GENERALIZED EMDEN-FOWLER EQUATIONS
OF SUBCRITICAL GROWTH

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Communicated by E. N. Dancer

Abstract

The existence of positive solutions, vanishing at infinity, for the semilinear eigenvalue problem $Lu = \lambda \cdot f(x, y)$ in $\mathbb{R}^N$ is obtained, where L is a strictly elliptic operator. The function f is assumed to be of subcritical growth with respect to the variable u.

1. Introduction

In a recent paper, Noussair and Swanson [3] study the semilinear problem

$$(1) \qquad Lu = \lambda \cdot f(x, u) \quad \text{in } \mathbb{R}^N \ (N \geq 3),$$

where $Lu = -\sum_{l,m=1}^{N} \partial_m(a_{lm}(x) \cdot \partial_l u)$ is a strictly elliptic operator, meaning that there exists a positive constant a_0 such that

$$\sum_{l,m=1}^{N} a_{lm}(x)\xi_l\xi_m \geq a_0|\xi|^2$$

holds for all $\xi \in \mathbb{R}^N$ and almost all $x \in \mathbb{R}^N$.

The function f is supposed to satisfy

$$(2) \qquad 0 \leq f(x, t) \leq \sum_{i=1}^{I} f_i(x) \cdot t^{\gamma i}$$

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for all $x \in \mathbb{R}^N$ and $t \geq 0$, where the constants γ_i

(3) $(i = 1, \ldots, I)$ satisfy $1 < \gamma_i < \dfrac{N+2}{N-2}$.

Furthermore, the authors of [3] assume

(A1) that the function f is locally Hölder continuous on $\mathbb{R}^N \times [0, \infty[$, and that the functions a_{lm} satisfy $a_{lm} \in C^{1+\alpha}_{\text{loc}}(\mathbb{R}^N)$ and $\|a_{lm}\|_{C^{0,\alpha}\overline{(\mathbb{R}^N)}} < \infty$ for a constant $0 < \alpha < 1$;

(A2) that there exists an open subset $\Omega \neq \varnothing$ of $\mathbb{R}^N$ such that $f(x, t) > 0$ holds on $\Omega \times]0, \infty[$;

(A3) that the functions f_i $(i = 1, \ldots, I)$ are nonnegative, bounded and continuous such that $f_i \in L^{q_i}(\mathbb{R}^N)$ holds for a constant $1 < q_i < \dfrac{2N}{2N-(n-2)\cdot(1+\gamma_i)}$;

(A4) and that $t \cdot f(x, t) \geq C \cdot F(x, t)$ holds for all $x \in \mathbb{R}^N$ and $t \geq 0$, where C is a positive constant and $F(x, t) = \int_0^t f(x, s)ds$.

Then, it is shown (see [3, Theorem 1]) that there is a positive solution pair (λ, u) of equation (1) such that $u \in L^Q(\mathbb{R}^N)$ holds for all $2N/(N-2) \leq Q \leq \infty$, and $|\nabla u(x)|$ as well $u(x)$ has uniform limit zero as $|x| \to \infty$. It should be mentioned that the authors of [3] do not require explicitly that the functions a_{lm} are uniformly Hölder continuous on $\mathbb{R}^N$. But, when they show that $|\nabla u(x)|$ vanishes uniformly at infinity (see [3, p. 58]), they use the estimate (8.86) in [2] which depends on $max\|a_{lm}\|_{C^{0,\alpha}}$; so the proof only works if $\|a_{lm}\|_{C^{0,\alpha}\overline{(\mathbb{R}^n)}} < \infty$.

The aim of this paper is to show that assumption (A4) is superfluous and that the assumptions (A2) and (A3) can be weakened considerably (see Corollary 1). Furthermore, we will prove the existence of a $\lambda > 0$ and a weak solution $u > 0$ of equation (1) if (A1) is replaced by the following condition:

(B1) The function $f: \mathbb{R}^N \times [0, \infty[\mapsto \mathbb{R}_+$ is measurable and, for $x \in \mathbb{R}^N$, the function $t \mapsto f(x, t)$ is continuous on $[0, +\infty[$. Furthermore, the functions a_{lm} satisfy $a_{lm} \in L^\infty(\mathbb{R}^N)$.

Instead of (A2), we will assume:

(B2) There exist an open subset $\Omega \neq \varnothing$ of $\mathbb{R}^N$ and constants $0 \leq \delta_0 < \delta_1 \leq \infty$ such that $f(x, t) > 0$ holds on $\Omega \times]\delta_0, \delta_1[$.

Assumption (A3) will be replaced by

(B3) The functions f_i are nonnegative and for each $i = 1, \ldots, I$ there exists a constant $0 < \varepsilon_i < \infty$ such that $f_i \in L^q(\mathbb{R}^N)$ holds for all $q \in [p_i, p_i + \varepsilon_i]$, where $p_i = \dfrac{2N}{2N-(N-2)\cdot(1+\gamma_i)}$.

Then, we will prove the following

THEOREM 1. *Suppose the operator L is strictly elliptic and the assumption (2), (3) and (B1)–(B3) are fulfilled. Then there exists a constant $\lambda > 0$ and a positive function u such that equation (1) holds in the weak sense. Moreover, the function u is locally Hölder continuous, satisfies $u \in L^Q(\mathbb{R}^N)$ for all $2N/(N-2) \leq Q \leq \infty$, and vanishes uniformly at infinity.*

COROLLARY 1. *Suppose that the assumptions of Theorem 1 are satisfied, that (A1) is fulfilled and $\varepsilon_i > N - p_i$ holds for $i = 1, \ldots, I$. Then, the function u satisfies $u \in C^2(\mathbb{R}^N)$, and the pair (λ, u) solves equation (1) in the classical sense. Moreover, $|\nabla u(x)| \to 0$ uniformly as $|x| \to \infty$.*

Our method of proof is different from that used by Noussair and Swanson. The authors of [3] minimize the functional

$$I_k(u) = \frac{1}{2} \cdot \int_{\mathbb{R}^N} \left(\sum_{l,m=1}^{N} a_{lm} \cdot \partial_l u \cdot \partial_m u + \frac{1}{k} \cdot u^2 \right) dx \qquad (k \in \mathbb{N})$$

subject to the constraints $u \in H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} F(x, u(x))dx = 1$, and show that the infimum is attained by a function $u_k \in H^1(\mathbb{R}^N)$ satisfying

$$\int \left(\sum_{l,m=1}^{N} a_{lm} \cdot \partial_l u_k \cdot \partial_m v + \frac{1}{k} \cdot u_k \cdot v \right) dx = \lambda_k \cdot \int f(x, u_k(x)) \cdot v(x) dx$$

for all $v \in H^1(\mathbb{R}^N)$ and a constant $\lambda_k > 0$.

Then, making essential use of (A4), the authors show that a subsequence of (λ_k, u_k) converges to a solution (λ, u) $(\lambda > 0, u > 0)$ of equation (1).

In contrast to this method, we will maximize the functional

$$J(u) = \int F(x, u(x))dx / (\|u\|_L^2 + \|u\|_L^{2^*})$$

on $\hat{H}^1 = \hat{H}^1(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) | \nabla u \in (L^2(\mathbb{R}^N))^N\}$, where

$$2^* = 2N/(N-2) \quad \text{and} \quad \|u\|_L^2 = \sum_{l,m=1}^{N} \int a_{lm} \cdot \partial_l u \cdot \partial_m u \, dx;$$

so, we need not assume (A4).

To prove that the function u is in Theorem 1 satisfies $u \in L^Q$ for all $2^* \leq Q \leq \infty$, we will use a method of Stampacchia [4] and not the device of Brézis and Kato [1]; so we need not assume that the functions f_i satisfy $f_i \in L^q(\mathbb{R}^N)$ for some $q < p_i$.

2. Preliminaries

By $L^p = L^p(\mathbb{R}^N)$ $(1 \le p \le \infty)$, we denote the usual Lebesgue space and $\| \cdot \|_p$ is the norm on L^p. For $u, v \in \widehat{H}^1$, we define

$$\langle u, v \rangle_L = \sum_{l, m=1}^{N} \int a_{lm}(x) \cdot \partial_l u(x) \cdot \partial_m v(x) dx$$

and $\|u\|_L = \langle u, u \rangle_L^{1/2}$.

Since L is strictly elliptic, we conclude from the Sobolev inequality that there is a constant $C_0 > 0$ such that

(4) $\|u\|_{2^*} \le C_0 \cdot \|u\|_L$ holds for all $u \in \widehat{H}^1$.

Hence, $\widehat{H}^1$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_L$. The function $u \in \widehat{H}^1$ is called a weak solution of equation (1), if $\langle u, v \rangle_L = \lambda \cdot \int f(x, u(x)) \cdot v(x) dx$ holds for all $v \in \widehat{H}^1$.

For $x \in \mathbb{R}^N$ and $t < 0$, we define $f(x, t) = 0$. Moreover, we set $F(x, t) = \int_0^t f(x, s) ds$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Then, it follows that

(5) $0 \le F(x, t) \le \sum_{i=1}^{I} (1 + \gamma_i)^{-1} \cdot f_i(x) \cdot t_+^{(1+\gamma_i)}$

holds on $\mathbb{R}^N \times \mathbb{R}$, where $t_+ = \max(t, 0)$.

From assumption (B2), we conclude that

(6) $F(x, t) > 0$ holds on $\Omega \times]\delta_0, \infty[$.

Further, from (4), (5) and Hölder's inequality, we obtain the following

LEMMA 1. *For $i = 1, \dots, I$, the constant p_i may be chosen as in* (B3). *Then, for $0 \le R < \infty$ and $u \in \widehat{H}^1$, we have*

$$\int_{|x| \ge R} F(x, u(x)) dx \le \sum_{i=1}^{I} (1 + \gamma_i)^{-1} \cdot \left(\int_{|x| \ge R} |f_i|^{p_i} dx \right)^{1/p_i} \cdot (C_0 \cdot \|u\|_L)^{1+\gamma_i}.$$

Lemma 1 shows that $J(u)$ is well defined for all $0 \ne u \in \widehat{H}^1$. Furthermore, since $2 < 1 + \gamma_i < 2^*$, Lemma 1 implies the existence of a constant $S_1 < \infty$ such that

(7) $J(u) \le S_1$ holds for all $0 \ne u \in \widehat{H}^1$.

Thus, $\mathscr{S} = \sup_{0 \ne u \in \widehat{H}^1} J(u)$ is a well defined real number.

The function $\varphi_0 \in C_0^\infty$ may be chosen such that $\operatorname{supp} \varphi_0 \subset \Omega$ and $\sup_x \varphi_0(x) > \delta_0$, where δ_0 is the constant from (B2). Then, according to (6), we see that

$$(8) \qquad\qquad S_0 = \frac{1}{2} \cdot J(\varphi_0) > 0 \quad \text{and} \quad \mathscr{S} > S_0.$$

3. Proof of the results

In the following, we always suppose that the assumptions of Theorem 1 are fulfilled.

PROOF OF THEOREM 1. Let $(u_n)_n \subset \hat{H}^1$ be a sequence such that $u_n \neq 0$ and $J(u_n) \to \mathscr{S}$ as $n \to \infty$.

According to (8), we may assume without restriction that

$$(9) \qquad\qquad J(u_n) \geq S_0 \quad \text{holds for all } n.$$

Since $J((u_n)_+) \geq J(u_n)$, we may assume further that $u_n \geq 0$. Then, Lemma 1 implies

$$S_0 \cdot (\|u_n\|_L^2 + \|u_n\|_L^{2^*}) \leq C \cdot \sum_{i=1}^l \|u_n\|_L^{1+\gamma_i}$$

for all n and a constant C.

Hence, there exist constants $0 < U_0 < U_1 < \infty$ such that

$$(10) \qquad\qquad U_0 \leq \|u_n\|_L \leq U_1 \quad \text{holds for all} \quad n.$$

So, we can find a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, and u, $0 \leq u \in \hat{H}^1$, such that $u_n \underset{w}{\to} u$ in $\hat{H}^1$. According to Lemma 1 and (10), for each $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ such that $\int_{|x| \geq R_\varepsilon} F(x, u(x)) dx \leq \varepsilon$, and

$$(11) \qquad\qquad \int_{|x| \geq R_\varepsilon} F(x, u_n(x)) dx \leq \varepsilon$$

holds for all n.

Using the Rellich-Kondrachov theorem and the fact that $f_i \in L^{p_i+\varepsilon_i}$, it can be shown that

$$(12) \qquad\qquad \int_{|x| < R_\varepsilon} F(x, u_n(x)) dx \to \int_{|x| < R_\varepsilon} F(x, u(x)) dx$$

as $n \to \infty$. But (11) and (12) then imply

$$(13) \qquad\qquad \int F(x, u_n(x)) dx \to \int F(x, u(x)) dx.$$

From (9), (10) and (13) it follows that

$$\int F(x, u(x))dx \geq S_0 \cdot (U_0^2 + U_0^{2^*}) > 0$$

and therefore, that $u \neq 0$.

From the uniform boundedness principle, we obtain

(14)
$$\|u\|_L^2 + \|u\|_L^{2^*} \leq \liminf(\|u_n\|_L^2 + \|u_n\|_L^{2^*}).$$

Now, (13) and (14) imply $J(u) \geq \limsup J(u_n) = \mathscr{S}$ and, consequently, that $J(u) = \mathscr{S}$.

For any $v \in \hat{H}^1$, we can find an $\varepsilon_0 = \varepsilon_0(v)$, such that $\|u + \varepsilon \cdot v\|_L > 0$ holds for all $|\varepsilon| < \varepsilon_0$. For $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, we define $\eta(\varepsilon) = J(u + \varepsilon \cdot v)$. But $\eta'(0) = 0$ then implies

(15)
$$\langle u, v \rangle_L = \lambda \cdot \int f(x, u(x)) \cdot v(x)dx, \quad \text{where}$$

$$\lambda = \left(\|u\|_L^2 + \|u\|_L^{2^*} \right) \cdot \left(\int F(x, u(x))dx \cdot (2 + 2^* \cdot \|u\|_L^{2^*-2}) \right)^{-1} > 0.$$

For each $i = 1, \ldots, I$, there exists a constant ε_i^*, satisfying $0 < \varepsilon_i^* < \frac{2N}{1+\gamma_i} - (N-2)$, such that

$$p_i + \varepsilon_i = 2N/(2N - (N - 2 + \varepsilon_i^*) \cdot (1 + \gamma_i)).$$

For $k > 0$, let $u_k = (u - k)_+$ and $A(k) = \{x | u(x) > k\}$. Then, it is well known that $u_k \in \hat{H}^1$, that $\partial_l u_k = \partial_l u$ holds on $A(k)$ and that $\partial_l u_k = 0$ on $\mathbb{R}^N \backslash A(k)$. Inserting $v = u_k$ in (15), we conclude from (B3) and (4):

(16)
$$\left(\int_{A(k)} (u - k)^{2^*} dx \right)^{2/2^*} \leq C_0^2 \cdot \sum_{i=1}^I \|f_i\|_{p_i + \varepsilon_i}$$

$$\cdot \left(\int_{A(k)} |u(x)|^{2N/((N-2)+\varepsilon_i^*)} dx \right)^{((N-2)+\varepsilon_i^*) \cdot (1+\gamma_i)/2N}$$

$$\leq C_0^2 \cdot \sum_{i=1}^I \|f_i\|_{p_i + \varepsilon_i} \cdot \|u\|_{2^*}^{1+\gamma_i} \cdot (\text{meas } A(k))^{\varepsilon_i^* \cdot (1+\gamma_i)/2N}.$$

Furthermore, for $h > k > 0$, we obtain

(17)
$$\left(\int_{A(k)} (u - k)^{2^*} dx \right)^{2/2^*} \geq (h - k)^2 \cdot (\text{meas } A(h))^{2/2^*}.$$

In the following, the constant $k_0 > 0$ may be chosen such that meas $A(k_0)$ ≤ 1; and the constant γ_0 may be defined by $\gamma_0 = \min\{\gamma_1, \ldots, \gamma_I\}$. Furthermore, ε_0^* may satisfy $0 < \varepsilon_0^* \leq \min \{\varepsilon_1^*, \ldots, \varepsilon_I^*\}$, $\varepsilon_0^* < 2 \cdot (N-2)/(1+\gamma_0)$ and

$$n \cdot \varepsilon_0^* \neq 2 \cdot (N-2)/(1+\gamma_0) \quad \text{for all } n \in \mathbb{N}.$$

Then, there exists a constant $n_0 \in \mathbb{N}$ such that

$$n_0 \cdot \varepsilon_0^* < 2 \cdot (N-2)/(1+\gamma_0) \quad \text{and} \quad (n_0 + 1) \cdot \varepsilon_0^* > 2 \cdot (N-2)/(1+\gamma_0).$$

Hence, for $h > k \geq k_0$, we conclude from (16) and (17):

$$\text{meas } A(h) \leq C \cdot (h-k)^{-2^*} \cdot (\text{meas } A(k))^{(1+\gamma_0) \cdot \varepsilon_0^* / 2 \cdot (N-2)},$$

where the constant C is independent of h and k. By part (iii) of [4, Lemma 4.1], it follows that

(18) $\text{meas } A(h) \leq K \cdot h^{-\mu}$

holds for a constant $K > 0$ and all $h \geq k_0$, where

$$\mu = 4N/(2 \cdot (N-2) - (1+\gamma_0) \cdot \varepsilon_0^*).$$

For $p \geq 2^*$, we have

$$\int (u(x))^p dx = \int_{\{u \leq k_0\}} (u(x))^p dx + \int_0^\infty p \cdot \sigma^{p-1} \cdot \text{meas}(\{u > k_0\} \cap \{u > \sigma\}) d\sigma$$
$$\leq k_0^{p-2^*} \cdot \|u\|_{2^*}^{2^*} + k_0^p \cdot \text{meas } \{u > k_0\}$$
$$+ \int_{k_0}^\infty p \cdot \sigma^{p-1} \text{meas } \{u > \sigma\} d\sigma$$

(see [2, Lemma 9.7]).

Thus, (18) shows that $u \in L^p$ for all $2^* \leq p < \mu$. In particular, we see that $u \in L^{p_1}$ for

$$p_1 = 4N/(2 \cdot (N-2) - 2 \cdot \varepsilon_0^*) = 2N/((N-2) - \varepsilon_0^*).$$

Moreover, by induction, it can be shown that

$$u \in L^{p_n} \quad \text{holds for all } n = 1, \ldots, n_0,$$

where $p_n = 2N/((N-2) - n \cdot \varepsilon_0^*)$.

Using the fact that $u \in L^{p_{n_0}}$ and proceeding as above, we obtain

$$\text{meas } A(h) \leq C \cdot (h-k)^{-2^*} \cdot (\text{meas } A(k))^{(n_0+1) \cdot (1+\gamma_0) \cdot \varepsilon_0^* / 2 \cdot (N-2)}$$

for all $h > k \geq k_0$. But now, part (i) of [4, Lemma 4.1] implies $u \in L^\infty$; so, we have $u \in L^Q$ for all $2^* \leq Q \leq \infty$.

The function $h: \mathbb{R}^N \times \mathbb{R} \to [0, \infty[$ may be defined by $h(x, t) = f(x, t)/t$ if $x \in \mathbb{R}^N$ and $t > 0$, and by $h(x, t) = 0$ otherwise. Then, by (2), we obtain that

$$(19) \qquad 0 \le h(x, t) \le \sum_{i=1}^I f_i(x) \cdot t_+^{\gamma_i - 1}$$

holds for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

The constant α_0 may be chosen such that $1 < \alpha_0 < \gamma_0$, and p_0 may be defined by $p_0 = 2N/(2N - (N - 2) \cdot (1 + \alpha_0))$. Further, by $r_i(i = 1, \dots, I)$, we denote the constant

$$r_i = 2^* \cdot (\gamma_i - 1)/(\gamma_i - \alpha_0).$$

Since $u \in L^{r_i}$, we obtain by Hölder's inequality:

$$\|f_i \cdot u^{\gamma_i - 1}\|_{p_0} \le \|f_i\|_{p_i} \cdot \|u\|_{r_i}^{\gamma_i - 1} \quad \text{for all } i = 1, \dots, I.$$

Hence, (19) implies $h(\cdot, u(\cdot)) \in L^{p_0}$.

From (15), it follows that

$$\langle u, v \rangle_L - \lambda \cdot \int h(x, u(x)) \cdot u(x) \cdot v(x) dx = 0$$

holds for all $v \in \widehat{H}^1$.

Then, because $p_0 > N/2$, we conclude from [4, Corollary 8.1] that u is positive. Since u is bounded and $f(x, u(x)) = h(x, u(x)) \cdot u(x)$, we see that $f(\cdot, u(\cdot)) \in L^{p_0}$. Then, using [2, Theorem 8.24] and proceeding as in [3, p. 58], it can be shown that u is locally Hölder continuous and $u(x) \to 0$ uniformly as $|x| \to \infty$.

PROOF OF COROLLARY 1. Now, we assume additionally that (A1) is satisfied and $p_i + \varepsilon_i > N$ holds for $i = 1, \dots, I$. The function u and the constant λ may be chosen as in Theorem 1. Then, the function $g(x) = \lambda \cdot f(x, u(x))$ is locally Hölder continuous on $\mathbb{R}^N$. In particular, we see that $g \in L_{\text{loc}}^2$. Hence, [2, Theorem 8.8] implies $u \in W_{\text{loc}}^{2,2}$ and

$$- \sum_{l, m=1}^N (a_{lm} \cdot \partial_{ml} u + \partial_m a_{lm} \cdot \partial_l u) = g \quad \text{a.e. in } \mathbb{R}^N.$$

Now, we apply [2, Theorem 9.19] and conclude that $u \in C_{\text{loc}}^2$. Furthermore, we see that (λ, u) solves equation (1) in the classical sense.

The constant q_0 may be chosen such that

$$N < q_0 \le \min\{p_1 + \varepsilon_1, \dots\dots, p_I + \varepsilon_I\},$$

and we let $B_\rho(x)$ denote the ball in $\mathbb{R}^N$ with centre x and radius ρ. Then, using [2, Problem 4.8(b)] and the remark after (4.46) in [2, p. 70], and proceeding as in the proof of [2, Theorem 8.32], it can be shown there exists a constant C, independent of x, such that

$$(20) \qquad \|u\|_{C^{1,\nu}(B_1(x))} \leq C \cdot \left(\|u\|_{C^0(B_2(x))} + \|g\|_{L^{q_0}(B_2(x))} \right),$$

where $0 < \nu < \min\{\alpha, (q_0 - N)/q_0\}$. Since u vanishes uniformly at infinity and

$$\|g\|_{L^{q_0}(B_2(x))} \leq K \cdot \sum_{i=1}^{I} \|f_i\|_{L^{p_i+\varepsilon_i}(B_2(x))}$$

holds for a constant K which is independent of x, we conclude from (20) that $|\nabla u(x)|$ has uniform limit zero as $|x| \to \infty$.

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