 - 2\sqrt{a_0}\sqrt{a_1} + a_0}{a_1 - a_0}$$
$$= \frac{2\sqrt{a_0}\sqrt{a_1} - 2a_0}{a_1 - a_0}$$
$$= \frac{2\sqrt{a_0}\left(\sqrt{a_1} - \sqrt{a_0}\right)}{a_1 - a_0}.$$

Hence

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{a_1 - a_0}{\sqrt{a_0}\sqrt{a_1 - a_0}}$$
$$= \frac{\sqrt{a_1 - a_0}}{\sqrt{a_0}} = \sqrt{\frac{a_1}{a_0} - 1}.$$

By Lemmas 3 and 4, we have

$$\gamma\left(a_2, a_3\right) = \frac{a_3\theta}{\sqrt{\frac{a_3}{a_2} - 1}} = \frac{a_1\left(2\theta\right)}{\sqrt{\frac{a_1}{a_0} - 1}} = \gamma\left(a_0, a_1\right),$$

establishing Lemma 2.

A simple case of (1), which is reflected in the title of [1], is $\gamma(1, 2) = \frac{\pi}{2}$.

A corresponding result for the alternating iteration of geometric and arithmetic means (also considered in [1]) can be deduced by substituting $b_n = 1/a_n$.

Reference

1. Erik Vigren, π is a mean of 2 and 4, *Math. Gaz.* **108** (July 2024), pp. 331-334.

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