DOMAIN PERTURBATIONS OF THE BIHARMONIC OPERATOR

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1. Introduction. Eigenvalue problems for the biharmonic operator (iterated Laplacian) $L = \Delta\Delta$ will be studied on bounded plane domains. Our purpose is to obtain asymptotic variational formulae for eigenvalues and eigenfunctions under the deformation of removing an ϵ-disk and adjoining additional boundary conditions on the new boundary component thereby introduced, valid on a positive interval $0 < \epsilon \leqslant \epsilon_0$. Eigenvalue problems can be considered in connection with each of the following sets of homogeneous boundary conditions:

1. $u = \dfrac{\partial u}{\partial n} = 0,$

2. $u = \Delta u = 0,$

3. $\dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0,$

4. $\Delta u = \dfrac{\partial \Delta u}{\partial n} = 0,$

where $\partial/\partial n$ represents differentiation in the exterior normal direction to the boundary. Representation formulae for Green's functions associated with each of the four problems have been given, for example, by Gould **(6)** and Weinstock **(9)**.

Boundary conditions 1 pertain to the clamped vibrating plate, which has been widely treated. Detailed references are given in **(6)**. The eigenvalues are proportional to the squares of the natural vibration frequencies of such a plate. Our first result (section 3) gives asymptotic estimates for the *perturbed* eigenvalues and eigenfunctions of this problem, obtained when additional boundary conditions 1 are imposed on the boundary of a deleted ϵ-disk.

An essential feature of the variational problem for boundary conditions 1, which does not occur for the analogous second-order problems **(8)**, is that the perturbed eigenvalues μ_ϵ do not converge in general to the (classical) eigenvalues for the unperturbed domain. Accordingly, it is one of our main purposes to define a (non-classical) eigenvalue problem whose eigenvalues λ can be matched with the eigenvalues μ_ϵ in a one-to-one manner such that $\mu_\epsilon \to \lambda$ as $\epsilon \to 0$ (section 2).

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The eigenvalue problem for L with boundary conditions 2 is the problem for a supported vibrating plate. It is well known and easy to verify that the eigenvalues of problem 2 are the squares of the eigenvalues of the operator $-\Delta$ with null boundary conditions (fixed-edge membrane problem **(6**, p. 105**)**. Thus asymptotic variational formulae for problem 2 can be obtained from the author's results **(8)** for second-order operators (section 4).

Likewise the eigenvalues of L with boundary conditions 3 are the squares of the positive eigenvalues of $-\Delta$ with the boundary condition $\partial u/\partial n = 0$ (free-edge membrane problem). Variational formulae for the latter eigenvalues and corresponding eigenfunctions will be obtained in section 5 by methods analogous to those in **(8)**.

The eigenvalue problem with boundary conditions 4 (natural conditions) is the so-called free plate problem. It is known **(9)** that the positive eigenvalues are identical with those of problem 1. (Every harmonic function is a solution of problem 4 for $\lambda = 0$.)

We shall not consider mixed boundary conditions here: e.g. $u = 0$ on part of the boundary and $\partial u/\partial n = 0$ on the remaining part. Also we shall restrict ourselves to the plane, although our methods, involving functional analysis rather than complex function theory, are capable of generalization to Euclidean space E^n or to Riemannian space.

2. Basic and perturbed domains. Let M be an (open connected) bounded plane domain whose boundary B consists of a finite number of arcs having continuous curvature. Throughout, $\partial/\partial n$ denotes differentiation in the exterior normal direction $\boldsymbol{n}$ to B. The basic domain $\mathfrak{D}^j$ for L is defined to be the set of all complex-valued functions on $\bar{M}$ that are of class $C^4[M] \cap C^j[\bar{M}]$ and satisfy boundary conditions j on B, $j = 1, 2, 3$. The corresponding *basic eigenvalue problem* for each j is

$$Lx = \lambda x, \qquad x \in \mathfrak{D}^j. \tag{2.1}$$

The variation of the eigenvalues and eigenfunctions will be obtained when the domain $\mathfrak{D}^j$ is perturbed to a domain $\mathfrak{D}_\epsilon^j$, defined as follows. Let p_0 be a fixed (but arbitrary) point in M. Let $|p - p_0|$ denote the Euclidean distance between p and p_0. Let γ be the circle $|p - p_0| = \epsilon$, and let N_ϵ be the disk $|p - p_0| < \epsilon$. Let $M_\epsilon = M \diagdown \bar{N}_\epsilon$. The parameter ϵ measures the smallness of the deleted disk, $0 < \epsilon \leqslant \epsilon_0$. Then the perturbed domain $\mathfrak{D}_\epsilon^j$ is defined to be the set of all complex-valued functions on $\bar{M}_\epsilon$ that are of class $C^4[M_\epsilon] \cap C^j[\bar{M}_\epsilon]$ and satisfy the boundary conditions j on $B \cup \gamma$. The *perturbed eigenvalue problem* for each j is

$$Ly = \mu y, \qquad y \in \mathfrak{D}_\epsilon^j. \tag{2.2}$$

Functions in $\mathfrak{D}_\epsilon^j$ are defined only on $\bar{M}_\epsilon$, but they will be extended to M by defining them to be zero inside γ.

It will be shown that the eigenvalues μ_ϵ corresponding to the perturbed domain $\mathfrak{D}_\epsilon^1$ do not converge in general to the eigenvalues for $\mathfrak{D}^1$, but rather to the eigenvalues of the following non-classical problem:

$$Lx = \lambda x, \qquad x \in \mathfrak{D}, \tag{2.3}$$

where $\mathfrak{D}$ denotes the set of all complex-valued functions u in M that are of class $C^4[M \setminus \{p_0\}] \cap C^1[\bar{M}]$, satisfy boundary conditions 1 on B, and satisfy $u(p_0) = 0$ with $\Delta u(p)$ having (at worst) the singularity $\log|p - p_0|$ at p_0.

If $A(p, q)$ denotes the standard biharmonic Green's function for $\mathfrak{D}^1$ (**6**, p. 101; **9**, p. 9; **4**), we define the "Green's function" for $\mathfrak{D}$ to be

$$A(p, q; p_0) = A(p, q) - A(p, p_0)A(q, p_0)/A(p_0, p_0),$$

which has the following properties:

(i) $A(p, q; p_0) = \partial A(p, q; p_0)/\partial n_p = 0 \qquad \text{if } p \in B;$
(ii) $A(p_0, q; p_0) = 0;$
(iii) as a function of either argument p or q, $A(p, q; p_0)$ is of class

$$C^4[M \setminus \{p_0\}] \cap C^1[\bar{M}];$$

(iv) $A(p, q; p_0) = A(q, p; p_0) \quad (p, q \in M);$
(v) $\Delta_p A(p, q; p_0)$ has logarithmic singularities at $p = p_0$ and at $p = q$.

For arbitrary $f \in C^4[M \setminus \{p_0\}] \cap C^1[\bar{M}]$, we obtain from the Rayleigh–Green identity and the above properties that

$$f(p) = \frac{A(p, p_0)f(p_0)}{A(p_0, p_0)} + \int_M A(p, q; p_0)Lf(q)\, dq$$
$$+ \int_B \left[f(q) \frac{\partial \Delta A(p, q; p_0)}{\partial n_q} - \frac{\partial f}{\partial n_q} \Delta_q A(p, q; p_0) \right] ds_q.$$

Suppose $f = x \in \mathfrak{D}$ and $Lx = \lambda x$. Then $f = \partial f/\partial n = 0$ on B and $f(p_0) = 0$. Hence x satisfies the integral equation

$$x(p) = \lambda \textstyle\int_M A(p, q; p_0)x(q)dq \tag{2.4}$$

with symmetric L^2 kernel.

Conversely, any solution of this integral equation satisfies boundary conditions 1 on B, $x(p_0) = 0$, and

$$\Delta x(p) \sim [\Delta_p A(p, p_0)/A(p_0, p_0)] \textstyle\int_M A(q, p_0)x(q)\, dq$$

as $p \to p_0$. Thus $\Delta x(p)$ has the required singularity at $p = p_0$. Differentiation of (2.4) by a well-known procedure leads to the differential equation $\Delta\Delta x(p) = \lambda x(p)$, $p \neq p_0$.

Since integral operators with symmetric L^2 kernels are known to be symmetric, completely continuous transformations on Hilbert space, problem (2.3) has a complete orthonormal sequence of eigenfunctions and corresponding eigenvalues λ_n such that $\lambda_n \to \infty$ as $n \to \infty$. Since functions in $\mathfrak{D}_\epsilon^1$ can be continued to M such that they belong to $\mathfrak{D}$, it follows from the standard maximum–minimum principle for eigenvalues that $\lambda_n \leqslant \mu_{n\epsilon}$ for each $n = 1, 2, \ldots$ and each ϵ **(2)**.

3. Asymptotic estimates for problem 1. The purpose is to obtain asymptotic estimates for the difference between the eigenvalues of (2.2) (when $j = 1$) and (2.3). The following notation will be used:

$$\phi(\epsilon) = \epsilon \log(1/\epsilon), \qquad 0 < \epsilon < 1. \tag{3.1}$$

Let $\mathfrak{G}$, $\mathfrak{G}_\epsilon$ be the Hilbert spaces $L^2[M]$, $L^2[M_\epsilon]$ respectively. The inner product in each space is given by

$$(u, v) = \int_M u\bar{v}\, dp$$

when it is agreed that functions defined in M_ϵ are extended to M by defining them to be zero inside γ. Let $\mathfrak{H}_\epsilon$ denote the space of single-valued harmonic functions in M_ϵ, and let Q_ϵ denote the projection mapping from $\mathfrak{G}_\epsilon$ into the subspace $\mathfrak{H}_\epsilon$.

Let $A_\epsilon(p, q)$ be the Green's functions for L on $\mathfrak{D}_\epsilon^1$, i.e. corresponding to boundary conditions 1 on $B \cup \gamma$ **(6**, p. 101**)**. Let A, A_ϵ be the linear integral operators whose kernels are the respective Green's functions $A(p, q; p_0)$, $A_\epsilon(p, q)$. These operators are known to be symmetric, completely continuous transformations on $\mathfrak{H}$, $\mathfrak{H}_\epsilon$ respectively, whose eigenvalues α_i, β_i are reciprocals of the eigenvalues λ_i, μ_i of (2.3), (2.2), $i = 1, 2, \ldots$.

Let α be an arbitrary eigenvalue of A, of multiplicity m ($m \geqslant 1$), and let $\mathfrak{A}_\alpha$ be the corresponding (m-dimensional) eigenspace. Let $\mathfrak{A}_{\alpha\epsilon}$ be the space consisting of the functions of $\mathfrak{A}_\alpha$ restricted to $\bar{M}_\epsilon$. We may assume an ϵ-neighbourhood $(0, \epsilon_0]$ to have been selected initially on which

$$\dim \mathfrak{A}_{\alpha\epsilon} = \dim \mathfrak{A}_\alpha = m.$$

LEMMA *There exist positive constants $\epsilon_1 < 1$ and c such that*

$$\|A_\epsilon x - \alpha x\| \leqslant c\phi(\epsilon)\, \|x\| \tag{3.2}$$

for every $x \in \mathfrak{A}_{\alpha\epsilon}$ and every ϵ satisfying $0 < \epsilon \leqslant \epsilon_1$.

Proof. It is sufficient to prove (3.2) in the case $\|x\| = 1$. For such $x \in \mathfrak{A}_{\alpha\epsilon}$, the function $f = A_\epsilon x - \alpha x$ is a solution of the Dirichlet problem $Lf = 0$ in M_ϵ, $f = \partial f/\partial n = 0$ on B, $f = x$ on γ, $\partial f/\partial n = \partial x/\partial n$ on γ **(5)**. It then follows from the Rayleigh–Green representation formula that

$$f(p) = \int_\gamma [x(q)\partial \Delta_q A_\epsilon(p, q)/\partial n_q - \Delta_q A_\epsilon(p, q)\partial x/\partial n_q]\, ds_q. \tag{3.3}$$

It is known **(6**, p. 101; **9**, p. 17**)** that

$$\Delta_q A_\epsilon(p, q) = -G_\epsilon(p, q) + \int_{M_\epsilon} a_\epsilon(q, r) G_\epsilon(p, r)\, dr, \tag{3.4}$$

where $G_\epsilon(p, r)$ is the standard harmonic Green's function for M_ϵ corresponding to null boundary conditions, and $a_\epsilon(q, r)$ is the reproducing kernel in the space

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$\mathfrak{H}_\epsilon$ of harmonic functions **(1; 6)**. The integral term on the right side of (3.4) is therefore the projection of $G_\epsilon(p, q)$ into $\mathfrak{H}_\epsilon$. It follows from (3.3) and (3.4) that $f(p)$ may be decomposed into the following three terms:

$$f_1(p) = \int_\gamma \left[G_\epsilon(p, q) \frac{\partial x}{\partial n_q} - x(q) \frac{\partial G_\epsilon(p, q)}{\partial n_q} \right] ds_q;$$

$$f_2(p) = \int_{M_\epsilon} \int_\gamma x(q) G_\epsilon(p, r) \frac{\partial a_\epsilon(q, r)}{\partial n_q} ds_q \, dr; \tag{3.5}$$

$$f_3(p) = -\int_{M_\epsilon} \int_\gamma G_\epsilon(p, r) a_\epsilon(q, r) \frac{\partial x}{\partial n_q} ds_q \, dr.$$

The existence of the boundary integrals follows upon transformation by Green's formula (see below). Since a standard theorem on harmonic functions gives $f_1(p) = O(\epsilon \log \epsilon)$ uniformly in M_ϵ, it will be sufficient in the proof of (3.2) to establish that $f_2(p) = O(\epsilon)$ and $f_3(p) = O(\epsilon \log \epsilon)$ uniformly in M_ϵ.

Since $U(q) = x(q)/\epsilon$ is a bounded function of q and ϵ for $q \in \gamma$, $0 < \epsilon \leqslant \epsilon_0$, we can construct a function $u \in C^2[M]$ independent of ϵ satisfying the boundary conditions $u = \partial u/\partial n = 0$ on B, $u = U$ on γ, and $\partial u/\partial n = 0$ on γ. Since $a_\epsilon(q, r)$ is harmonic in M_ϵ, we obtain from Green's symmetric formula

$$f_2(p) = -\epsilon \int_{M_\epsilon} \int_{M_\epsilon} \Delta u(q) a_\epsilon(q, r) G_\epsilon(p, r) \, dq \, dr.$$

By the reproducing property of $a_\epsilon(q, r)$,

$$f_2(p) = -\epsilon \int_{M_\epsilon} Q_\epsilon[\Delta u(r)] G_\epsilon(p, r) \, dr,$$

where Q_ϵ denotes the projection operator onto $\mathfrak{H}_\epsilon$. Then the Schwartz inequality yields

$$|f_2(p)| \leqslant \epsilon \, ||\Delta u|| \, ||G_\epsilon(p, r)||. \tag{3.6}$$

In order to prove that $||G_\epsilon(p, r)||$ is a bounded function of p and ϵ, we observe that

$$H_\epsilon(p, r) = G_\epsilon(p, r) - G(p, r) + G(p_0, r) \, h_\epsilon(p)$$

is a regular harmonic function of $p \in M_\epsilon$, where h_ϵ is the harmonic measure with boundary values 1 on γ and 0 on B, and $G(p, r)$ is the harmonic Green's function for M. Since $H_\epsilon(p, r)$ has the boundary values 0 on B and $O(\epsilon)$ on γ, it follows from the maximum principle **(3)** that $H_\epsilon(p, r)$ is uniformly $O(\epsilon)$ in M_ϵ. Since $||h_\epsilon|| = O(1/\log \epsilon)$, it follows that $||G_\epsilon(p, r)||$ is bounded for $p \in M_\epsilon$, $0 < \epsilon \leqslant \epsilon_0$. The inequality (3.6) then shows that $f_2(p) = O(\epsilon)$ in M_ϵ.

To prove that $f_3(p) = O(\epsilon \log \epsilon)$, consider the function defined by

$$g(p, p_0) = -(2\pi/\log \epsilon)[G(p, p_0) + \psi(p)],$$

where ψ is the biharmonic function in M with boundary values such that $g = \partial g/\partial n = 0$ on B. Let

$$h(p) = \epsilon \log \epsilon [A(p, p_0)/A(p_0, p_0) - g(p, p_0)].$$

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Then $Lh = 0$, $h = \partial h/\partial n = 0$ on B, $h = O(\epsilon)$ on γ, and $\partial h/\partial n = 1 + O(\epsilon \log \epsilon)$ on γ. Hence we obtain analogues of the representation (3.3) and the decomposition (3.5) of h into $h_1 + h_2 + h_3$. We find that $h_1 = O(\epsilon \log \epsilon)$ and $h_2 = O(\epsilon)$ in the same manner as above. Furthermore,

$$\begin{aligned} h_3(p) &= -\int_{M_\epsilon}\int_\gamma a_\epsilon(q,r)G_\epsilon(p,r)\,\partial h/\partial n_q\,ds_q\,dr \\ &= -\int_{M_\epsilon}\int_\gamma a_\epsilon(q,r)G_\epsilon(p,r)\,ds_q\,dr + O(\epsilon\log\epsilon) \end{aligned}$$

by an argument similar to that used above for $f_2(p)$. If x_0' denotes the limit of $\partial x/\partial n$ as $q \to p_0$, then since $x \in \mathfrak{D}$, (3.5) yields

$$\begin{aligned} f_3(p) &= -\int_{M_\epsilon}\int_\gamma [x_0' + O(\epsilon\log\epsilon)]a_\epsilon(q,r)G_\epsilon(p,r)\,ds_q\,dr \\ &= x_0'\,h_3(p) + O(\epsilon\log\epsilon) = O(\epsilon\log\epsilon). \end{aligned}$$

This completes the proof of the lemma.

THEOREM 1. *Corresponding to each eigenvalue* λ *of the basic problem* (2.3), *of multiplicity* m, *there exist positive numbers* ϵ_1 *and* c_1 *such that exactly* m *eigenvalues* μ_i *of the perturbed problem* (2.2) *lie in the interval* $[\lambda, \lambda + c_1\phi(\epsilon)]$ *provided* $0 < \epsilon \leqslant \epsilon_1$. *Furthermore, there exist orthonormal eigenfunctions* x_i *associated with* λ *and corresponding orthonormal* y_i *associated with the* μ_i *such that the uniform estimates*

$$y_i(p) = x_i(p) + O(\phi), \tag{3.7}$$

$$\Delta y_i(p) = \Delta x_i(p) + O(\phi)\log|p - p_0| \tag{3.8}$$

are valid, $p \in M_\epsilon$, $0 < \epsilon \leqslant \epsilon_1$, $i = 1, 2, \ldots, m$.

Proof. Let $P(\delta)$ be the projection operator onto the subspace $\mathfrak{F}_{\epsilon\delta}$ of $\mathfrak{G}_\epsilon$ spanned by all the eigenfunctions of A_ϵ whose corresponding eigenvalues lie in the interval $(\alpha - \delta, \alpha)$, $\delta > 0$. Then for any $w \in \mathfrak{G}_\epsilon$, the inequality

$$||w - P(\delta)w|| \leqslant \delta^{-1}\,||A_\epsilon w - \alpha w|| \tag{3.9}$$

is valid **(7**, p. 34**)**. In particular, w can be replaced by any $x \in \mathfrak{A}_{\alpha\epsilon}$. Then (3.9) combined with the lemma (3.2) yields

$$||x - P(\delta)x|| \leqslant c\phi(\epsilon)\delta^{-1}\,||x||. \tag{3.10}$$

This implies **(7**, p. 35**)** that there are at least m eigenvalues β_i ($i = 1, 2, \ldots, m$) of A_ϵ in the interval $[\alpha - c\phi(\epsilon), \alpha]$. Since λ, μ_i are reciprocals of α, β_i, there are at least m eigenvalues μ_i of the perturbed problem (2.2) in the interval $[\lambda, \lambda + c_1\phi(\epsilon)]$, where c_1 is another positive constant.

To prove there are exactly m, let λ^i denote the ith *distinct* eigenvalue, of multiplicity m_i, $\lambda^1 < \lambda^2 < \ldots$. Since the eigenvalues do not accumulate and since $\phi(\epsilon) \to 0$ as $\epsilon \to 0$, a positive interval $(0, \epsilon_1]$ can be selected so that $c_1\phi(\epsilon)$ is less than the minimum of all the differences $\lambda^{i+1} - \lambda^i$ for ϵ on this interval; and accordingly at least m_i eigenvalues μ are included in the subinterval $[\lambda^i, \lambda^i + c_1\phi(\epsilon)]$ of $[\lambda^i, \lambda^{i+1})$, $i = 1, 2, \ldots$. Since $\lambda_i \leqslant \mu_i$ for each i is known as a general consequence of $M_\epsilon \subset M$, an easy induction argument shows that exactly m_i eigenvalues μ are included in $[\lambda^i, \lambda^i + c_1\phi(\epsilon)]$.

In order to prove the second part of the theorem, we select the number δ in (3.9) as the distance between α and the nearest eigenvalue $\alpha^i \neq \alpha$. Then $\mathfrak{F}_{\epsilon\delta}$ is m-dimensional by the first part of the theorem, and $P(\delta)x = 0$ implies $x = 0$ by (3.10). Hence there exist m uniquely determined linearly independent eigenfunctions $z_i \in \mathfrak{A}_{\alpha\epsilon}$ which $P(\delta)$ maps into the orthonormal eigenfunctions y_i, and by (3.10), $||z_i - y_i|| = O(\phi)$, $i = 1, 2, \ldots, m$. We use the Schwartz inequality to obtain

$$|(z_i, z_j) - (y_i, y_j)| \leqslant ||y_i||\,||z_j - y_j|| + ||z_j||\,||z_i - y_i||.$$

Hence

$$(z_i, z_j) = \delta_{ij} + O(\phi), \qquad i, j = 1, 2, \ldots, m. \tag{3.11}$$

Since the z_i are linearly independent, an orthonormal sequence $\{x_i\}$ can be obtained by the Schmidt process as linear combinations of the z_i. Because of (3.11) it turns out that $||x_i - z_i|| = O(\phi)$, and hence $||x_i - y_i|| = O(\phi)$.

Omitting the subscript i, we select a typical x in the set $\{x_i\}$ and the corresponding y in $\{y_i\}$. It follows from the uniform estimate

$$|y(p) - \lambda A_\epsilon x(p)| = |A_\epsilon(\mu y - \lambda x)(p)| \leqslant ||A_\epsilon||\,(\mu\,||y - x|| + |\mu - \lambda|\,||x||)$$

and the first part of the theorem that there exists a constant c such that

$$|y(p) - \lambda A_\epsilon x(p)| \leqslant c\phi(\epsilon) \qquad (p \in M_\epsilon;\ 0 < \epsilon \leqslant \epsilon_1). \tag{3.12}$$

The function g defined by

$$g(p) = \lambda A_\epsilon x(p) - x(p) + f(p),$$

where f is given by (3.3), is a solution of the Dirichlet problem $Lg = 0$, $g = \partial g/\partial n = 0$ on $B \cup \gamma$, and hence g is identically zero. The following estimate is then a direct consequence of (3.12):

$$y(p) = x(p) - f(p) + O(\phi), \qquad 0 < \epsilon \leqslant \epsilon_1.$$

Since $f(p) = O(\phi)$ uniformly in M_ϵ by the lemma, (3.7) is established. The proof of (3.8) is similar and will be omitted.

As a consequence of Theorem 1, we shall establish the following asymptotic variational formulae for eigenvalues:

$$\mu_{\epsilon i} - \lambda_i = \int_\gamma \left[\bar{x}_i(p)\,\frac{\partial \Delta f_i(p)}{\partial n} - \Delta f_i(p)\,\frac{\partial \bar{x}_i}{\partial n}\right] ds + O(\phi^2) \tag{3.13}$$

$$(i = 1, 2, \ldots;\ 0 < \epsilon \leqslant \epsilon_1),$$

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where f_i is the solution of the Dirichlet problem $Lf = 0, f = \partial f/\partial n = 0$ on B, $f = x_i$ on γ, $\partial f/\partial n = \partial x_i/\partial n$ on γ.

The subscripts i will be deleted in the proof of (3.13). Since $\Delta\Delta x = \lambda x$ and $\Delta\Delta y = \mu y$, the Rayleigh–Green formula gives

$$(3.14)\qquad (\lambda - \mu) \int_{M_\epsilon} x(p)\bar{y}(p)\,dp = \int_{B\cup\gamma} \left(\bar{y}\frac{\partial\Delta x}{\partial n} + \Delta\bar{y}\frac{\partial x}{\partial n} - x\frac{\partial\Delta\bar{y}}{\partial n} - \Delta x\frac{\partial\bar{y}}{\partial n}\right) ds.$$

By Theorem 1, $y(p) = x(p) + O(\phi)$ on M_ϵ. Since $y = \partial y/\partial n = 0$ on $B \cup \gamma$, $x = \partial x/\partial n = 0$ on B, and $||x|| = 1$,

$$(\lambda - \mu)[1 + O(\phi)] = -\int_\gamma \left(\Delta\bar{y}\frac{\partial x}{\partial n} - x\frac{\partial\Delta\bar{y}}{\partial n}\right) ds.$$

Similarly,

$$-\mu \int_{M_\epsilon} f(p)\bar{y}(p)\,dp = -\int_\gamma \left(\Delta\bar{y}\frac{\partial x}{\partial n} - x\frac{\partial\Delta\bar{y}}{\partial n}\right) ds.$$

Since $\mu = \lambda + O(\phi)$ and $y(p) = x(p) + O(\phi)$, the following asymptotic variational formula is obtained:

$$\mu - \lambda = \lambda \textstyle\int_{M_\epsilon} \bar{x}(p)f(p)\,dp + O(\phi^2).$$

An identity similar to (3.14) then yields the result (3.13).

4. Asymptotic estimates for problem 2. The analogue of Theorem 1 will now be obtained directly from the corresponding known result **(8)** for the fixed-edge membrane problem for the operator $-\Delta$.

Let $\mathfrak{B}$ be the set of all complex-valued functions on $\bar{M}$ that are of class $C^2[M]$, continuous on $\bar{M}$, and zero on B. The basic eigenvalue problem corresponding to this boundary condition is (fixed-edge membrane problem)

$$(4.1)\qquad -\Delta x = \nu x, \qquad x \in \mathfrak{B}.$$

Let $\mathfrak{B}_\epsilon$ be the set of all complex-valued functions on $\bar{M}_\epsilon$ that are of class $C^2[M_\epsilon]$, continuous on $\bar{M}_\epsilon$, and zero on $B \cup \gamma$. The perturbed problem corresponding to these boundary conditions is

$$(4.2)\qquad -\Delta y = \sigma y, \qquad y \in \mathfrak{B}_\epsilon.$$

The basic and perturbed problems for $L = \Delta\Delta$ corresponding to boundary conditions 2 are

$$(4.3)\qquad Lx = \lambda x, \qquad x \in \mathfrak{D}^2,$$

$$(4.4)\qquad Ly = \mu y, \qquad y \in \mathfrak{D}_\epsilon^2.$$

For every m-fold degenerate eigenvalue λ of (4.3), there is an m-fold degenerate eigenvalue ν of (4.1) such that $\lambda = \nu^2$, and the eigenfunctions x_i associated with ν are identical with the eigenfunctions associated with λ. Likewise $\mu_i = \sigma_i^2$ is the ith eigenvalue of (4.4) corresponding to the ith eigenvalue σ_i of (4.2), and the associated eigenfunctions are identical. According to (**8**),

$$\sigma_i - \nu = -[|x_i(p_0)|^2 + O(\psi)] \textstyle\int_\gamma \nabla h \cdot \boldsymbol{n}\, ds = O(\psi), \tag{4.5}$$

where

$$\psi(\epsilon) = [\log(1/\epsilon)]^{-1} \qquad (0 < \epsilon < 1;\ i = 1, 2, \ldots, m),$$

and h is the solution of the Dirichlet problem $\Delta h = 0$ in M_ϵ, $h = 0$ on B, and $h = 1$ on γ. Furthermore, the following uniform estimates are valid:

$$y_i(p) = x_i(p) - x_i(p_0)h(p) + O(\psi), \qquad p \in M_\epsilon, \quad 0 < \epsilon \leqslant \epsilon_1, \quad i = 1, 2, \ldots . \tag{4.6}$$

It follows from (4.5) that $\mu_i - \lambda = (\sigma_i - \nu)(\sigma_i + \nu) = O(\psi)$, and hence

$$\begin{aligned} \mu_i - \lambda &= (\sigma_i - \nu)[2\nu + (\sigma_i - \nu)] \\ &= [-2\sqrt{\lambda}\,|x_i(p_0)|^2 + O(\psi)] \textstyle\int_\gamma \nabla h \cdot \boldsymbol{n}\, ds, \\ &\qquad 0 < \epsilon \leqslant \epsilon_0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{4.7}$$

THEOREM 2. *Corresponding to each eigenvalue* λ *of* (4.3), *of multiplicity* m, *there exist positive numbers* ϵ_1 *and* c_1 *such that exactly* m *eigenvalues* μ_i *of* (4.4) *lie in the interval* $[\lambda, \lambda + c_1\psi(\epsilon)]$ *provided* $0 < \epsilon \leqslant \epsilon_1$. *The asymptotic variational formulae* (4.7) *hold for the eigenvalues and the uniform estimates* (4.6) *hold for corresponding orthonormal eigenfunctions.*

5. Asymptotic estimates for problem 3. Let $\mathfrak{B}'$ be the set of complex-valued functions u on $\bar{M}$ that are of class $C^2[M] \cap C^1[\bar{M}]$ and satisfy the boundary condition $\partial u/\partial n = 0$ on B. Let $\mathfrak{B}_\epsilon'$ be the set of complex-valued functions on $\bar{M}_\epsilon$ that are of class $C^2[M_\epsilon] \cap C^1[\bar{M}_\epsilon]$ and satisfy the same boundary condition on $B \cup \gamma$. The basic and perturbed problems for $-\Delta$ (free-edge membrane problems) are

$$-\Delta x = \nu x, \qquad x \in \mathfrak{B}'; \tag{5.1}$$

$$-\Delta y = \sigma y, \qquad y \in \mathfrak{B}_\epsilon'. \tag{5.2}$$

Let x be an arbitrary eigenfunction associated with an m-fold degenerate eigenvalue ν of (5.1). Let N_ϵ be the linear integral operator whose kernel $N_\epsilon(p, q)$ is the Neumann function (Green's function of the second kind) for the operator $J = -\Delta + 1$ in M_ϵ. Consider the function $f = N_\epsilon x - \rho x$ in M_ϵ, where $\rho = 1/(\nu + 1)$. Then $Jf = 0$ in M_ϵ, $\partial f/\partial n = 0$ on B, and $\partial f/\partial n = -\rho \partial x/\partial n$ on γ. It follows from the standard representation formula for solutions of the differential equation $Jf = 0$ (**3**, p. 160) that

$$f(p) = \rho \int_\gamma N_\epsilon(p, q) \frac{\partial x}{\partial n_q} ds_q. \tag{5.3}$$

The inequality

$$\int_\gamma N_\epsilon(p, q)\, ds_q \leqslant c\epsilon N(p, p_0)$$

is valid, where c is independent of ϵ, and $N(p, p_0)$ is the Neumann function for J in M. Hence (5.3) yields

$$|f(p)| \leqslant c\rho\epsilon[\max_\gamma|\partial x/\partial n|]N(p, p_0).$$

Since $N(p, p_0) \in L^2[M]$, the inequality

$$(5.4) \qquad ||f|| = ||N_\epsilon x - \rho x|| \leqslant c_1 \epsilon\, ||x||$$

is obtained immediately as an analogue of (3.2).

THEOREM 3. *Corresponding to each positive eigenvalue ν of* (5.1), *of multiplicity m, there exist positive numbers ϵ_1 and c such that exactly m eigenvalues σ_i of* (5.2) *are in the interval $[\nu, \nu + c\epsilon]$ provided $0 < \epsilon \leqslant \epsilon_1$. There are orthonormal eigenfunctions x_i associated with ν and corresponding orthonormal y_i associated with σ_i such that*

$$(5.5) \qquad y_i(p) = x_i(p) - h_i(p) + O(\epsilon), \qquad p \in M_\epsilon, \quad 0 < \epsilon \leqslant \epsilon_1, \quad i = 1, 2, \ldots, m,$$

where h_i is the solution of the boundary-value problem

$$(5.6) \qquad Jh = 0 \text{ in } M_\epsilon, \qquad \partial h/\partial n = 0 \text{ on } B, \qquad \partial h/\partial n = \partial x_i/\partial n \text{ on } \gamma.$$

Furthermore, the following asymptotic variation formulae are valid:

$$(5.7) \qquad \sigma_i = \nu - [\bar{x}_i(p_0) + O(\phi)]\int_\gamma \nabla x_i \cdot \boldsymbol{n}\, ds.$$

Proof. The first statement follows from (5.4) similarly to Theorem 1. One obtains also $||x_i - y_i|| = O(\epsilon)$ by the method of Theorem 1.

To prove (5.5), observe first that there is a number c independent of ϵ such that

$$(5.8) \qquad \begin{aligned} |(y_i - (\nu + 1)N_\epsilon x_i)(p)| &= |N_\epsilon\{(\sigma_i + 1)y_i - (\nu + 1)x_i\}(p)| \\ &\leqslant ||N_\epsilon(p, q)||\, [(\sigma_i - \nu)\, ||y_i|| + (\nu + 1)\, ||x_i - y_i||] \\ &\leqslant c\epsilon. \end{aligned}$$

Since the function $f = (\nu + 1)N_\epsilon x_i - x_i + h_i$ is the unique solution of the Neumann problem (**3**, p. 153) $Jf = 0$ in M_ϵ, $\partial f/\partial n = 0$ on $B \cup \gamma$, f is identically zero. The uniform estimates (5.5) then follow from (5.8).

To prove (5.7), we first obtain

$$\sigma_i - \nu = -\int_\gamma \bar{y}_i \nabla x_i \cdot \boldsymbol{n}\, ds[1 + O(\epsilon)]$$

in a standard way from Green's identity (**8**, p. 18). Use of (5.5) then leads to (5.7).

The basic and perturbed problems for L corresponding to boundary conditions 3 are given by (2.1), (2.2) (with $j = 3$). By the methods of section 4 we obtain the following analogue of Theorem 2.

THEOREM 4. *Corresponding to each eigenvalue* λ *of* (2.1) (*with* $j = 3$), *of multiplicity* m, *there exist positive numbers* ϵ_1 *and* c_1 *such that exactly* m *eigenvalues* μ_i *of* (2.2) *lie in the interval* $[\lambda, \lambda + c_1 \epsilon]$ *provided that* $0 < \epsilon \leqslant \epsilon_1$. *The uniform estimates* (5.5) *for eigenfunctions and the asymptotic formulae*

$$\mu_i = \lambda + [-2\sqrt{\lambda}\ \bar{x}_i(p_0) + O(\phi)] \int_\gamma \nabla x_i \cdot \boldsymbol{n}\, ds$$

for eigenvalues are valid, $i = 1, 2, \ldots, m$.

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