EXISTENCE AND UNIQUENESS IN THE THEORY OF BENDING OF ELASTIC PLATES

by CHRISTIAN CONSTANDA

1. Introduction

Kirchhoff's kinematic hypothesis that leads to an approximate two-dimensional theory of bending of elastic plates consists in assuming that the displacements have the form [1]

$$u_\alpha = x_3 v_\alpha(x_1, x_2), \quad \alpha = 1, 2,$$

$$u_3 = v_3(x_1, x_2). \tag{1.1}$$

In general, the Dirichlet and Neumann problems for the equilibrium equations obtained on the basis of (1.1) cannot be solved by the boundary integral equation method both inside and outside a bounded domain because the corresponding matrix of fundamental solutions does not vanish at infinity [2]. However, as we show in this paper, the method is still applicable if the asymptotic behaviour of the solution is suitably restricted.

2. Notation and formulae

Let (x_1, x_2, x_3) be the Cartesian coordinates of a generic point in R^3. We consider a homogeneous and isotropic plate occupying a region $\Omega \times [-h_0/2, h_0/2]$, where $h_0 = \text{const}$ and Ω is a domain in the (x_1, x_2)-plane, whose boundary $\partial\Omega$ is a closed Lyapunov curve.

Unless otherwise stipulated, throughout the paper Latin and Greek suffixes take the values $1, 2, 3$ and $1, 2$, respectively, the convention of summation over repeated indices is adopted, and $(\ldots)_{,\alpha} = \partial(\ldots)/\partial x_\alpha$.

Also, we denote by M^* the transpose of a matrix M, and by H_k and J_k $(k = 0, 1)$ the averaging operators

$$H_k w(x_i) = (1/h_0) [x_3^k w(x_i)]_{x_3 = -h_0/2}^{x_3 = h_0/2},$$

$$J_k w(x_i) = (1/h_0) \int_{-h_0/2}^{h_0/2} x_3^k w(x_i)\, dx_3.$$

Let t_{ij} be the stresses, u_i the displacements, f_i the body forces, and λ and μ the elastic coefficients of the material. If we set

$$F_\alpha = -(J_1 f_\alpha + H_1 t_{3\alpha}), \quad F_3 = -(J_0 f_3 + H_0 t_{33}),$$

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then the linear equilibrium equations obtained by using (1.1) can be written in the form

$$L(\partial_x)v(x) = F(x), \tag{2.1}$$

where $L(\partial_x) = L(\partial/\partial x_1, \partial/\partial x_2) = (L_{ij}(\partial_x))$,

$$L(\xi_1, \xi_2) = \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix}, \tag{2.2}$$

$v^* = (v_1, v_2, v_3)$, $F^* = (F_1, F_2, F_3)$, $\Delta = \xi_\alpha\xi_\alpha$, and $h^2 = h_0^2/12$.

In connection with the system (2.1) we consider the boundary stress operator $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2) = (T_{ij}(\partial_x))$, where

$$T(\xi_1, \xi_2) = \begin{pmatrix} h^2(\lambda+2\mu)n_1\xi_1 + h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1 + h^2\lambda n_1\xi_2 & 0 \\ h^2\lambda n_2\xi_1 + h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1 + h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu(n_1\xi_1 + n_2\xi_2) \end{pmatrix} \tag{2.3}$$

and n_α are the components of the unit outward normal to $\partial\Omega$.

In what follows we assume that λ and μ satisfy the inequalities

$$\lambda + \mu > 0, \quad \mu > 0,$$

which ensure that the system (2.1) is elliptic and that the internal energy of the plate (measured on the unit of its middle plane) $h_0E(v, v)$, where

$$2E(v, v) = h^2[\lambda v_{\alpha,\alpha}v_{\beta,\beta} + \mu(v_{\alpha,\beta} + v_{\beta,\alpha})v_{\alpha,\beta}] + \mu(v_{3,\alpha} + v_\alpha)(v_{3,\alpha} + v_\alpha), \tag{2.4}$$

is positive.

From (2.1)–(2.4) we easily obtain the Betti formula

$$\int_\Omega v^* Lv\, d\sigma + \int_\Omega 2E(v, v)\, d\sigma = \int_{\partial\Omega} v^* Tv\, ds \tag{2.5}$$

and the relations

$$\int_\Omega L_{3i}v_i\, d\sigma = \int_{\partial\Omega} T_{3i}v_i\, ds,$$

$$\int_\Omega L_{\alpha i} - x_\alpha L_{3i})v_i\, d\sigma = \int_{\partial\Omega} (T_{\alpha i} - x_\alpha T_{3i})v_i\, ds, \tag{2.6}$$

where v is a smooth (3×1)-matrix.

3. Fundamental solutions

The matrix of fundamental solutions for the system (2.1) is [2]

$$D(x, y) = B(\partial_x)t(x, y), \tag{3.1}$$

where

$$t(x, y) = t(r) = a[r^2 \ln r + 4h^2 \ln r + 4h^2 K_0(r/h)],$$

$$a = [8\pi h^2 \mu^2 (\lambda + 2\mu)]^{-1}, \tag{3.2}$$

$$r = |x - y| = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2},$$

$K_0(\xi)$ is the modified Bessel function of order zero, and the entries of the matrix $B(\partial_x) = (B_{ij}(\partial/\partial x_1, \partial/\partial x_2))$ are

$$B_{\alpha\alpha}(\xi) = h^2 \mu(\lambda + 2\mu)\Delta^2 - h^2 \mu(\lambda + \mu)\Delta\xi_\alpha^2 - \mu^2\xi_\alpha^2, \quad (\alpha \text{ not summed})$$

$$B_{33}(\xi) = h^4 \mu(\lambda + 2\mu)\Delta^2 - h^2 \mu(\lambda + 3\mu)\Delta + \mu^2,$$

$$\tag{3.3}$$

$$B_{12}(\xi) = B_{21}(\xi) = -\mu\xi_1\xi_2[h^2(\lambda + \mu)\Delta + \mu],$$

$$B_{\alpha 3}(\xi) = -B_{3\alpha}(\xi) = \mu^2\xi_\alpha(h^2\Delta - 1).$$

We also consider the matrix $P(x, y) = (P_{ij}(x, y))$, where

$$P_{ij}(x, y) = (T(\partial_y)D(y, x))_{ji}. \tag{3.4}$$

We denote Ω by Ω_{in} and set $\Omega_{ex} = R^2 \backslash \{\Omega_{in} \cup \partial\Omega\}$. As in the classical theory (see, for example, [3]), we can derive the relations

$$\int_{\partial\Omega} T_{ik}(\partial_x)D_{kj}(x, y)\,ds_x = \phi(y)\delta_{ij},$$

$$\int_{\partial\Omega} x_\alpha T_{3k}(\partial_x)D_{kj}(x, y)\,ds_x = y_\alpha\phi(y)\delta_{3j}, \tag{3.5}$$

where

$$\phi(y) = \begin{cases} -1, & y \in \Omega_{in}, \\ -1/2, & y \in \partial\Omega, \\ 0, & y \in \Omega_{ex} \end{cases}$$

and δ_{ij} are the Kronecker delta, and the Somigliana formula

$$v(x) = \int_{\partial\Omega} [D(x, y)T(\partial_y)v(y) - P(x, y)v(y)]\,ds_y$$

$$- \int_{\Omega_{in}} D(x, y)L(\partial_y)v(y)\,ds_y, \quad x \in \Omega_{in}, \tag{3.6}$$

where v is a smooth (3×1)-matrix.

Let $\mathscr{A}$ be the set of (3×1)-matrices v in Ω_{ex} having an asymptotic expansion of the

form

$$v_1(r,\theta) = \frac{1}{r}[\alpha_0 \sin\theta + 2\alpha_1 \cos\theta - \alpha_0 \sin 3\theta + (\alpha_2 - \alpha_1)\cos 3\theta]$$

$$+ \frac{1}{r^2}[(2\beta_1 + \gamma_1)\sin 2\theta + \gamma_2 \cos 2\theta - 2\beta_1 \sin 4\theta + 2\beta_2 \cos 4\theta]$$

$$+ \frac{1}{r^3}[2\rho_1 \sin 3\theta + 2\pi_1 \cos 3\theta + 3(\rho_2 - \rho_1)\sin 5\theta + 3(\pi_2 - \pi_1)\cos 5\theta] + O\left(\frac{1}{r^4}\right),$$

$$v_2(r,\theta) = \frac{1}{r}[2\alpha_2 \sin\theta + \alpha_0 \cos\theta + (\alpha_2 - \alpha_1)\sin 3\theta + \alpha_0 \cos 3\theta]$$

$$+ \frac{1}{r^2}[(2\beta_2 + \gamma_2)\sin 2\theta - \gamma_1 \cos 2\theta + 2\beta_2 \sin 4\theta + 2\beta_1 \cos 4\theta]$$

$$+ \frac{1}{r^3}[2\pi_2 \sin 3\theta - 2\rho_2 \cos 3\theta + 3(\pi_2 - \pi_1)\sin 5\theta + 3(\rho_1 - \rho_2)\cos 5\theta] + O\left(\frac{1}{r^4}\right),$$

$$v_3(r,\theta) = -(\alpha_1 + \alpha_2)\ln r - [\alpha_1 + \alpha_2 + \alpha_0 \sin 2\theta + (\alpha_1 - \alpha_2)\cos 2\theta]$$

$$+ \frac{1}{r}[(\beta_1 + \gamma_1)\sin\theta + (\beta_2 + \gamma_2)\cos\theta - \beta_1 \sin 3\theta + \beta_2 \cos 3\theta]$$

$$+ \frac{1}{r^2}[\sigma_1 \sin 2\theta + \sigma_2 \cos 2\theta + (\rho_2 - \rho_1)\sin 4\theta + (\pi_2 - \pi_1)\cos 4\theta] + O\left(\frac{1}{r^3}\right),$$

where $\alpha_0,\ \alpha_v,\ \beta_v,\ \gamma_v,\ \rho_v,\ \pi_v,\ \sigma_v = $ const, and $\mathscr{A}'$ the set of matrices $v' = v + C$, where $v \in \mathscr{A}$ and

$$C^* = (c_1, c_2, c - c_1 x_1 - c_2 x_2), \quad c, c_\alpha = \text{const.} \tag{3.7}$$

Remark. In view of (1.1), the matrix (3.7) represents a rigid displacement, and it is obvious that for $v \in \mathscr{A}'$ the internal energy density (2.6) remains finite as $r \to \infty$.

4. Elastic potentials

We define the single layer potential

$$V(x) = \int_{\partial\Omega} D(x,y)z(y)\,ds_y$$

and the double layer potential

$$W(x) = \int_{\partial\Omega} P(x,y)z(y)\,ds_y,$$

where the (3×1)-matrix z is an unknown density. Proceeding as in [4], we can prove the following assertions:

Theorem 1. $L(\partial_x)V(x)=0$, $L(\partial_x)W(x)=0$, $x \notin \partial\Omega$.

Theorem 2.   (i) *If $z(x)$ is continuous on $\partial\Omega$, then $V(x)$ is continuous in R^2.*

(ii) *If $z(x)$ is Hölder continuous on $\partial\Omega$, then $W(x)$ tends to finite limits as $x \to x_0 \in \partial\Omega$ both from Ω_{in} and from Ω_{ex}, these limits being*

$$W_{in}(x_0) = -\frac{1}{2}z(x_0) + \int_{\partial\Omega} P(x_0,y)z(y)\,ds_y,$$

$$W_{ex}(x_0) = \frac{1}{2}z(x_0) + \int_{\partial\Omega} P(x_0,y)z(y)\,ds_y.$$

(iii) *If $z(x)$ is Hölder continuous on $\partial\Omega$, then $T(\partial_x)V(x)$ tends to finite limits as $x \to x_0 \in \partial\Omega$ both from Ω_{in} and from Ω_{ex}, these limits being*

$$(TV)_{in}(x_0) = \frac{1}{2}z(x_0) + \int_{\partial\Omega} T(\partial_{x_0})D(x_0,y)z(y)\,ds_y,$$

$$(TV)_{ex}(x_0) = -\frac{1}{2}z(x_0) + \int_{\partial\Omega} T(\partial_{x_0})D(x_0,y)z(y)\,ds_y. \tag{4.1}$$

Also, we have [2]

Theorem 3.   *If $z(x)$ is continuous on $\partial\Omega$, then*

(i) $W \in \mathscr{A}$;

(ii) $V \in \mathscr{A}$ *if and only if*

$$\int_{\partial\Omega} z_3\,ds = \int_{\partial\Omega} (z_\alpha - x_\alpha z_3)\,ds = 0. \tag{4.2}$$

5. Boundary value problems

Let $A(x)$, $B(x)$, $R(x)$, and $S(x)$ be (3×1)-matrices prescribed on $\partial\Omega$. For the homogeneous system (2.1), that is,

$$L(\partial_x)v(x)=0, \tag{5.1}$$

we consider the following Dirichlet and Neumann problems:

(I) Find v satisfying (5.1) in Ω_{in} and

$$v(x)=A(x), \quad x \in \partial\Omega. \tag{5.2}$$

(II) Find v satisfying (5.1) in Ω_{in} and

$$T(\partial_x)v(x) = B(x), \quad x \in \partial\Omega. \tag{5.3}$$

(III) Find $v \in \mathscr{A}'$ satisfying (5.1) in Ω_{ex} and

$$v(x) = R(x), \quad x \in \partial\Omega. \tag{5.4}$$

(IV) Find $v \in \mathscr{A}$ satisfying (5.1) in Ω_{ex} and

$$T(\partial_x)v(x) = S(x), \quad x \in \partial\Omega. \tag{5.5}$$

Theorem 4. (i) *The interior Dirichlet problem* (I) *has a unique solution for any Hölder continuous matrix* $A(x)$. *This solution can be expressed as a double layer potential.*

(ii) *The exterior Neumann problem* (IV) *has a unique solution for any Hölder continuous matrix* $S(x)$ *if and only if*

$$\int_{\partial\Omega} S_3 \, ds = \int_{\partial\Omega} (S_\alpha - x_\alpha S_3) \, ds = 0. \tag{5.6}$$

The solution can be expressed as a single layer potential.

Proof. First, suppose that $v \in \mathscr{A}$ is a solution of (IV). Let K_R be a circle whose radius R is sufficiently large so that $(\Omega \cup \partial\Omega) \subset K_R$. Applying (2.6) to v in $\Omega_{\text{ex}} \cap K_R$, we arrive at the equalities

$$\int_{\partial\Omega} S_3 \, ds = \int_{\partial\Omega} T_{3i}v_i \, ds = -\int_{\Omega_{\text{ex}} \cap K_R} L_{3i}v_i \, d\sigma + \int_{\partial K_R} T_{3i}v_i \, ds = \int_{\partial K_R} T_{3i}v_i \, ds,$$

$$\int_{\partial\Omega} (S_\alpha - x_\alpha S_3) \, ds = \int_{\partial\Omega} (T_{\alpha i} - x_\alpha T_{3i})v_i \, ds$$

$$= -\int_{\Omega_{\text{ex}} \cap K_R} (L_{\alpha i} - x_\alpha L_{3i})v_i \, d\sigma + \int_{\partial K_R} (T_{\alpha i} - x_\alpha T_{3i})v_i \, ds.$$

Since $T_{\alpha i}v_i = O(1/R^2)$ and $T_{3i}v_i = O(1/R^3)$ for $v \in \mathscr{A}$ [2], the relations (5.6) are obtained immediately by letting $R \to \infty$.

Suppose now that (5.6) hold. Seeking the solutions of (I) and (IV) in the form of W and V, respectively, we interpret (5.2) and (5.5) as

$$W_{\text{in}}(x) = A(x), \quad x \in \partial\Omega,$$

$$(TV)_{\text{ex}}(x) = S(x), \quad x \in \partial\Omega,$$

and from Theorem 2(ii, iii) it follows that the unknown density $z(x)$ must satisfy the

integral equation

$$-\frac{1}{2}z(x)+\int_{\partial\Omega}P(x,y)z(y)\,ds_y=A(x),\quad x\in\partial\Omega \qquad (D_{in})$$

(in the case of (I)), and

$$-\frac{1}{2}z(x)+\int_{\partial\Omega}T(\partial_x)D(x,y)z(y)\,ds_y=S(x),\quad x\in\partial\Omega \qquad (N_{ex})$$

(in the case of (IV)).

In view of (3.4), (D_{in}) and (N_{ex}) are mutually adjoint, and using (3.1)–(3.4) it is not difficult to show that their index [5] is zero, which implies that Fredholm's theorems are applicable to these equations.

Multiplying $(N_{ex})_3$ and the combination $(N_{ex})_\alpha-x_\alpha\times(N_{ex})_3$ by ds_x, integrating over $\partial\Omega$, and taking (3.5) into account, we obtain

$$\int_{\partial\Omega}z_3\,ds=-\int_{\partial\Omega}S_3\,ds,\quad \int_{\partial\Omega}(z_\alpha-x_\alpha z_3)\,ds=-\int_{\partial\Omega}(S_\alpha-x_\alpha S_3)\,ds. \qquad (5.7)$$

Suppose the existence of a non-trivial solution $z^0(x)$ to the homogeneous equation (N_{ex}^0), that is,

$$-\frac{1}{2}z^0(x)+\int_{\partial\Omega}T(\partial_x)D(x,y)z^0(y)\,ds_y=0,\quad x\in\partial\Omega.$$

Then, by Theorem 2(iii), the single layer potential

$$V^0(x)=\int_{\partial\Omega}D(x,y)z^0(y)\,ds_y$$

satisfies

$$(TV^0)_{ex}(x)=-\frac{1}{2}z^0(x)+\int_{\partial\Omega}T(\partial_x)D(x,y)z^0(y)\,ds_y=0,\quad x\in\partial\Omega,$$

and from (5.7) we find that

$$\int_{\partial\Omega}z_3^0\,ds=\int_{\partial\Omega}(z_\alpha^0-x_\alpha z_3^0)\,ds=0.$$

By Theorem 3(ii), $V^0\in\mathscr{A}$. Since, according to Theorems 1 and 2(iii), we also have

$$L(\partial_x)V^0(x)=0,\quad x\in\Omega_{ex},$$

$$(TV^0)_{ex}(x)=0,\quad x\in\partial\Omega,$$

the uniqueness theorem for (IV) [2] implies that $V^0(x)\equiv0,\ x\in\Omega_{ex}$. Theorem 2(i) now

yields $V^0(x) = 0$, $x \in \partial\Omega$, and since $L(\partial_x)V^0(x) = 0$, $x \in \Omega_{\text{in}}$, from the uniqueness theorem for (I) we conclude that $V^0(x) \equiv 0$, $x \in R^2$. Finally, from (4.1) we obtain

$$z^0(x) = (TV^0)_{\text{in}}(x) - (TV^0)_{\text{ex}}(x) = 0, \quad x \in \partial\Omega.$$

This shows that (N_{ex}^0) has only the trivial solution. Hence, so does the homogeneous equation (D_{in}^0), which means that both (D_{in}) and (N_{ex}) are uniquely solvable. To complete the proof we only need to check that in the case of (IV) the solution V belongs to $\mathscr{A}$. But this is established immediately, since the relations (4.2) hold for the density $z(x)$ of V by virtue of (5.6) and (5.7).

Theorem 5. *The interior Neumann problem* (II) *is solvable for any Hölder continuous matrix* $B(x)$ *if and only if*

$$\int_{\partial\Omega} B_3 \, ds = \int_{\partial\Omega} (B_\alpha - x_\alpha B_3) \, ds = 0. \tag{5.8}$$

The solution can be expressed as a single layer potential and is unique up to a rigid displacement (3.7).

Proof. We seek the solution in the form of V and interpret (5.3) as

$$(TV)_{\text{in}}(x) = B(x), \quad x \in \partial\Omega.$$

By Theorem 2(iii), this means that $z(x)$ must satisfy the integral equation

$$\frac{1}{2}z(x) + \int_{\partial\Omega} T(\partial_x)D(x, y)z(y) \, ds_y = B(x), \quad x \in \partial\Omega, \tag{N_{in}}$$

for which we can again verify that the index is zero.

It is easy to check that a matrix C of the form (3.7) satisfies

$$L(\partial_x)C(x) = 0, \quad x \in \Omega_{\text{in}},$$

$$T(\partial_x)C(x) = 0, \quad x \in \partial\Omega.$$

Therefore, by (3.6), we have the representation

$$C(x) = - \int_{\partial\Omega} P(x, y)C(y) \, ds_y, \quad x \in \Omega_{\text{in}}.$$

Since the right-hand side above is a double layer potential, letting $x \to x_0 \in \partial\Omega$ and making use of Theorem 2(ii) we obtain

$$\frac{1}{2}C(x_0) + \int_{\partial\Omega} P(x_0, y)C(y) \, ds_y = 0, \tag{5.9}$$

which means that C is a solution of the adjoint homogeneous equation. Clearly, $v^{(i)}$, where

$$v^{(1)*} = (1, 0, -x_1), \quad v^{(2)*} = (0, 1, -x_2), \quad v^{(3)*} = (0, 0, 1),$$

are linearly independent solutions of the latter, so the homogeneous equation (N_{in}^0) has at least three linearly independent solutions $\mu^{(i)}$. Just as in [4], we can prove that $\{\mu^{(i)}\}$ is a complete system of such solutions for (N_{in}^0), hence, so is $\{v^{(i)}\}$ for (5.9). This implies that (N_{in}) is solvable if and only if

$$\int_{\partial\Omega} v^{(i)*} B \, ds = 0,$$

which is equivalent to (5.8).

The last part of the statement of the theorem is obvious.

Remark. Taking into account (1.1) and the averaging procedure employed to derive the simplified two-dimensional bending equations, it can be shown that in the case of both Neumann problems (II) and (IV) the conditions (5.6) and (5.8) imply that the total force and moment across $\partial\Omega$ must be zero.

Theorem 6. *The exterior Dirichlet problem* (III) *has a unique solution for any Hölder continuous matrix* $R(x)$.

Proof. We assume that the systems of functions $\{\mu^{(i)}\}$ and $\{v^{(i)}\}$ mentioned in the proof of Theorem 5 have been bi-orthonormalized (by means of a procedure similar to that used in [4]) and seek the solution of (III) in the form $W + C_i v^{(i)}$, where W is a double layer potential of unknown density z and

$$C_i = \int_{\partial\Omega} \mu^{(i)*} R \, ds. \tag{5.10}$$

Then Theorem 2(ii) yields

$$\frac{1}{2} z(x) + \int_{\partial\Omega} P(x, y) z(y) \, ds_y = R(x) - C_i v^{(i)}(x), \quad x \in \partial\Omega. \tag{D_{ex}}$$

In view of (3.4), this equation is adjoint to (N_{in}), hence its index is zero. From the definition of the C_i in (5.10) we see that

$$\int_{\partial\Omega} \mu^{(k)*}(R - C_i v^{(i)}) \, ds = 0,$$

which implies that (D_{ex}) is solvable. Also, by Theorem 3(i), $W \in \mathscr{A}$, and since $C_i v^{(i)}$ is a rigid displacement of the form (3.7), it follows that $W + C_i v^{(i)} \in \mathscr{A}'$, as required.

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UNIVERSITY OF STRATHCLYDE
GLASGOW
SCOTLAND