COMMUTATOR LENGTH OF ABELIAN-BY-NILPOTENT GROUPS

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Abstract. Let G be a group and $G' = [G, G]$ be its commutator subgroup. Denote by $c(G)$ the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. The exact values of $c(G)$ are computed when G is a free nilpotent group or a free abelian-by-nilpotent group. If G is a free nilpotent group of rank $n \geq 2$ and class $c \geq 2$, $c(G) = [n/2]$ if $c = 2$ and $c(G) = n$ if $c > 2$. If G is a free abelian-by-nilpotent group of rank $n \geq 2$ then $c(G) = n$.

1. Introduction. For an element g in the derived subgroup $G' = [G, G]$ of a group G we write $c(g)$ to denote the least integer such that g can be written as a product of $c(g)$ commutators and we put

$$c(G) = \sup\{c(g); g \in G'\}.$$

Let $F_{n,t} = \langle x_1, \ldots, x_n \rangle$ and $M_{n,t} = \langle x_1, \ldots, x_n \rangle$ be respectively the free nilpotent group of rank n and class t and the free metabelian nilpotent group of rank n and class t. P. W. Stroud, in his Ph.D. thesis [**3**] in 1966, proved that for all t, every element of the commutator subgroup $F'_{n,t}$ can be expressed as a product of n commutators. In 1985 H. Allambergenov and V. A. Romankov [**1**] proved that $c(M_{n,t})$ is precisely n provided $n \geq 2$, $t \geq 4$ or $n \geq 3$, $t \geq 3$. They did this by producing an element d_n in $\gamma_t(M_{n,t})$ that cannot be written as a product of fewer than n commutators. For the case $n = 2$, $c = 3$ they proved that every element of $\gamma_3(M_{2,3})$ is a commutator, and claimed that $c(M_{2,3})$ is one. We will show that the element $[x_1, x_2]^2$ cannot be written as a commutator in the group $M_{2,3} = F_{2,3} = \langle x_1, x_2 \rangle$. This is done in Theorem 1. Thus $c(F_{2,3}) = 2$ and $c(F_{n,t}) = c(M_{n,t}) = n$, for all $n \geq 2$ and $t \geq 3$.

In [**2**] C. Bavard and G. Meigniez considered the same problem for the n-generator free metabelian group M_n. They show that the minimum number $c(M_n)$ of commutators required to express an arbitrary element of the derived subgroup M'_n satisfies the inequality

$$[n/2] \leq c(M_n) \leq n,$$

where $[n/2]$ is the greatest integer part of $n/2$. Since $F_{n,3}$ groups are metabelian, the result of Allambergenov and Romankov [**1**] shows that $c(M_n) \geq n$, for $n \geq 3$, and Theorem 1 of this paper deals with the remaining case $n = 2$. We have $c(M_n) = n$, for all $n \geq 2$. Finally, we extend results in [**1**] and [**2**] to the larger class of abelian-by-nilpotent groups and show in Theorem 2 that $c(G) = n$ if G is a (non-abelian) free abelian-by-nilpotent group of rank n.

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The main results of this paper are as follows.

THEOREM 1. *Let $F_{2,3} = \langle x_1, x_2 \rangle$ be the free nilpotent group of class 3 on free generators x_1, x_2. Then $c(F_{2,3}) = 2$.*

THEOREM 2. *Let $G = \langle x_1, \ldots, x_n \rangle$ be a non-abelian free abelian-by-nilpotent group freely generated by $x_1, \ldots, x_n$. Then $c(G) = n$. If A is an abelian normal subgroup of G and G/A is nilpotent, then every element of g' is a product of n commutators $[x_1, g_1]^{a_1}[x_2, g_2]^{a_2} \ldots [x_n, g_n]^{a_n}$, for suitable $g_1, \ldots, g_n$ in G and $a_1, \ldots, a_n$ in A.*

If $F_{n,2} = \langle x_1, \ldots, x_n \rangle$ is the free nilpotent group of class two, then $c(F_{n,2}) = n/2$, if n is even and $(n-1)/2$, if n is odd. This result appears in [**1**].

We know of no example of a finite group G of rank n where $c(G) > n$. Nor do we know of any example of a solvable group G of rank n where $c(G) > n$.

4. Proofs. We begin by establishing a technical result required in the proof of Theorem 1.

LEMMA 1. *The following system of three equations in variables $s_1, s_2, r_1, r_2, \alpha$ and β has no integer solution:*

$$r_2 s_1 - r_1 s_2 = 2, \tag{1}$$

$$\frac{s_1 r_2(r_2 - 1)}{2} - \frac{r_1 s_2(s_2 - 1)}{2} + r_2 s_2(s_1 - r_1) - \alpha r_2 + \beta s_2 = 0, \tag{2}$$

$$\frac{r_2 s_2(s_1 - 1)}{2} - \frac{r_1 s_2(r_1 - 1)}{2} - \alpha r_1 + \beta s_1 = 0. \tag{3}$$

Proof. Put $c_1 = \alpha r_2 - \beta s_2$, $c_2 = \alpha r_1 - \beta s_1$. Then

$$\alpha = \frac{\begin{vmatrix} c_1 & -s_2 \\ c_2 & -s_1 \end{vmatrix}}{\begin{vmatrix} r_2 & -s_2 \\ r_1 & -s_1 \end{vmatrix}} = \frac{s_1 c_1 - s_2 c_2}{2}, \qquad \beta = \frac{\begin{vmatrix} r_2 & c_1 \\ r_1 & c_2 \end{vmatrix}}{-2} = \frac{r_1 c_1 - r_2 c_2}{2}.$$

Hence we need $s_1 c_1 - s_2 c_2$ and $r_1 c_1 - r_2 c_2$ to be even:

$$c_1 = \frac{s_1 r_2(r_2 - 1)}{2} - \frac{r_1 s_2(s_2 - 1)}{2} + r_2 s_2(s_1 - r_1),$$

$$c_2 = \frac{r_2 s_1(s_1 - 1)}{2} - \frac{r_1 s_2(r_1 - 1)}{2}.$$

Hence we have

$$\begin{aligned} 2c_1 &= s_1 r_2^2 - s_1 r_2 - r_1 s_2^2 + r_1 s_2 + 2 s_1 r_2 s_2 - 2 r_1 r_2 s_2, \\ &= r_2(s_1 r_2 - r_1 s_2) - (s_1 r_2 - r_1 s_2) - s_2(r_1 s_2 - s_1 r_2) - r_1 r_2 s_2 + s_1 r_2 s_2, \\ &= -2 + 2(r_2 + s_2) - r_2 s_2(r_1 - s_1). \end{aligned}$$

Also we have

$$2c_2 = s_1^2 r_2 - s_1 r_2 - r_1^2 s_2 + r_1 s_2 = -2 + s_1^2 r_2 - r_1^2 s_2.$$

Hence we need to satisfy the following conditions:

$$r_2 s_1 - r_1 s_2 = 2, \tag{1}$$

$$2c_1 = -2 + 2(r_2 + s_2) - r_2 s_2 (r_1 - s_1), \tag{4}$$

$$2c_2 = -2 + s_1^2 r_2 - r_1^2 s_2, \tag{5}$$

$$s_1 c_1 + s_2 c_2 \equiv r_1 c_1 + r_2 c_2 \equiv 0 \pmod 2. \tag{6}$$

Case 1. $r_1 s_2 = 2k$, for some integer k. Then

$$c_1 = -1 + (r_2 + s_2) - k r_2 + (1 + k) s_2$$
$$c_2 = -1 + (1 + k) s_1 - k r_1.$$

Further

$$0 \equiv s_1 c_1 + s_2 c_2 \equiv s_1 + s_1 s_2 + s_2 \pmod 2 \tag{7}$$

and

$$0 \equiv r_1 c_1 + r_2 c_2 \equiv r_1 + r_1 r_2 + r_2 \pmod 2. \tag{8}$$

From (7) and (8), it follows that r_1, r_2, s_1 and s_2 are all even. But then $r_1 s_2 - r_2 s_1$ is divisible by 4, contradicting (1).

Case 2. $r_1 s_2$ is odd. It follows from (1) that r_1, r_2, s_1, s_2 are all odd.

$$\begin{aligned} s_1 c_1 + s_2 c_2 &= \frac{1}{2} s_1^2 r_2 (r_2 - 1) - \frac{1}{2} s_1 r_1 s_2 (s_2 - 1) + s_1 s_2 r_2 (s_1 - r_1) \\ &\quad + \frac{1}{2} s_2 r_2 s_1 (s_1 - 1) - \frac{1}{2} s_2^2 r_1 (r_1 - 1) \\ &= \frac{1}{2} (s_1 r_2 - s_2 r_1)(s_1 r_2 + s_2 r_1) \\ &\quad - \frac{1}{2} s_1 (s_1 r_2 - r_1 s_2) - \frac{1}{2} s_1 s_2 (r_1 s_2 - r_2 s_1) \\ &\quad - \frac{1}{2} s_2 (r_2 s_1 - s_2 r_1) + s_1 s_2 r_2 (s_1 - r_1) \\ &= (s_1 r_2 + s_2 r_1) - s_1 + s_1 s_2 - s_2 + s_1 s_2 r_2 (s_1 - r_1), \end{aligned}$$

which is odd and contradicts (7).

We shall use the following well known identities for groups which are nilpotent of class 3.

LEMMA 2. *Let $G = \langle x, y \rangle$ be nilpotent of class 3. Then, for all integers r,s the following hold*:

$$[x^r, y] = [x, y]^r [x, y, x]^{r(r-1)/2},$$
$$[x^r, y^s] = [x, y]^{rs} [x, y, x]^{rs(r-1)/2} [x, y, y]^{rs(s-1)/2}.$$

Proofs of Theorem 1. Let h,g be any two elements of $F_{2,3} \setminus \gamma_3(F_{2,3})$. We study the form of the element $[h, g]$. Since $\gamma_3(F_{2,3})$ lies in the center of $F_{2,3}$ we may express h as

$x_1^{r_1}x_2^{r_2}[x_2,x_1]^\beta$ and g as $x_1^{s_1}x_2^{s_2}[x_2,x_1]^\alpha$. Put $z=[x_2,x_1]$, $y_1=z^\beta$ and $y_2=z^\alpha$. Then

$$\begin{aligned}[h,g] &= [x_1^{r_1}x_2^{r_2}, x_1^{s_1}x_2^{s_2}][x_1^{r_1}x_2^{r_2}, y_2][y_1, x_1^{s_1}x_2^{s_2}] \\ &= [x_1^{r_1}x_2^{r_2}, x_2^{s_2}][x_1^{r_1}x_2^{r_2}, x_1^{s_1}][x_1^{r_1}x_2^{r_2}, x_1^{s_1}, x_2^{s_2}][x_1^{r_1}, y_2][x_2^{r_2}, y_2][y_1, x_2^{s_2}][y_1, x_1^{s_1}] \\ &= [x_1^{r_1}, x_2^{s_2}][x_1^{r_1}, x_2^{s_2}, x_2^{r_2}][x_2^{r_2}, x_1^{s_1}][x_2^{r_2}, x_1^{s_1}, x_2^{s_2}][x_1, z]^{\alpha r_1} \\ &\quad \times [x_2, z]^{\alpha r_2}[z, x_2]^{\beta s_2}[z, x_1]^{\beta s_1} \\ &= [x_1, x_2]^{r_1 s_2}[x_1, x_2, x_1]^{s_2 r_1(r_1-1)/2}[x_1, x_2, x_2]^{r_1 s_2(s_2-1)/2}[x_1, x_2, x_2]^{r_1 r_2 s_2} \\ &\quad \times [x_2, x_1]^{r_2 s_1}[x_2, x_1, x_2]^{s_1 r_2(r_2-1)/2}[x_1, x_1, x_1]^{r_2 s_1(s_1-1)/2}[x_2, x_1, x_2]^{r_2 s_1 s_2} \\ &\quad \times [x_2, x_1, x_1]^{-\alpha r_1+\beta s_1}[x_1, x_1, x_2]^{-\alpha r_2+\beta s_2} \\ &= [x_2, x_1]^\lambda [x_2, x_1, x_2]^\mu [x_2, x_1, x_1]^\nu,\end{aligned}$$

where

$$\begin{aligned}\lambda &= r_2 s_1 - r_1 s_2, \\ \mu &= \frac{s_1 r_2(r_2-1)}{2} - \frac{r_1 s_2(s_2-1)}{2} - r_1 r_2 s_2 + r_2 s_1 s_2 + \beta s_2 - \alpha r_2, \\ \nu &= \frac{r_2 s_1(s_1-1)}{2} - \frac{s_2 r_1(r_1-1)}{2} + \beta s_1 - \alpha r_1.\end{aligned}$$

Since $[x_2,x_1]$, $[x_2,x_1,x_2]$ and $[x_2,x_1,x_1]$ are the basic commutators and the group under consideration is the free nilpotent class 3 group, it follows that if $[h,g]=[x_2,x_1]^2$ then $\lambda=2$, $\mu=\nu=0$. But, by Lemma 1, there are no integers α, β, r_1, s_1, r_2, s_2 for this set of equations to hold and we conclude that $c(F_{2,3})\geq 2$. Since $c(F_{2,3})\leq 2$ by [**1**] or [**2**], we obtain the equality.

The proof of Theorem 2 makes use of the following two results. The first is elementary; the second is a result of Peter Stroud [**4**]. We shall include the proofs since Stroud's result never got published, except in his Ph.D. thesis, due to his untimely death. In the case of a finite group G, Brian Hartley [**3**] has given a bound for $c(G)$ in terms of the Fitting length of G. His proof incorporates Stroud's proof given below.

LEMMA 3. *Let A be a normal subgroup of $G=\langle x_1,\ldots,x_n\rangle$. If A is abelian or A lies in the second center $\zeta_2(G)$ of G, then every element of $[G,A]$ has the form $\prod_{i=1}^{n}[x_i,a_i]$, where $a_i\in A$.*

Proof. Consider $[g,d]$, where $d\in A$ and $g=x_{i_1}^{\varepsilon_1}\ldots x_{i_r}^{\varepsilon_r}$ where $\varepsilon_i\in\{1,-1\}$. Write $x_{i_1}=x$, $\varepsilon_1=\varepsilon$ and $g=x^\varepsilon y$. Then $[g,d]=[x^\varepsilon y,d]=[x,d^\varepsilon][y,d]$, if $A\leq\zeta_2(G)$, and $[g,d]=[x^\varepsilon,d][x^\varepsilon,d,y][y,d]=[x^\varepsilon,d][y,d[d,x^\varepsilon]]$, if A is abelian. If $\varepsilon=-1$, then use $[x^{-1},d]=[x,xd^{-1}x^{-1}]$.

Iterate the process r times to obtain $[g,d]=\prod_{j=1}^{r}[x_{ij},d_j]$ with $d_j\in A$. Finally use the identity $[x,d_1][x,d_2]=[x,d_1d_2]$ to see that every $\Pi[g_i,d_i]$, $d_i\in A$ has the form $\prod_{i=1}^{n}[x_i,a_i]$, where $a_i\in A$.

LEMMA (P. Stroud). *Let $G=\langle x_1,\ldots,x_n\rangle$ be a nilpotent group. Then every element of the commutator subgroup G' is a product of n commutators $[x_1,g_1]\ldots[x_n,g_n]$, for suitable g_i in G.*

Proof. Use induction on the nilpotency class of G. If G is abelian, then $G' = 1$ and the result is clear. Next let $G \in \mathcal{N}_{r+1}$, nilpotent of class $r+1$, and assume the result for groups in the class $\mathcal{N}_r$. Let $\Gamma = \gamma_{r+1}(G) = [\gamma_r(G), G]$. Then an element g of G' has the form $g = [x_1, h_1] \ldots [x_n, h_n]d$, for some $d \in \Gamma$. By Lemma 3, we have

$$g = [x_1, h_1] \ldots [x_n, h_n][x_1, a_1] \ldots [x_n, a_n] = \prod_{i=1}^{n} [x_i, h_1 a_i].$$

Proof of Theorem 2. By hypothesis, there exists a normal abelian subgroup A of G such that G/A is nilpotent. By Lemma 3, $[A, G] = \{[x_1, a_1] \ldots [x_n, a_n];\ a_i \in A\}$ and, since $G/[A, G]$ is nilpotent, using Stroud's result, every element $g \in G'$ has the form

$$g = \left(\prod_{i=1}^{n} [x_i, g_i]\right)\left(\prod_{i=1}^{n} [x_i, a_i]\right), \text{ with } a_i \in A,$$

$$= \prod_{i=1}^{n} ([x_i, a_i][x_i, g_i]^{d_i}), \text{ for suitable } d_i \in A.$$

Now $[x_i, g_i a_i] = [x_i, a_i][x_i, g_i]^{a_i} = ([x_i, a_i][x_i, g_i])^{a_i}$.

Thus $[x_i, a_i][x_i, g_i] = [x_i, g_i a_i]^{a_i^{-1}}$ and $g = \prod_{i=1}^{n} [x_i, g_i a_i]^{d_i a_i^{-1}}$, with $d_i a_i^{-1} \in A$. Thus $c(G) \leq n$ and every element of G' has the required form. Since G is free abelian-by-nilpotent and non-abelian, the free metabelian group on n-generators is a quotient of G and hence so is the free nilpotent-class-three group on n generators. By Theorem 1 for the case when $n = 2$ and by [**1**] for $n > 2$ we have $c(G) \geq n$. This shows that $c(G) = n$ and completes the proof.

REFERENCES

1. Kh. S. Allambergenov and V. A. Romankov, Product of commutators in groups, *Dokl. Akad. Nauk. UzSSR* (1984), No. 4, 14–15 (Russian).

2. C. Bavard and G. Meigniez, Commutateurs dans les groupes métabéliens, *Indag. Math. N.S.* **3**(2) (1992), 129–135.

3. B. Hartley, Subgroups of finite index in profinite groups, *Math Z.* **168** (1979), 71–76.

4. P. Stroud, *Ph.D. Thesis* (Cambridge, 1966).

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