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EXISTENCE THEOREMS ON THE DIRICHLET PROBLEM FOR THE EQUATION $\Delta u + f(x,u) = 0$

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In this note we consider the Dirichlet problem $\Delta u + f(x,u) = 0$ in Ω, $u = 0$ on $\partial\Omega$; here Ω is a bounded domain in $\mathbb{R}^n$ ($n \geqq 3$), with smooth boundary $\partial\Omega$. We prove the existence of strong solutions to the previous problem, which are positive if f satisfies a suitable condition. As a consequence we find that the problem with $f(x,u) = |u|^{((n+2)/(n-2))} + g(x,u)$, may have positive solutions even if g is not a lower-order perturbation of $|u|^{((n+2)/(n-2))}$. Next, we examine the case $f(x,u) = |u|^{((n+2)/(n-2))} + h(x)$.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqq 3$, with a $C^{1,1}$-boundary $\partial\Omega$, and let f be a real-valued function defined on $\Omega \times \mathbb{R}$. If $q \in]1, +\infty[$, we set $X_\infty(\Omega) = \{u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) : \Delta u \in L^\infty(\Omega)\}$. Owing to known results (see, for instance, [**5**, Theorem 9.15]), we see that the set $X_\infty(\Omega)$ is independent of q. Moreover, it is easy to see that $X_\infty(\Omega)$, equipped with the norm $\|u\|_{X_\infty(\Omega)} = \|u\|_\infty + \|\Delta u\|_\infty$, is a Banach space. Finally, by Lemma 9.17 of [**5**] and Theorem 6.2 of [**1**], we have that $X_\infty(\Omega) \subset C^1(\bar{\Omega})$ and the natural injection is compact.

Consider the problem

$$\text{(P)} \begin{cases} \Delta u + f(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A function $u : \Omega \to \mathbb{R}$ is said to be a strong solution to (P) if $u \in X_\infty(\Omega)$ and, for almost every $x \in \Omega$, one has $\Delta u(x) + f(x, u(x)) = 0$. A strong solution u to problem (P) such that $u(x) > 0$ in Ω is said to be a positive solution to (P).

In this note we establish existence theorems concerning strong solutions (Theorem 1) and positive solutions (Theorem 2) to problem (P). We next observe that, under the assumptions of Theorem 2, the problem $\Delta u + |u|^{((n+2)/(n-2))} + g(x,u) = 0$, $u = 0$ on $\partial\Omega$, may have positive solutions even if g is not a lower-order perturbation of $|u|^{((n+2)/(n-2))}$ (Remark 2). Finally, as a consequence of Theorem 1, we obtain a result (Theorem 3) which improves, for significant values of n, Theorem 3.3 of [**8**].

Our notation is standard and, in any case, we refer to [**1**] and [**5**].

Results

Owing to Theorem 9.15 of [5], we see that Δ is a one-to-one operator from $X_\infty(\Omega)$ onto $L^\infty(\Omega)$. Moreover, because of Remark 1 of [7], for every $u \in X_\infty(\Omega)$ we have

$$\|u\|_\infty \leqq B\|\Delta u\|_\infty,$$

where

$$B=\begin{cases} \dfrac{1}{2n\pi}\left[\Gamma\left(1+\dfrac{n}{2}\right)|\Omega|\right]^{2/n} & \text{if } |\Omega|>1 \\ \\ \dfrac{1}{2n\pi}\left[\Gamma\left(1+\dfrac{n}{2}\right)\right]^{2/n} & \text{if } |\Omega|\leqq 1; \end{cases} \tag{1}$$

($|\Omega|$ is the Lebesgue measure of Ω and Γ is the Gamma-function).

We have the following results.

Theorem 1. *Let f be a real-valued function defined on $\Omega\times\mathbb{R}$. Assume that*

(i_1) *for almost every $x\in\Omega$ the function $z\to f(x,z)$ is continuous,*

(i_2) *for every $z\in\mathbb{R}$ the function $x\to f(x,z)$ is measurable,*

(i_3) *there exists $r>0$ such that the function*

$$x\to M(x)=\sup_{|z|\leqq Br}|f(x,z)|,$$

where B is given by (1), *belongs to $L^\infty(\Omega)$ and its norm in this space is less than or equal to r.*

Then, problem (P) *has at least one strong solution $u\in X_\infty(\Omega)$.*

Proof. Let $q\in\,]\frac{n}{2},+\infty[$. Consider the set

$$K=\{v\in L^q(\Omega):|v(x)|\leqq M(x) \quad \text{a.e. in } \Omega\}.$$

Obviously, K is a nonempty, convex, closed and bounded subset of $L^q(\Omega)$. Therefore, because of reflexivity of $L^q(\Omega)$, it is weakly compact (see, for instance [3, Theorem 13, p. 422 and Corollary 8, p. 425]).

Now, let $\psi: X_\infty(\Omega)\to L^\infty(\Omega)$ be the operator defined by $\psi(u)=-\Delta u$ for every $u\in X_\infty(\Omega)$. Bearing in mind the definition of $X_\infty(\Omega)$, Theorem 9.15 of [5] ensures that ψ is a one-to-one operator from $X_\infty(\Omega)$ onto $L^\infty(\Omega)$.

For every $v\in K$ and every $x\in\Omega$, we set

$$G(v)(x)=f(x,\psi^{-1}(v)(x)).$$

Let us prove that $G(K)\subseteq K$. To this end, fix $v\in K$ and observe that, by Remark 1 of [7] and (i_3), we have

$$\operatorname*{ess\,sup}_{x\in\Omega}|\psi^{-1}(v)(x)|\leqq B\operatorname*{ess\,sup}_{x\in\Omega}|v(x)|\leqq B\operatorname*{ess\,sup}_{x\in\Omega}M(x)\leqq Br.$$

Then,

$$|f(x,\psi^{-1}(v)(x))|\leqq \sup_{|z|\leqq Br}|f(x,z)| \quad \text{a.e. in } \Omega.$$

This implies $G(v)\in K$.

Now, let us prove that the operator G is weakly sequentially continuous. Let $v\in K$ and let $\{v_h\}$ be a sequence in K weakly converging to v in $L^q(\Omega)$. Due to Lemma 9.17 of [5], $\{\psi^{-1}(v_h)\}$ converges weakly to $\psi^{-1}(v)$ in $W^{2,q}(\Omega)$. Therefore, since $q>(n/2)$, the Rellich–Kondrachov Theorem (see [1, Theorem 6.2]) guarantees that the sequence $\{\psi^{-1}(v_h)\}$ converges to $\psi^{-1}(v)$ in $C^0(\bar{\Omega})$. So, by (i_1), $\{G(v_h)\}$ converges to $G(v)$ almost everywhere in Ω. Bearing in mind that, for almost every $x\in\Omega$ and every $h\in\mathbb{N}$ one has

$$|G(v_h)(x)|\leqq M(x),$$

the Lebesgue Dominated Convergence Theorem yields $\lim_{h\to+\infty}G(v_h)=G(v)$ in $L^q(\Omega)$. Consequently, $\{G(v_h)\}$ converges weakly to $G(v)$ in $L^q(\Omega)$.

We now have proved that the function $G:K\to K$ verifies all the assumptions of Theorem 1 of [2]. Then, there is $v\in K$ such that $v=G(v)$. The function $u(x)=\psi^{-1}(v)(x)$, $x\in\Omega$, satisfies our conclusion. □

Remark 1. We explicitly observe that the preceding theorem can be regarded as an extension to the case $p=+\infty$ of Theorem 2.3 of [6].

Theorem 2. *Let f be a real-valued function defined on $\Omega\times\mathbb{R}$. Assume that all the hypotheses of Theorem 1 hold. Moreover suppose that*

(i_4) *for almost every $x\in\Omega$ one has*

$$m(x)=\inf_{0\leqq z\leqq Br}f(x,z)>0.$$

Then, problem (P) *has at least one positive solution $u\in X_\infty(\Omega)$.*

Proof. Let us sketch the proof. We define

$$K_+=\{v\in L^q(\Omega):m(x)\leqq v(x)\leqq M(x)\quad \text{a.e. in }\Omega\},$$

$$L^\infty_+(\Omega)=\{u\in L^\infty(\Omega):u(x)>0\quad \text{a.e. in }\Omega\},$$

$$X_\infty^+(\Omega)=\{u\in X_\infty(\Omega):u(x)>0 \quad \text{in } \Omega\}.$$

Due to the Strong Maximum Principle (see [5, Theorem 9.6]), the operator $\psi_+:X_\infty^+(\Omega)\to L_+^\infty(\Omega)$ defined by $\psi_+(u)=-\Delta u$ for all $u\in X_\infty^+(\Omega)$ is a one-to-one mapping from $X_\infty^+(\Omega)$ onto $L_+^\infty(\Omega)$. Now, for every $v\in K_+$ and every $x\in\Omega$, we set $G_+(v)(x)=f(x,\psi_+^{-1}(v)(x))$. The same arguments used in the proof of Theorem 1, show that Theorem 1 of [2] can be applied to G_+ to conclude the proof. □

Remark 2. Consider the problem

$$(\mathrm{P}_+)\begin{cases}-\Delta u=u^p+g(x,u) & \text{in } \Omega\\ u>0 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $p=(n+2)/(n-2)$ is critical from the viewpoint of Sobolev embedding. In the paper [4] this problem is studied assuming that the function $g:\Omega\times[0,+\infty[\to\mathbb{R}$ is a lower-order perturbation of u^p, namely

$$(*)\quad \lim_{u\to+\infty}\frac{g(x,u)}{u^p}=0.$$

We emphasize that Theorem 2 may be applied also when the preceding condition is not satisfied. In fact, consider the equation $-\Delta u=|u|^p+\lambda(1+|u|)^p$, where λ is a positive real parameter. Of course, in this case, condition (*) is not satisfied. Nevertheless, Theorem 2 can be used when, for instance, $n=3$, $|\Omega|\leqq 1$, and $\lambda\in]0,((1-B)(2^5B))[$ (by choosing $r=1/B$).

Now we set

$$C(n)=\frac{\dfrac{4}{n+2}\left(\dfrac{n-2}{n+2}\right)^{((n-2)/4)}}{B^{((n+2)/4)}},$$

where B is given by (1), and

$$B(n)=\frac{\dfrac{4}{n+2}\left(\dfrac{n-2}{n+2}\right)^{((n-2)/4)}}{\left(\dfrac{n-1}{n-2}\dfrac{1}{\sqrt{n}}\right)^{((n+2)/2)}}.$$

It is a simple matter to see that, for significant choices of n (as $n=3,4,5,6,7$ and others), the constant $C(n)$ is considerably larger than $B(n)$. Therefore, the following result improve, for these n, Theorem 3.3 of [8].

Theorem 3. *Let $h \in L^{\infty}(\Omega)$. Suppose that*

$$\|h\|_{\infty} \leqq C(n).$$

Then, the problem

$$(\mathrm{P}') \quad \begin{cases} \Delta u + |u|^{((n+2)/(n-2))} + h(x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one strong solution $u \in X_{\infty}(\Omega)$.

Proof. Let us apply Theorem 1. To this end, choose, for every $x \in \Omega$ and for every $z \in \mathbb{R}$

$$f(x,z) = |z|^{((n+2)/(n-2))} + h(x) \quad \text{and} \quad r = \left(\frac{n-2}{n+2}\right)^{((n-2)/4)} \left(\frac{1}{B}\right)^{((n+2)/4)}.$$

Of course, the function f so defined satisfies all the assumptions of Theorem 1. Indeed

$$\sup_{|z| \leqq Br} |f(x,z)| \leqq |Br|^{((n+2)/(n-2))} + \|h\|_{\infty} \leqq$$

$$B^{((n+2)/(n-2))} \left[\left(\frac{n-2}{n+2}\right)^{((n-2)/4)} \left(\frac{1}{B}\right)^{((n+2)/4)} \right]^{((n+2)/(n-2))} + C(n) \leqq \left(\frac{n-2}{n+2}\right)^{((n-2)/4)} \left(\frac{1}{B}\right)^{((n+2)/4)}.$$

By Theorem 1, there exists $u \in X_{\infty}(\Omega)$ such that $\Delta u(x) + |u(x)|^{((n+2)/(n-2))} + h(x) = 0$ almost everywhere in Ω, $u(x) = 0$ on $\partial\Omega$ and this completes the proof. □

Remark 3. We observe that Theorem 3.3 of [8] yields weak solutions to problem (P′), while the preceding result gives strong solutions, belonging to $X_{\infty}(\Omega)$.

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