Third Meeting, January 11th, 1895.

WM. PEDDIE, Esq , D.Sc., F.R.S.E., Vice-President, in the Chair.

Properties connected with the Angular Bisectors of a Triangle.

By J. S. MACKAY, M.A., LL.D.

NOTATION.

When points and lines are not specifically designated in the course of the following pages it will be understood that the notation for them is that recommended in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I. pp. 6–11 (1894). It may be convenient to repeat all that is necessary for the present purpose.

A′ B′ C′ = mid points of the sides BC CA AB

D E F = points of contact of sides with incircle

D_1 E_1 F_1 = „ „ „ „ „ „ first excircle.

And so on.

H = orthocentre of ABC

I = incentre of ABC

I_1 I_2 I_3 = 1st 2nd 3rd excentres of ABC

L M N = feet of interior angular bisectors of ABC

L′ M′ N′ = „ „ exterior „ „ „ „

O = circumcentre of ABC

U U′ = ends of that diameter of the circumcircle which is perpendicular to BC. U is on the opposite side of BC from A.

Similarly for V V′ and W W′.

X Y Z = feet of the perpendiculars from A B C.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

The various points

$$KK', \quad PP', \quad QQ', \quad SS', \quad TT'$$

are defined as they occur.

α	β	γ	$=$ AI	BI	CI	
α_1	β_1	γ_1	$= AI_1$	BI_1	CI_1	
α_2	β_2	γ_2	$= AI_2$	BI_2	CI_2	
α_3	β_3	γ_3	$= AI_3$	BI_3	CI_3	
$\alpha_1 - \alpha$	$\beta_2 - \beta$	$\gamma_3 - \gamma$	$= I_1I$	I_2I	I_3I	
$\alpha_2 + \alpha_3$	$\beta_3 + \beta_1$	$\gamma_1 + \gamma_2$	$= I_2I_3$	I_3I_1	I_1I_2	
h_1	h_2	h_3	= the perpendiculars AX BY CZ			
l_1	l_2	l_3	= the interior angular bisectors of A B C			
λ_1	λ_2	λ_3	= ,, exterior ,, ,, ,, ,, ,, ,,			
	r		= radius of the incircle			
r_1	r_2	r_3	= radii of the 1st 2nd 3rd excircles			
	R		= radius of the circumcircle			
	s		= semiperimeter of ABC			
s_1	s_2	s_3	$= s - a$	$s - b$	$s - c$	
u_1	v_1	w_1	$=$ BL	CM	AN	
u_1'	v_1'	w_1'	$=$ BL′	CM′	AN′	
u_2	v_2	w_2	$=$ CL	AM	BN	
u_2'	v_2'	w_2'	$=$ CL′	AM′	BN′	

§ 1.

If from either end of the base of a triangle a perpendicular be drawn to the bisector of the interior or exterior vertical angle, the distance of the foot of this perpendicular from the mid point of the base is equal to half the difference or half the sum of the sides of the triangle.*

FIGURE 7.

Let BP BP′, the perpendiculars from B on AL AL′, the interior and the exterior bisectors of $\angle$ A, meet AC in B_1 B_2.

* Compare Leybourn's *Mathematical Repository*, old series, I. 284 (1799), II. 24 (1801).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

Then $\quad BP = B_1P \qquad BP' = B_2P'$

and $\quad CB_1 = AC - AB \qquad CB_2 = AC + AB.$

Now since $P\,P'$ are the mid points of $BB_1\ BB_2$ and A' the mid point of BC,

therefore $\quad A'P = \frac{1}{2}CB_1 \qquad A'P' = \frac{1}{2}CB_2$

$$= \tfrac{1}{2}(AC - AB) \qquad = \tfrac{1}{2}(AC + AB)$$

Similarly if $Q\ Q'$ be the feet of the perpendiculars from C on $AL\ AL'$,

$$A'Q = \tfrac{1}{2}(AC - AB) \qquad A'Q' = \tfrac{1}{2}(AC + AB).$$

It is not easy to assign authorities to the properties given in the following pages. Some of these properties occur incidentally in the solutions of problems on the construction of triangles, and are there spoken of, or assumed without being spoken of, as well known theorems. A large collection of them will be found in four articles entitled "Useful Propositions in Geometry" by M. A. Harrison, which appeared in Leybourn's *Mathematical Repository*, old series, I. 283-5, 367-9, II. 23-5, 234-7 (1799-1801). In these articles no mention is made of properties connected with the bisector of the exterior vertical angle.

It has been conjectured that "M. A. Harrison" is a pseudonym, adopted either by J. H. Swale or John Lowry.

(1) $\quad \angle ABB_1 = AB_1B = BAL' = B_2AL' = AP'A' = AQ'A'$

$$= \tfrac{1}{2}(B + C)$$

$$\angle CBB_1 = BL'A = \tfrac{1}{2}(B - C)$$

(2) $A'PP'$ is a straight line parallel to AC. Hence $P\ P'$ are situated on $C'A'$ one of the sides of the triangle $A'B'C'$, which is complementary to ABC.

Similarly, if from B perpendiculars be drawn to the bisectors of the interior and exterior angles at C, the feet of these perpendiculars will also be situated* on $C'A'$.

(3) If perpendiculars be drawn from each vertex of a triangle to the interior and the exterior bisectors of the angles at the other vertices, the twelve points of intersection thus obtained will range, four

* Arthur Lascases in the *Nouvelles Annales*, XVIII. 171 (1859).

and four, on three straight lines, which by their mutual intersections will form the triangle complementary to the given triangle. *

The proof of this is obvious enough from what precedes; but the following demonstration will be found interesting.

FIGURE 8.

Let ABC be a triangle, I I_1 I_2 I_3 the incentre and the ex-centres.

The four lines I_2B I_3I_1 I_3C I_1I_2 are the interior and the exterior bisectors of the angles B and C. Now these four lines, taken three and three, form the four triangles

$$I_3I_1C \quad I_2IC \quad I_2I_1B \quad I_3IB$$

Hence, by a theorem due to Wallace,† the circumcircles of these four triangles all pass through the same point A; and by one of Steiner's theorems ‡ the feet of the perpendiculars let fall from A on the four straight lines are collinear.

Let A_1 A_2 A_3 A_4 be the feet of the perpendiculars.
Then AA_1BA_2 is a rectangle;
therefore A_1A_2 passes through C′ the mid point of AB.
Similarly A_3A_4 ,, ,, B′ ,, ,, ,, ,, AC;
therefore the straight line $A_1A_2A_3A_4$ bisects AB and AC.

(4) $$A_3A_4 = b, \quad A_1A_2 = c, \quad A_2A_4 = s$$
$$A_1A_3 = s_1, \quad A_2A_3 = s_2, \quad A_1A_4 = s_3$$

(5) The four points A_1 A_2 A_3 A_4 lie, two and two, on the circumferences of the six circles which have for diameters the distances of A from I I_1 I_2 I_3 B C.

(6) If the circles be denoted by their diameters, the circles AI AI_1 touch each other at A, and have I_2I_3 for common tangent; they also touch the circle II_1 the former at I and the latter at I_1.

---

* T. T. Wilkinson in the *Lady's and Gentleman's Diary* for 1862, p. 74. The demonstration given is also due to him, as well as part of (4). See the *Diary* for 1863, pp. 54-5.

† Leybourn's *Mathematical Repository*, new series, Vol. I., p. 22 of the Questions (1804).

‡ Gergonne's *Annales* XVIII. 302 (1828) or Steiner's *Gesammelte Werke*, I. 223

The circles AI_2 AI_3 touch each other at A, and have AI_1 for common tangent; they also touch the circle $I_2 I_3$ the former at I_2 and the latter at I_3.

(7) The radical axis of the circles

AB	AI	AI_2	is	AA_1
AB	AI_3	AI_1	,,	AA_2
AC	AI	AI_3	,,	AA_3
AC	AI_1	AI_2	,,	AA_4

(8)
$$AP : AL = \tfrac{1}{2}(AC + AB) : AC$$
$$AQ : AL = \tfrac{1}{2}(AC + AB) : AB$$

FIGURE 9.

Since PA′ is parallel to AC,

therefore triangles ACL PA′L are similar;

therefore
$$AL : PL = AC : PA'$$
$$= AC : \tfrac{1}{2}(AC - AB)$$

therefore
$$AL - PL : AL = \tfrac{1}{2}(AC + AB) : AC$$

(9)
$$AP' : AL' = \tfrac{1}{2}(AC - AB) : AC$$
$$AQ' : AL' = \tfrac{1}{2}(AC - AB) : AB$$

(10)
$$PQ : AL = AC^2 - AB^2 : 2AC \cdot AB$$
$$P'Q' : AL' = AC^2 - AB^2 : 2AC \cdot AB$$

FIGURE 9.

Since triangles ACL PA′L are similar

therefore
$$PL : AL = PA' : AC$$
$$= \tfrac{1}{2}(AC - AB) : AC$$

Since triangles ABL QA′L are similar

therefore
$$QL : AL = QA' : AB$$
$$= \tfrac{1}{2}(AC - AB) : AB$$

therefore $$\frac{PL+QL}{AL} = \frac{AC-AB}{2AC} + \frac{AC-AB}{2AB}$$

$$= \frac{AC \cdot AB - AB^2}{2AC \cdot AB} + \frac{AC^2 - AC \cdot AB}{2AC \cdot AB}$$

$$= \frac{AC^2 - AB^2}{2AC \cdot AB}$$

(11)* $$ABC = AQ \cdot BP = AP \cdot CQ$$
$$= AQ' \cdot BP' = AP' \cdot CQ'$$

For triangles AXL, BPL are similar

therefore $$AX : BP = AL : BL$$
$$= QL : A'L$$
$$= AQ : BA'$$

therefore $$AQ \cdot BP = AX \cdot BA'$$
$$= ABC$$

The last two expressions for ABC may be derived from the first two, since

$$AP = BP' \quad AQ = CQ' \quad BP = AP' \quad CQ = AQ'.$$

(12) The values of the following angles may be registered for reference:

$$ABP = ACQ \quad = 90^\circ - \tfrac{1}{2}A$$
$$BDF = BFD = BD_2F_2 = BF_2D_2 = 90^\circ - \tfrac{1}{2}B$$
$$CDE = CED = CD_3E_3 = CE_3D_3 = 90^\circ - \tfrac{1}{2}C$$

$$ABP' = ACQ' \quad = \tfrac{1}{2}A$$
$$BD_1F_1 = BF_1D_1 = BD_3F_3 = BF_3D_3 = \tfrac{1}{2}B$$
$$CD_1E_1 = CE_1D_1 = CD_2E_2 = CE_2D_2 = \tfrac{1}{2}C$$

In triangles BDP CQD

$$PBD = \tfrac{1}{2}(B-C), \quad BDP = 90^\circ + \tfrac{1}{2}C, \quad DPB = 90^\circ - \tfrac{1}{2}B$$
$$= DCQ \qquad\qquad = CQD \qquad\qquad = QDC$$

* Part of (11) is given in Hind's *Trigonometry*, 4th ed., p. 304 (1841).

In triangles BD_1P CQD_1

$$PBD_1 = \tfrac{1}{2}(B - C), \quad BD_1P = \tfrac{1}{2}C, \quad D_1PB = 180^\circ - \tfrac{1}{2}B$$
$$= D_1CQ \qquad = CQD_1 \qquad = QD_1C$$

In triangles BD_2P' $CQ'D_2$

$$P'BD_2 = \tfrac{1}{2}A + B, \quad BD_2P' = \tfrac{1}{2}C, \quad D_2P'B = 90^\circ - \tfrac{1}{2}B$$
$$= D_2CQ' \qquad = CQ'D_2 \qquad = Q'D_2C$$

In triangles BD_3P' $CQ'D_3$

$$P'BD_3 = \tfrac{1}{2}A + C, \quad BD_3P' = 90^\circ - \tfrac{1}{2}C, \quad D_3P'B = \tfrac{1}{2}B$$
$$= D_3CQ' \qquad = CQ'D_3 \qquad = Q'D_3C$$

$$AEP = AFP = AQE_1 = AQF_1 = 90^\circ + \tfrac{1}{2}C$$
$$APE = APF = AE_1Q = AF_1Q = \tfrac{1}{2}B$$

$$AE_2P' = AF_2P' = AQ'E_3 = AQ'F_3 = \tfrac{1}{2}C$$
$$AP'E_2 = AP'F_2 = AE_3Q' = AF_3Q' = \tfrac{1}{2}B$$

(13)* $$AP \cdot AQ = BP' \cdot CQ' = s\, s_1$$
$$BP \cdot CQ = AP' \cdot AQ' = s_2 s_3$$

The similar triangles AEP AQE_1 give

$$AE : AP = AQ : AE_1$$

therefore $$AP \cdot AQ = AE \cdot AE_1$$
$$= ss_1 \quad ;$$

and $$AP = BP' \quad AQ = CQ'.$$

The other equalities may be derived from the similar triangles BDP CQD, and the fact that

$$BP = AP' \quad CQ = AQ'.$$

(14) $$AP \cdot AQ \cdot BP \cdot CQ = AP' \cdot AQ' \cdot BP' \cdot CQ'$$
$$= \Delta^2$$

* (13) Half of this is given in Hind's *Trigonometry*, 4th ed., p. 304 (1841).

(15) Let D, D_1, D_2, D_3 be the points where the incircles and the excircles touch BC.

FIGURE 9.

It is known that

$$A'D = A'D_1 = \tfrac{1}{2}(AC - AB), \quad A'D_2 = A'D_3 = \tfrac{1}{2}(AC + AB);$$

hence D D_1 P Q lie on a circle with centre A′
and D_2 D_3 P′ Q′ ,, ,, ,, ,, ,, ,, ,,

(16) The incircle and first excircle of ABC cut the circle DPD_1Q orthogonally, and the second and third excircles cut $D_2Q'P'D_3$ orthogonally.

For D D_1 is perpendicular to I D and I_1D_1;
and D_2D_3 ,, ,, ,, I_2D_2 ,, I_3D_3.

(17) $$IP \cdot IQ = r^2, \quad I_1P \cdot I_1Q = r_1^2$$
$$I_2P' \cdot I_2Q' = r_2^2, \quad I_3P' \cdot I_3Q' = r_3^2$$

(18) If I I_1 be considered as one pair of a system of coaxal circles, then P Q are the limiting points of the system; and P′ Q′ are the limiting points of the coaxal system of which I_2 I_3 form one pair.

For DD_1 is a common tangent to the circles I I_1, and A′ is its mid point; therefore the radical axis of I I_1 passes through A′.

Now since the circle whose diameter is DD_1 has its centre at A′ and cuts I I_1 orthogonally, therefore it passes through the limiting points of the system I I_1; and the limiting points of the system I I_1 are situated on the line II_1.

(19) APBP′ is a rectangle, and PP′ bisects AB. Hence if AX be perpendicular to BC, the circle on AB as diameter passes through* P P′ X.

Similarly the circle on AC as diameter* passes through Q Q′ X.

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1856, p. 49.

(20) Triangles XPQ, XP′Q′ are inversely similar* to ABC.

FIGURE 9.

Since A P B X are concyclic

therefore $\angle APX = \angle ABX$

therefore $\angle XPQ = \angle ABC$

Since A C Q X are concyclic

therefore $\angle AQX = \angle ACX$

therefore triangle XPQ is similar to ABC

In like manner XP′Q′ is similar to ABC

(21) The directly similar triangles XPQ XP′Q′ have their homologous sides mutually perpendicular.

(22) The incentre and the excentres of triangles XPQ XP′Q′ are situated on BX and AX.

Since A P B X are concyclic

therefore $\angle BXP = \angle BAP = \frac{1}{2}A$

Since A C Q X are concyclic

therefore $\angle CXQ = \angle CAQ = \frac{1}{2}A$

therefore BX bisects $\angle PXQ$

therefore BX contains the incentre and one excentre of XPQ.

Now AX is perpendicular to BX

therefore AX contains the other excentres.

In like manner it may be proved that AX contains the incentre and one excentre of triangle XP′Q′, and that BX contains the other excentres.

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1856, p. 49. The first part of the theorem, however, is given in Leybourn's ***Mathematical Repository***, old series, II. 25 (1801).

(23) To determine the incentre and the excentres of the triangles XPQ XP′Q′.

Since AX ID are parallel

therefore AL : IL = XL : DL.

But in the similar triangles ABC XPQ
AL and XL are homologous lines ;
therefore IL and DL are homologous lines,
and I D homologous points ;
therefore D is the incentre of XPQ.

Since $\angle$ DPD_1 is right, D_1 is the first excentre.

The other excentres are the points where DP and DQ intersect AX.

In like manner it may be proved that D_3 and D_2 are the third and second excentres of triangle XP′Q′ and that the incentre and first excentre are the points where D_3Q' and D_3P' intersect AX.

(24) The circumcircles* of XPQ XP′Q′ pass through A′.

Since $\angle A'XQ = \angle CAQ$

$= \angle A'PQ$

because A′P and CA are parallel ;
therefore A′ P X Q are concyclic.

In like manner A′ Q′ P′ X are concyclic.

(25) The diameters* of the circles XPQ XP′Q′ are respectively equal to AU′ AU.

For the diameter of XPQ is the perpendicular drawn from A′ to PQ and terminated by AX ; and this perpendicular along with A′U′ U′A AX produced forms a parallelogram.

(26) The diameters of the circles XPQ XP′Q′ coincide with the radical axes of the circles I I_1 and I_2 I_3.

---

* The first parts of (24) and (25) are found in Leybourn's ***Mathematical Repository***, old series, II. 24, 235 (1801).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(27) The circle XPQ cuts orthogonally the system of circles $I\ I_1$; and the circle $XP'Q'$ cuts orthogonally the system $I_2\ I_3$.

For the circle XPQ passes through the limiting points P Q of the system $I\ I_1$ and has its centre on the radical axis of the same system.

(28) The centres of the circles XPQ and $XP'Q'$ and the nine-point centre of triangle ABC are collinear.

For they are situated on the straight line which bisects $A'X$ perpendicularly.

(29) The sum of the areas of the circles XPQ $XP'Q'$ is equal to the area of the circle ABC.

For the areas of circles are proportional to the squares of their diameters

and $$AU'^2 + AU^2 = UU'^2.$$

(30) $$XPQ + XP'Q' = ABC.^*$$

(31)† $$ABC : XPQ = UU' : U'K$$
$$ABC : XP'Q' = UU' : UK$$

For $$ABC : XPQ = UU'^2 : AU'$$
$$= UU' : U'K$$

For another proof see §4, (11).

(32) $$XP \quad XP' \quad XQ \quad XQ'$$

are respectively parallel to

$$CU \quad CU' \quad BU \quad BU'$$

For $$\angle XPQ = \angle ABC$$
$$= \angle AUC.$$

---

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1855, p. 71.

† Parts of (28), (29), (30), (31), are found in Leybourn's *Mathematical Repository*, old series, II. 236, 25 (1801).

(33) The triangles

$$A'BP \quad A'CQ \quad A'BP' \quad A'CQ'$$

are respectively similar to

$$QAX \quad PAX \quad Q'AX \quad P'AX .$$

For $A'P$ is parallel to CA ;

therefore $\angle BA'P = C$

$= \angle AQX$;

and $\angle BPA' = 90° + \frac{1}{2}A$

$= \angle AXQ$.

Or it may be proved that

$$\angle A'BP = \angle QAX$$

since the sides of the one are perpendicular to the sides of the other.

(34) The triangles

$$A'UP \quad A'UQ \quad A'U'P' \quad A'U'Q'$$

are respectively similar to

$$QCX \quad PBX \quad Q'CX \quad P'BX.$$

For $\angle A'UP = \angle QCX$

since the sides of the one are perpendicular to the sides of the other ;

and $\angle A'PU = \frac{1}{2}A$

$= \angle QXC$.

(35) The following triads of points are collinear :*

$$P\ D\ E ;\quad P\ D_1\ E_1 ;\quad P'\ D_2\ E_2 ;\quad P'\ D_3\ E_3$$

$$Q\ D\ F ;\quad Q\ D_1\ F_1 ;\quad Q'\ D_2\ F_2 ;\quad Q'\ D_3\ F_3.$$

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1857, p. 51.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

FIGURE 10.

The points B D P I are concyclic ;
therefore ∠ BDP is the supplement of ∠ BIP.
Because the isosceles triangles CDE, UBI have

$$\angle C = \angle U$$

therefore $\angle CDE = \angle BIU$;
therefore ∠ BDP is the supplement of ∠ CDE ;
therefore DP coincides with DE.

(36) The following quintets of points are concyclic :

B	D	I	F	P ;	C	D	I	E	Q
B	D_1	I_1	F_1	P ;	C	D_1	I_1	E_1	Q
B	D_2	I_2	F_2	P′ ;	C	D_2	I_2	E_2	Q′
B	D_3	I_3	F_3	P′ ;	C	D_3	I_3	E_3	Q′

the diameters of the various circles being

BI	BI_1	BI_2	BI_3
CI	CI_1	CI_2	CI_3

(37) Since C L B L′ form a harmonic range,
and CQ BP AL′ are parallel,
therefore Q L P A form a harmonic range.*

FIGURE 9.

Similarly for Q′ A P′ L′.

(38) If at L a perpendicular be drawn to AL meeting AB AC at P Q, then AP or AQ is a harmonic mean† between AB AC.

* Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 58-9 (1890).
† Rev. R. Townsend in *Mathematical Questions from the Educational Times*, XIV. 76 (1870).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

FIGURE 11.

If B_1 be the image of B in AL,
then AL bisects $\angle BLB_1$;
therefore LQ bisects $\angle CLB_1$;
therefore $L(BAB_1Q)$ is a harmonic pencil.
Now this pencil is cut by the transversal AC;
therefore A B_1 Q C form a harmonic range
and AQ is the harmonic mean between AB_1 AC.

Similarly for L′.

(39) $$AU : IU = AB + AC : BC$$

FIGURE 12.

Draw IP IQ parallel to AB AC.

The quadrilaterals ABUC IPUQ are similar;
therefore $AU : IU = AB + AC : IP + IQ.$

Now $\angle UBL = \angle UAC = \angle UAB = \angle UIP$;
and $UB = UI$;
therefore triangles UBL UIP are congruent,
and $BL = IP.$
Similarly $CL = IQ$;
therefore $AU : IU = AB + AC : BL + CL.$

(40) If on A′K as diameter a circle is described, and from O the circumcentre of ABC a perpendicular is drawn to A′K meeting this circle at P, then*

$$AB^2 + AC^2 = 4PU^2.$$

* John Whitley in the ***Gentleman's Mathematical Companion*** for 1803, p. 38.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

FIGURE 13.

The triangle PUU′ is isosceles;

therefore $$PA'^2 = PU^2 - UA' \cdot A'U' = PU^2 - A'B^2;$$

and $$PK^2 = PU^2 - UK \cdot KU' = PU^2 - AK^2.$$

Now $$AB^2 + AC^2 = 2A'B^2 + 2A'A^2$$
$$= 2A'B^2 + 2AK^2 + 2A'K^2$$
$$= 2A'B^2 + 2AK^2 + 2PA'^2 + 2PK^2$$
$$= 4PU^2$$

§ 2.

If from the mid point of the base of a triangle a perpendicular be drawn to the bisector of the interior or exterior vertical angle, this perpendicular will cut off from the sides segments equal to half the sum* or half the difference of the sides.

FIGURE 14.

Let the perpendiculars from A to AU AU′ meet AC at S S′, and AB at T T′.

Draw BB_1 BB_2 parallel to the perpendiculars.

Because A′ is the mid point of BC,
therefore S „ „ „ „ „ B_1C.
Now $B_1C = AC - AB$;
therefore $CS = \frac{1}{2}(AC - AB)$;
therefore $AS = \frac{1}{2}(AC + AB)$.

* Part of this is found in Leybourn's *Mathematical Repository*, old series, I. 284 (1799).

Similarly $\quad BT = AT' = AS' = \frac{1}{2}(AC - AB)$

and $\quad AT = BT' = CS' = \frac{1}{2}(AC + AB)$

(1) $\quad \angle ATS = \angle AST \quad = \angle ABB_1 = \frac{1}{2}(B + C)$

$\angle BA'T = \angle CA'S \quad = \angle CBB_1 = \frac{1}{2}(B - C)$

(2) $\quad AS^2 + CS^2 = AT^2 + BT^2 = \frac{1}{2}(b^2 + c^2)$

(3) $\quad AS^2 - CS^2 = AT^2 - BT^2 = bc$

(4) $\quad AS \cdot CS = AT \cdot BT \quad = \frac{1}{4}(AC^2 - AB^2)$

$= \frac{1}{4}(CX^2 - BX^2) = A'B \cdot A'X$

(5) $\quad AS : CS = AT : BT \quad = b + c : b - c$

(6) $\quad SS' = AB = c, \quad TT' \quad = AC = b$

Instead of drawing perpendiculars to the two bisectors of the vertical angle either from the ends of the two sides or from the mid point of the base, if perpendiculars be drawn to the sides from certain points in the two bisectors of the vertical angle, values will be obtained for half the sum and half the difference of the sides.

FIGURE 9.

From U U′ let the perpendiculars US U′S′ be drawn to AC, and UT U′T′ to AB.

(7)* $\quad AS = AT = CS' = BT' = \frac{1}{2}(AC + AB)$

For the right-angled triangles UAS UAT are congruent;

therefore $\quad AS = AT \quad US = UT.$

Hence the right-angled triangles UCS UBT are congruent,

and $\quad CS = BT.$

* Parts of (7) and (8) are found in Leybourn's *Mathematical Repository*, old series, I. 283-4 (1799).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

Similarly $AS' = AT' \quad CS' = BT'$.

Now $\frac{1}{2}(AC + AB) = \frac{1}{2}\{(AS + CS) + (AT - BT)\}$
$= \frac{1}{2}\ (AS + AT) = AS = AT$;

and also $= \frac{1}{2}\{(CS' + AS') + (BT' - AT')\}$
$= \frac{1}{2}\ (CS' + BT') = CS' = BT'$

(8) $CS = BT = AS' = AT' = \frac{1}{2}(AC - AB)$

For $\frac{1}{2}(AC - AB) = \frac{1}{2}\{(AS + CS) - (AT - BT)\}$
$= \frac{1}{2}\ (CS + BT) = CS = BT$;

and also $= \frac{1}{2}\{(CS' + AS') - (BT' - AT')\}$
$= \frac{1}{2}\ (AS' + AT') = AS' = AT'$.

(9) $\angle U'BC = U'CB = U'UB = U'UC$
$= U'AS' = U'AT' = AUS = AUT$
$= \frac{1}{2}(B + C)$

(10) $\angle U'BA = U'CA = U'UA = BUT = CUS$
$= \frac{1}{2}(B - C)$

For half the sum of two magnitudes increased by half their difference gives the greater.

(11) If BP CQ be drawn perpendicular to AU,
$\angle ABP = 90^\circ - \frac{1}{2}A = \frac{1}{2}(B + C)$.

But $\angle AUS \qquad = \frac{1}{2}(B + C)$;

therefore BP US intersect* on the circle ABC .

Similarly CQ UT „ „ „ „ „ .

A like statement holds good for BP′ U′S′,
and for CQ′ U′T′.

* Leybourn's *Mathematical Repository*, old series, I. 285 (1799).

(12)* If

$$\left.\begin{array}{ll} BP & US \\ CQ & UT \\ BP' & U'S' \\ CQ' & U'T' \end{array}\right\} \text{meet on the circumcircle at} \left\{\begin{array}{l} B_2 \\ C_2 \\ B_2' \\ C_2' \end{array}\right.$$

$$4US \cdot SB_2 = 4UT \cdot TC_2$$
$$= 4U'S' \cdot S'B_2' = 4U'T' \cdot T'C_2' = AC^2 - AB^2$$

For $\quad US \cdot SB_2 = AS \cdot SC$

$$= \tfrac{1}{2}(AC + AB) \cdot \tfrac{1}{2}(AC - AB)$$

(13) The incentre I of ABC is situated on AU. If with centre U and radius UI a circle be described, it will pass through B and C and will cut AC AB again at B_1 C_1 such that $\quad B_1S = CS \quad C_1T = BT$.

Hence $\quad B_1C = BC_1 = AC - AB$;

and B P B_1 are collinear, and so are C Q C_1 .

(14)
$$U'B_2 = UB_2' = AB_1 = AB$$
$$U'C_2 = UC_2' = AC_1 = AC$$

(15) B_2 B_2' are symmetrically situated with respect to O, and so are C_2 C_2'

(16) By reference to § 1, (12) it will be seen that

$$\angle AEP = AQE_1 \;;$$

hence $\quad E$ E_1 P Q are concyclic.

Similarly $\quad F$ F_1 P Q ,, ,,

The diameters of these two circles are EE_1 FF_1, and their centres are S T .

(17) In like manner E_2 E_3 P' Q' are concyclic,

and also $\quad F_2$ F_3 P' Q' ,, ,,

The diameters of these two circles are E_2E_3 F_2F_3, and their centres are S' T'.

---

* Leybourn's *Mathematical Repository*, old series, I. 368 (1799).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(18) All the four circles are equal to each other, and their diameters are equal to BC.

The first two cut the circles $I \; I_1$ orthogonally,
and the second „ „ „ „ $I_2 \; I_3$ „

Compare § 1, (16).

§ 3.

To find values for the rectangles contained by various segments of the base BC.

FIGURE 9.

The values of the segments here given will be found useful in the verification of properties (1)–(12).

$$BX = \frac{a^2 - b^2 + c^2}{2a} \quad CX = \frac{a^2 + b^2 - c^2}{2a} \quad A'X = \frac{b^2 - c^2}{2a}$$

$$A'D = A'D_1 = \frac{b-c}{2}$$

$$A'D_2 = A'D_3 = \frac{b+c}{2}$$

$$BL = \frac{ca}{b+c} \qquad BL' = \frac{ca}{b-c}$$

$$CL = \frac{ab}{b+c} \qquad CL' = \frac{ab}{b-c}$$

$$A'L = \frac{a(b-c)}{2(b+c)} \qquad A'L' = \frac{a(b+c)}{2(b-c)}$$

$$LD = \frac{s_1(b-c)}{b+c} \qquad LD_1 = \frac{s(b-c)}{b+c}$$

$$L'D_2 = \frac{s_3(b+c)}{b-c} \qquad L'D_3 = \frac{s_2(b+c)}{b-c}$$

$$DX = \frac{s_1(b-c)}{a} \qquad D_1X = \frac{s(b-c)}{a}$$

$$D_2X = \frac{s_3(b+c)}{a} \qquad D_3X = \frac{s_2(b+c)}{a}$$

$$LX = \frac{2ss_1(b-c)}{a(b+c)} \qquad L'X = \frac{2s_2s_3(b+c)}{a(b-c)}$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(1) $$A'X \cdot A'L = A'D^2 = A'D_1^2 = \tfrac{1}{4}(b-c)^2$$

Because A C Q X are concyclic

therefore $\angle A'XQ = \frac{1}{2}A = \angle A'QL$;

therefore triangles A′XQ A′QL are similar ;

therefore $A'X : A'Q = A'Q : A'L$;

therefore
$$A'X \cdot A'L = A'Q^2$$
$$= A'D^2$$

(2) $$A'X \cdot A'L' = A'D_2^2 = A'D_3^2 = \tfrac{1}{4}(b+c)^2$$

This follows, in a manner analogous to the preceding, from the similarity of triangles A′XQ′ A′Q′L′.

The following method may be used for proving (1) and (2).

Since the points A I L I_1 form a harmonic range,
therefore their projections on BC will form a harmonic range ;
that is, X D L D_1 is a harmonic range.
Hence, since DD_1 is bisected at A′,

$$A'X \cdot A'L = A'D^2 = A'D_1^2.$$

Similarly, since I_2 A I_3 L′ form a harmonic range, so also will D_2 X D_3 L′.
Hence, since D_2D_3 is bisected at A′,
$$A'X \cdot A'L' = A'D_2^2 = A'D_3^2.$$

(3) $$A'X \cdot LX = DX \cdot D_1X = \frac{ss_1(b-c)^2}{a^2}$$

For
$$A'X \cdot LX = A'X^2 - A'X \cdot A'L$$
$$= A'X^2 - A'D^2$$
$$= DX \cdot D_1X$$

(4) $$A'X \cdot L'X = D_2X \cdot D_3X = \frac{s_2s_3(b+c)^2}{a^2}$$

For
$$A'X \cdot L'X = A'X \,.\, A'L' - A'X^2$$
$$= A'D_3^2 - A'X^2$$
$$= D_2X \cdot D_3X$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(5) $$A'X \cdot LD = A'D \cdot DX = \frac{s_1(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD &= A'X \cdot A'D - A'X \cdot A'L \\ &= A'X \cdot A'D - A'D^2 \\ &= A'D \cdot DX \end{aligned}$$

(6) $$A'X \cdot LD_1 = A'D_1 \cdot D_1X = \frac{s(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD_1 &= A'X \cdot A'D_1 + A'X \cdot A'L \\ &= A'X \cdot A'D_1 + A'D_1{}^2 \\ &= A'D_1 \cdot D_1X \end{aligned}$$

(7) $$A'X \cdot L'D_2 = A'D_2 \cdot D_2X = \frac{s_3(b+c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot L'D_2 &= A'X \cdot A'L' + A'X \cdot A'D_2 \\ &= A'D_2{}^2 + A'X \cdot A'D_2 \\ &= A'D_2 \cdot D_2X \end{aligned}$$

(8) $$A'X \cdot L'D_3 = A'D_3 \cdot D_3X = \frac{s_2(b+c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot L'D_3 &= A'X \cdot A'L' - A'X \cdot A'D_3 \\ &= A'D_3{}^2 - A'X \cdot A'D_3 \\ &= A'D_3 \cdot D_3X \end{aligned}$$

(9) $$A'L \cdot LX = DL \cdot LD_1 = \frac{ss_1(b-c)^2}{(b+c)^2}$$

For
$$\begin{aligned} A'L \cdot LX &= A'L \cdot A'X - A'L^2 \\ &= A'D^2 - A'L^2 \\ &= DL \cdot LD_1 \end{aligned}$$

(10) $$A'L' \cdot L'X = D_2L' \cdot L'D_3 = \frac{s_2s_3(b+c)^2}{(b-c)^2}$$

For
$$\begin{aligned} A'L' \cdot L'X &= A'L'^2 - A'L' \cdot A'X \\ &= A'L'^2 - A'D_3{}^2 \\ &= D_2L' \cdot L'D_3 \end{aligned}$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(11) $$DX \cdot D_1X = BD \cdot DC - BX \cdot XC$$

For $$BD \cdot DC - BX \cdot XC = (A'B^2 - A'D^2) - (A'B^2 - A'X^2)$$
$$= A'X^2 - A'D^2$$
$$= DX \cdot D_1X$$

(12) $$D_2X \cdot D_3X = BD_2 \cdot D_2C + BX \cdot XC$$

For $$BD_2 \cdot D_2C - BX \cdot XC = (A'D_2^2 - A'C^2) + (A'C^2 - A'X^2)$$
$$= A'D_2^2 - A'X^2$$
$$= D_2X \cdot D_3X$$

In *Mathematical Questions from the Educational Times*, XIII. 34 (1870), T. T. Wilkinson says regarding (1):

"This is one of the properties of Halley's diagram, which was partially discussed in the four numbers of the *Student*, published at Liverpool from 1797 to 1800. It there forms Prop. 8, and is due to *Non Sibi*, a name assumed by the first editor, Mr John Knowles. In the diagram as there considered, the properties of one side only are given; but when all the sides are considered, there seems to be no limit to the relations between the different parts of the figure. Some time ago I considered the 'angular properties' only; and after writing down about 130 of them, they seemed to arise more abundantly than ever."

Halley's diagram somewhat resembles Figure 15, and it obtained that name, among the non-academic geometers of England, from the statement of W. Jones in his *Synopsis Palmariorum Matheseos*, p. 245 (1706), that he received it "from the learned Mr Halley." Jones says that an endless variety of useful theorems may be deduced from it, and that by inspection only.

The property (1), however, is older than the *Student*; for it is spoken of as a well-known theorem in the *Ladies' Diary* for 1785.

(3) and (7) occur in M'Dowell's *Exercises on Euclid*, § 154 (1863); (9) and (11) are found in Leybourn's *Mathematical Repository*, old series, I. 369 (1799).

§ 4.

To find values for the rectangles contained by various segments of the diameter UU′.

FIGURE 15.

(1) $$A'U \cdot UK' = A'X \cdot A'L = \tfrac{1}{4}(b-c)$$

From the similar triangles UA′L AKU′

$$A'L : A'U = KU' : KA$$

that is, $$A'L : A'U = UK' : A'X$$

(2) $$A'U' \cdot U'K' = A'X \cdot A'L' = \tfrac{1}{4}(b+c)^2$$

From the similar triangles UA′L′ AKU

$$A'L' : A'U' = KU : KA$$

that is, $$A'L' : A'U' = U'K' : A'X$$

(3) $$A'K \cdot KU' = A'X \cdot LX$$

For $$\begin{aligned} A'X \cdot LX &= DX \cdot D_1X \\ &= A'X^2 - A'D^2 \\ &= KA^2 - A'D^2 \\ &= UK \cdot KU' - A'U \cdot KU' \\ &= A'K \cdot KU' \end{aligned}$$

(4) $$A'K \cdot KU = A'X \cdot L'X$$

For $$\begin{aligned} A'X \cdot L'X &= D_2X \cdot D_3X \\ &= A'D_2^2 - A'X^2 \\ &= A'D_2^2 - KA^2 \\ &= A'U' \cdot U'K' - UK \cdot KU' \\ &= A'K \cdot KU \end{aligned}$$

(5) $$A'K \cdot A'U = BD \cdot DC$$

For
$$\begin{aligned} BD \cdot DC &= A'C^2 - A'D^2 \\ &= A'U \cdot A'U' - A'U \cdot UK' \\ &= A'K \cdot A'U \end{aligned}$$

(6) $$A'K \cdot A'U' = BD_2 \cdot D_2C$$

For
$$\begin{aligned} BD_2 \cdot D_2C &= A'D_2^2 - A'C^2 \\ &= A'U' \cdot U'K' - A'U \cdot A'U' \\ &= A'K \cdot A'U' \end{aligned}$$

(7) $$A'K \cdot A'K' = BX \cdot XC$$

For
$$\begin{aligned} BX \cdot XC &= AX \cdot XR \\ &= A'K \cdot A'K' \end{aligned}$$

(8)* $$A'U \cdot UK = US^2$$

For
$$\begin{aligned} US^2 &= CU^2 - CS^2 \\ &= A'U \cdot UU' - A'U \cdot UK' \\ &= A'U \cdot U'K' \\ &= A'U \cdot UK \end{aligned}$$

(9) $$U'K \cdot ID = A'D \cdot DX$$

Since ID IL are respectively perpendicular to AK AU', therefore the right-angled triangles AKU' IDL are similar;

therefore $$U'K : AK = LD : ID$$

therefore
$$\begin{aligned} U'K \cdot ID &= AK \cdot LD \\ &= A'X \cdot LD \\ &= A'D \cdot DX \end{aligned}$$

(10)† $$U'K \cdot I_1D_1 = A'D_1 \cdot D_1X$$

The proof of this is similar to the preceding.

---

* For (1), (2), (3), (5), (8) see Leybourn's *Mathematical Repository*, old series, I. 285, 368, 367, 369, 368 (1799).

† (9) and (10) are given by T. T. Wilkinson in *Mathematical Questions from the Educational Times*, XXIV. 28 (1875).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(11) $ABC : XPQ = UU' : U'K$

and $ABC : XP'Q' = UU' : UK$

/

FIGURE 9.

Draw $A'X'$ perpendicular to PQ;
then X' is the mid point of PQ.

Because $\angle UCA' = \frac{1}{2}A = \angle A'PX'$,

therefore the right-angled triangles $UA'C$ $A'X'P$ are similar;

therefore $A'C^2 : X'P^2 = UC^2 : A'P^2$.

But
$$ABC : XPQ = BC^2 : PQ^2$$
$$= A'C^2 : X'P^2;$$

therefore
$$ABC : XPQ = UC^2 : A'P^2$$
$$= A'U \cdot UU' : A'U \cdot UK'$$
$$= UU' : UK'$$
$$= UU' : U'K$$

For another proof see § 1, (28).

In a similar manner it may be shown that

$$ABC : XP'Q' = UU' : UK.$$

(12) Because $U'K + UK = UU'$

another proof is obtained of the theorem that

$$ABC = XPQ + XP'Q'$$

(13) If the base BC and the vertical angle A be given, and if in AU AU' the bisectors of the interior and exterior angles at A, there be taken AP equal to half the sum, and AQ equal to half the difference of the sides, the loci of P and Q are two circles. If their radii be denoted by r' r'' and the radius of the circle inscribed in CUU' by r''', then

$$R = r' + r'' + r'''$$

Mr G. Robinson, jun., Hexham, in the *Lady's and Gentleman's Diary* for 1862, p. 74. Two solutions will be found in the *Diary* for 1863, pp. 49-50.

§ 5.

If through A' a perpendicular is drawn to BC, then $AD\ AD_1\ AD_2\ AD_3$ will intersect this perpendicular at $R_1\ R\ R_3\ R_2$ such that*

$$A'R = r \quad A'R_1 = r_1 \quad A'R_2 = r_2 \quad A'R_3 = r_3$$

FIGURE 16.

Let DI produced meet AD_1 at D'.

Since the line joining the extremities of two parallel and similarly directed radii of two circles passes through their external homothetic centre; and since A is the external homothetic centre of the circles I_1 and I, and I_1D_1 ID' are parallel; therefore ID' is a radius of the incircle I, and $DD' = 2r$.

Now since $A'D = A'D_1$, and $A'R$ is parallel to DD', therefore $A'R = \frac{1}{2}DD' = r$.

Similarly for the other equalities.

(1) Through B', the mid point of CA, a perpendicular to CA is drawn, and this perpendicular is intersected by

$BE \quad BE_1 \quad BE_2 \quad BE_3$

in the points $S_2 \quad S_3 \quad S \quad S_1$;

through C', the mid point of AB, a perpendicular to AB is drawn, and this perpendicular is intersected by

$CF \quad CF_1 \quad CF_2 \quad CF_3$

in the points $T_3 \quad T_2 \quad T_1 \quad T$ respectively; then

$$B'S = r \quad B'S_1 = r_1 \quad B'S_2 = r_2 \quad B'S_3 = r_3$$
$$C'T = r \quad C'T_1 = r_1 \quad C'T_2 = r_2 \quad C'T_3 = r_3.$$

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1863, p. 77, and for 1864, pp. 54-5.

(2) The four triangles RST $R_1S_1T_1$ $R_2S_2T_2$ $R_3S_3T_3$ are inversely similar to ABC; they have O, the circumcentre of ABC, for their common centre of homology, and OI OI_1 OI_2 OI_3 for the diameters of their circumcircles.

FIGURE 17.

Since $$A'R_1 = r_1 = I_1D_1$$
therefore $\angle I_1R_1A'$ is right.
Similarly $\angle I_1S_1B'$ and $\angle I_1T_1C'$ are right;
therefore the circle whose diameter is OI_1
passes through R_1 S_1 T_1.

Since R_1 S_1 O_1 T_1 are concyclic,

therefore
$$\begin{aligned} \angle S_1R_1T_1 &= 180^\circ - \angle S_1OT_1 \\ &= 180 - \angle B'OC' \\ &= A; \end{aligned}$$

and $$\angle R_1S_1T_1 = \angle R_1OT_1 = B,$$

since T_1O R_1O are respectively perpendicular to AB BC; therefore triangle $R_1S_1T_1$ is similar to ABC.

(3) Let the mid points of OI OI_1 OI_2 OI_3

be denoted by I' I_1' I_2' I_3'

then $I_1'I_2'I_3'I'$ is an orthic tetrastigm, similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

For the radius of the circumcircle of any of the four triangles of the orthic tetrastigm $I_1I_2I_3I$ is $2R$.

(4) Let AI BI CI meet the circumcircle of ABC in U V W, and let the points diametrically opposite to U V W be U' V' W'.

Then I is the orthocentre of the triangle UVW. Now since O is the circumcentre of UVW, therefore I' is the nine-point centre of the four triangles of the orthic tetrastigm $UVWI$.

In like manner since I_1 is the orthocentre, and O the circumcentre of the triangle $UV'W'$, I_1' is the nine-point centre of the four triangles of the orthic tetrastigm $UV'W'I_1$; and similarly for I_2' I_3'.

See *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 54–5 (1894).

(5) The sum of the circumcircles of the four RST triangles is three times the circumcircle of ABC.

Since circles are proportional to the squares of their diameters, the circumcircle of ABC is to the sum of the four RST circles as $4R^2$ is to $OI^2 + OI_1^2 + OI_2^2 + OI_3^2$.

Now
$$\begin{aligned}\Sigma(OI^2) &= 4R^2 + 2R(r_1 + r_2 + r_3 - r)\\ &= 4R^2 + 2R \cdot 4R\\ &= 12R^2.\end{aligned}$$

§ 6.

If UD UD_1 $U'D_2$ $U'D_3$ intersect AX at X_0 X_1 X_2 X_3, then*
$$XX_0 = r \quad XX_1 = r_1 \quad XX_2 = r_2 \quad XX_3 = r_3$$

FIGURE 18.

Through I draw a parallel to BC meeting UU′ in K_0 and AX in X_0; join UD DX_0.

Because $A'D^2 = A'X \cdot A'L$

therefore $A'D : A'X = A'L : A'D$

that is
$$\begin{aligned}A'D : K_0X_0 &= A'L : K_0I\\ &= UA' : UK_0\end{aligned}$$

therefore the points U D X_0 are collinear.

* The first of these properties is given by W. Dixon Rangeley in the *Gentleman's Diary* for 1822, p. 47; the first and second (without any hint as to the third and fourth) by W. H. Levy in the *Lady's and Gentleman's Diary* for 1849, p. 75.

Similarly, if through I_1 a parallel be drawn to BC meeting AX in X_1, it may be proved that $U\ D_1\ X_1$ are collinear.

Through I_3 draw a parallel to BC meeting UU′ in K_3 and AX in X_3; join $U'D_3$ D_3X_3.

Because $$A'D_3^2 = A'X \cdot A'L'$$

therefore $$A'D_3 : A'X = A'L' : A'D_3$$

that is $$A'D_3 : K_3X_3 = A'L' : K_3I_3$$
$$= U'A' : U'K_3;$$

therefore the points $U'\ D_3\ X_3$ are collinear.

Similarly for the points $U'\ D_2\ X_2$.

(1) $VE\ \ V'E_1\ \ VE_2\ \ V'E_3$ intersect BY at $Y_0\ \ Y_1\ \ Y_2\ \ Y_3$; and

$WF\ \ W'F_1\ \ W'F_2\ \ WF_3$ intersect CZ at $Z_0\ \ Z_1\ \ Z_2\ \ Z_3$ such that

$$YY_0 = r \quad YY_1 = r_1 \quad YY_2 = r_2 \quad YY_3 = r_3$$
$$ZZ_0 = r \quad ZZ_1 = r_1 \quad ZZ_2 = r_2 \quad ZZ_3 = r_3.$$

(2) The four triangles $X_0Y_0Z_0$ $X_1Y_1Z_1$ $X_2Y_2Z_2$ $X_3Y_3Z_3$ are inversely similar to ABC; they have H, the orthocentre of ABC, for their common centre of homology, and $HI\ HI_1\ HI_2\ HI_3$ for the diameters of their circumcircles.

FIGURE 19.

Since $$XX_1 = r_1 = I_1D_1$$
therefore $\angle I_1X_1X$ is right.
Similarly $\angle I_1Y_1Y$ and $\angle I_1Z_1Z$ are right;
therefore the circle whose diameter is HI
passes through $X_1\ Y_1\ Z_1$.

Since X_1 Y_1 H Z_1 are concyclic

therefore $\angle Y_1X_1Z_1 = 180° - \angle Y_1HZ_1$

$= A$;

and $\angle X_1Y_1Z_1 = \angle X_1HZ_1 = B$,

since Z_1H X_1H are respectively perpendicular to AB BC ;

therefore triangle $X_1Y_1Z_1$ is similar to ABC.

(3) The mid points of HI HI_1 HI_2 HI_3 form an orthic tetrastigm similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

(4) The sum of the circumcircles of the four triangles $X_0Y_0Z_0 \cdots\cdot$ is four times the sum of the circumcircles of the three triangles

AYZ XBZ XYC.

It will be seen from a subsequent Section that the values * of $HI^2 \cdots\cdot$ may be written

$$HI^2 = 4(R^2 - 2Rr) + bc + ca + ab - (a^2 + b^2 + c^2)$$

$$HI_1^2 = 4(R^2 + 2Rr_1) + bc - ca - ab - (a^2 + b^2 + c^2)$$

$$HI_2^2 = 4(R^2 + 2Rr_2) - bc + ca - ab - (a^2 + b^2 + c^2)$$

$$HI_3^2 = 4(R^2 + 2Rr_3) - bc - ca + ab - (a^2 + b^2 + c^2)$$

Hence $\Sigma(HI^2) = 4(12R^2 - a^2 - b^2 - c^2)$

$= 4(HA^2 + HB^2 + HC^2)$

See *Proceedings of the Edinburgh Mathematical Society*, Vol. I., p. 63 (1894).

The statement that

$$HA^2 + HB^2 + HC^2 = 12R^2 - a^2 - b^2 - c^2$$

may be proved as follows.

FIGURE 20.

* The value of HI^2 is given by William Mawson in the *Lady's and Gentleman's Diary* for 1843, p. 75 ; the other values are given by William Rutherford and Samuel Bills in the *Diary* for 1844, p. 52.

The triangles CBZ AHZ are similar ;

therefore $BC^2 : HA^2 = CZ^2 : AZ^2$;

therefore $BC^2 + HA^2 : BC^2 = CZ^2 + AZ^2 : CZ^2$

$= CA^2 : CZ^2$;

therefore $BC^2 + HA^2 = \frac{BC^2 \cdot CA^2}{CZ^2}$

$= 4R^2$

by a theorem of Brahmegupta.

Similarly $CA^2 + HB^2 = AB^2 + HC^2 = 4R^2$

For another proof see Feuerbach, *Eigenschaften...des...Dreiecks*, Section VI., Theorem 2.

§ 7.

If A′I $A'I_1$ $A'I_2$ $A'I_3$ intersect AX at

X_0 X_1 X_2 X_3 then*

$AX_0 = r$ $AX_1 = r_1$ $AX_2 = r_2$ $AX_3 = r_3$

FIGURE 21.

Join CU, and draw the radius of the incircle IE.

Then $\angle UCA' = \angle IAE$;

therefore triangles CUA′ AIE are similar ;

therefore $CU : UA' = AI : IE$.

Now $CU : UA' = IU : UA'$

$= AI : AX_0$;

therefore $AX_0 = IE = r$

Similarly $AX_1 = r_1$.

* The first of these properties occurs incidentally in William Walker's proof of a theorem in the *Gentleman's Mathematical Companion* for 1803, p. 50.

If CU' be joined, and I_3E_3 the radius of the third excircle be drawn, then triangles $CU'A'$ AI_3E_3 will be similar, and since

$$CU' = I_3U',$$

it may be shown that $AX_3 = I_3E_3 = r_3$.

Corresponding to the four X points situated on AX, there will be four Y points, Y_0 Y_1 Y_2 Y_3, situated on BY, and four Z points, Z_0 Z_1 Z_2 Z_3 situated on CZ.

Some of the properties of this collection of points will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 89–96 (1894).

§ 8.

Figure 23.

If the medians AA' BB' CC' be intersected by

the radii			at the points		
DI	EI	FI	L_0	M_0	N_0
D_1I_1	E_1I_1	F_1I_1	L_1	M_1	N_1
D_2I_2	E_2I_2	F_2I_2	L_2	M_2	N_2
D_3I_3	E_3I_3	F_3I_3	L_3	M_3	N_3

then*

$$DL_0 = \frac{2\Delta}{b+c} \qquad EM_0 = \frac{2\Delta}{c+a} \qquad FN_0 = \frac{2\Delta}{a+b}$$

$$D_1L_1 = -\frac{2\Delta}{b+c} \qquad E_1M_1 = -\frac{2\Delta}{a-c} \qquad F_1N_1 = -\frac{2\Delta}{a-b}$$

$$D_2L_2 = -\frac{2\Delta}{b-c} \qquad E_2M_2 = -\frac{2\Delta}{c+a} \qquad F_2N_2 = \frac{2\Delta}{a-b}$$

$$D_3L_3 = \frac{2\Delta}{b-c} \qquad E_3M_3 = \frac{2\Delta}{a-c} \qquad F_3N_3 = -\frac{2\Delta}{a+b}$$

the distances of the L points from BC being considered positive when L is on the same side of BC as A, and negative when

* The first three values are given by W. H. Levy in the *Lady's and Gentleman's Diary* for 1859, p. 51.

it is on the opposite side from A. A similar convention holds for the M and the N points.

FIGURE 22.

Let AI meet BC at L; draw LS LT perpendicular to AC AB, and AX perpendicular to BC.

Then $A'L : A'D = A'D : A'X$
$= L_0D : AX$.

Now
$$\begin{aligned} A'L : A'D &= A'D - A'L : A'X - A'D \\ &= LD : DX \\ &= LI : IA \\ &= LB : BA \\ &= LT : AX; \end{aligned}$$

therefore $L_0D = LT = LS$

Again $LS \cdot AC + LT \cdot AB = 2ABC$

therefore $L_0D(b + c) = 2\triangle$

Similarly for the other equalities

(1) L_0 M_0 N_0 lie on EF FD DE; and similarly for L_1 M_1 N_1 .....

A proof of this will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 57–8 (1894).

(2)
$$\frac{1}{DL_0} + \frac{1}{D_1L_1} + \frac{1}{D_2L_2} + \frac{1}{D_3L_3}$$
$$= \frac{1}{EM_0} + \frac{1}{E_1M_1} + \frac{1}{E_2M_2} + \frac{1}{E_3M_3}$$
$$= \frac{1}{FN_0} + \frac{1}{F_1N_1} + \frac{1}{F_2N_2} + \frac{1}{F_3N_3} = 0$$

(3)* $$\frac{1}{DL_0}+\frac{1}{EM_0}+\frac{1}{FN_0}=\frac{2}{r}$$

$$\frac{1}{D_1L_1}+\frac{1}{E_1M_1}+\frac{1}{F_1N_1}=-\frac{2}{h_1}$$

$$\frac{1}{D_2L_2}+\frac{1}{E_2M_2}+\frac{1}{F_2N_2}=-\frac{2}{h_2}$$

$$\frac{1}{D_3L_3}+\frac{1}{E_3M_3}+\frac{1}{F_3N_3}=-\frac{2}{h_3}$$

(4) The diagonals of the following pairs of parallelograms

$DL_0D_1L_1$	$D_2L_2D_3L_3$	intersect	at	A′
$EM_0E_2M_2$	$E_3M_3E_1M_1$	,,	,,	B′
$FN_0F_3N_3$	$F_1N_1F_2N_2$	,,	,,	C′

(5) The four LMN triangles are homologous, and their centre of homology is G the centroid of ABC.

(6)† $$A'U : A'U - IL_0 = A'U : I_1L_1 - A'U = b+c : a$$
$$A'U' : I_2L_2 + A'U' = A'U' : I_3L_3 - A'U' = b-c : a$$

FIGURE 23.

From I U draw IE US perpendicular to AC.

Then $AS=\frac{1}{2}(b+c) \quad AE=\frac{1}{2}(b+c-a)$.

Now $$A'U : IL_0 = UA : IA$$
$$= AS : AE$$
$$= b+c : b+c-a ;$$

therefore $$A'U : A'U - IL_0 = b+c : a$$

(7) $$2A'U = IL_0 + I_1L_1$$
$$2A'U' = I_2L_2 - I_3L_3$$

therefore $$4R = IL_0 + I_1L_1 + I_2L_2 - I_3L_3$$

* The first result in (3) is given by W. H. Levy in the *Lady's and Gentleman's Diary* for 1858, p. 71.

† Of the four proportions in (6) the first is given by John Ryley, Leeds, in the *Gentleman's Mathematical Companion*, for 1802, p. 59. The solution in the text is that of J. H. Swale, Liverpool.

(8) If through $I \ I_1 \ I_2 \ I_3$ parallels be drawn to BC, meeting UU' in $K \ K_1 \ K_2 \ K_3$, then*

$$UK = UK_1 = US$$
$$U'K_2 = U'K_3 = U'S'$$

where $US \ U'S'$ are perpendicular to AC.

For the right-angled triangles CUS IUK are congruent,

since $$UC = UI,$$

and $$\angle CUS = \tfrac{1}{2}(B - C) = \angle IUK.$$

§ 9.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with each other.

The notation

$$AI = \alpha \quad AI = \beta \quad CI = \gamma, \quad \text{etc.},$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77, and adopted by Thomas Weddle in his admirable papers entitled "Symmetrical Properties of Plane Triangles," which appeared in the same publication (1843, 1845, 1848).

Neither Davies nor Weddle makes use of the equivalents for II_1, etc., namely $\alpha_1 - \alpha$, etc. Although the employment of these equivalents somewhat lengthens the formulae, it seems to me that it renders their symmetry a little more apparent.

In connection with the ascription, in the historical notes, of the great majority of the following formulae to Weddle, it is right to call attention to a letter of T. S. Davies in the *Lady's and Gentleman's Diary* for 1849, pp. 90-1, in which he states that when he undertook to arrange and systematise those properties of the triangle communicated to him, several sets of papers came into his hands, the most ample and elegant of which were those of Messrs Weddle and J. W. Elliott. The letter continues :

"I feel it to be due to him [Mr Elliott] to say that the names both of Mr Weddle and Mr Elliott might fairly have been prefixed to the far greater number of the properties, whilst each gentleman would have had a few properties designated as peculiar to himself."

I might have considerably shortened the lists of the formulae by giving only the leading identities, and referring the reader to Mr Lemoine's scheme of *continuous transformation*. I have done so here and there, but in general I have

---

* The property that $UK = US$ is referred to as well known in the *Gentleman's Mathematical Companion* for 1803, p. 50.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

either left or made the list complete in order that the reader who consults it may find ready to his hand the particular scrap of information of which he is in search. A table of logarithms which gave the logarithms of the prime numbers only would certainly be of some use, but merely to a select few.

Mr R. F. Muirhead suggests to me $t_1\ t_2\ t_3$ as a convenient mathematical translation of Mr Lemoine's *transformation continue en A, en B, en C.* Here is the t_1 applicable to the formulae which follow.

$$\begin{array}{cccccccccccccc} a & b & c & s & s_1 & s_2 & s_3 & r & r_1 & r_2 & r_3 & h_1 & h_2 & h_3 \end{array}$$

change into

$$\begin{array}{cccccccccccccc} a & -b & -c & -s_1 & -s & s_3 & s_2 & r_1 & r & -r_3 & -r_2 & -h_1 & h_2 & h_3 \end{array}$$

$$\begin{array}{ccccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{R} & \Delta & l_1 & l_2 & l_3 & \lambda_1 & \lambda_2 & \lambda_3 \end{array}$$

change into

$$\begin{array}{ccccccccccc} -\mathrm{A} & 180^\circ-\mathrm{B} & 180^\circ-\mathrm{C} & -\mathrm{R} & -\Delta & -l_1 & -\lambda_2 & -\lambda_3 & \lambda_1 & -l_2 & -l_3 \end{array}$$

$$\begin{array}{cccccccccccc} \alpha & \alpha_1 & \alpha_2 & \alpha_3 & \beta & \beta_1 & \beta_2 & \beta_3 & \gamma & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

change into

$$\begin{array}{cccccccccccc} -\alpha_1 & -\alpha & -\alpha_3 & -\alpha_2 & \beta_1 & \beta & -\beta_3 & -\beta_2 & \gamma_1 & \gamma & -\gamma_3 & -\gamma_2 \end{array}$$

$$\left.\begin{array}{ll} \alpha^2 = (r_2-r)(r_3-r) & \alpha_1^2 = (r_3+r_1)(r_1+r_2) \\ \beta^2 = (r_3-r)(r_1-r) & \beta_1^2 = (r_1+r_2)(r_1-r) \\ \gamma^2 = (r_1-r)(r_2-r) & \gamma_1^2 = (r_1-r)(r_3+r_1) \\ \alpha_2^2 = (r_2-r)(r_1+r_2) & \alpha_3^2 = (r_3+r_1)(r_3-r) \\ \beta_2^2 = (r_1+r_2)(r_2+r_3) & \beta_3^2 = (r_3-r)(r_2+r_3) \\ \gamma_2^2 = (r_2+r_3)(r_2-r) & \gamma_3^2 = (r_2+r_3)(r_3+r_1) \end{array}\right\} \quad (1)$$

Weddle remarks that these twelve equations along with the three in (55) of the r formulae* give the values of all the products of every two of the six quantities

$$r_1-r \quad r_2-r \quad r_3-r \quad r_2+r_3 \quad r_3+r_1 \quad r_1+r_2$$

* *Proceedings of the Edinburgh Mathematical Society,* Vol. XII., p. 98 (1894).

$$\left.\begin{array}{lll}
\alpha = \dfrac{\sqrt{bcrr_1}}{r_1} & \beta = \dfrac{\sqrt{carr_2}}{r_2} & \gamma = \dfrac{\sqrt{abrr_3}}{r_3} \\[2ex]
\alpha_1 = \dfrac{\sqrt{bcrr_1}}{r} & \beta_2 = \dfrac{\sqrt{carr_2}}{r} & \gamma_3 = \dfrac{\sqrt{abrr_3}}{r} \\[2ex]
\alpha_2 = \dfrac{\sqrt{bcr_2r_3}}{r_3} & \beta_3 = \dfrac{\sqrt{car_3r_1}}{r_1} & \gamma_1 = \dfrac{\sqrt{abr_1r_2}}{r_2} \\[2ex]
\alpha_3 = \dfrac{\sqrt{bcr_2r_3}}{r_2} & \beta_1 = \dfrac{\sqrt{car_3r_1}}{r_3} & \gamma_2 = \dfrac{\sqrt{abr_1r_2}}{r_1}
\end{array}\right\} \qquad (2)$$

$$\left.\begin{array}{ll}
\dfrac{\alpha}{r} = \dfrac{\alpha_1}{r_1} = \dfrac{\alpha_2}{s_3} = \dfrac{\alpha_3}{s_2} & \dfrac{\alpha}{s_1} = \dfrac{\alpha_1}{s} = \dfrac{\alpha_2}{r_2} = \dfrac{\alpha_3}{r_3} \\[2ex]
\dfrac{\beta}{r} = \dfrac{\beta_1}{s_3} = \dfrac{\beta_2}{r_2} = \dfrac{\beta_3}{s_1} & \dfrac{\beta}{s_2} = \dfrac{\beta_1}{r_1} = \dfrac{\beta_2}{s} = \dfrac{\beta_3}{r_3} \\[2ex]
\dfrac{\gamma}{r} = \dfrac{\gamma_1}{s_2} = \dfrac{\gamma_2}{s_1} = \dfrac{\gamma_3}{r_3} & \dfrac{\gamma}{s_3} = \dfrac{\gamma_1}{r_1} = \dfrac{\gamma_2}{r_2} = \dfrac{\gamma_3}{s}
\end{array}\right\} \qquad (3)$$

$$\alpha\alpha_1 = \alpha_2\alpha_3 = bc \quad \beta\beta_2 = \beta_3\beta_1 = ca \quad \gamma\gamma_3 = \gamma_1\gamma_2 = ab \qquad (4)$$

$$\alpha\alpha_1\alpha_2\alpha_3 = b^2c^2 \quad \beta\beta_1\beta_2\beta_3 = c^2a^2 \quad \gamma\gamma_1\gamma_2\gamma_3 = a^2b^2 \qquad (5)$$

$$\alpha\beta\gamma\alpha_1\beta_1\gamma_1\alpha_2\beta_2\gamma_2\alpha_3\beta_3\gamma_3 = (abc)^4 \qquad (6)$$

$$\left.\begin{array}{l}
\alpha\beta_1\gamma_3 = \alpha\beta_2\gamma_1 = \alpha_1\beta\gamma_2 = \alpha_1\beta_3\gamma \\
= \alpha_2\beta\gamma_3 = \alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2 = \alpha_3\beta_2\gamma = abc
\end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l}
\alpha\beta\gamma : abc = abc : \alpha_1\beta_2\gamma_3 \\
\alpha_1\beta_1\gamma_1 : abc = abc : \alpha\beta_3\gamma_2 \\
\alpha_2\beta_2\gamma_2 : abc = abc : \alpha_3\beta\gamma_1 \\
\alpha_3\beta_3\gamma_3 : abc = abc : \alpha_2\beta_1\gamma
\end{array}\right\} \qquad (8)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$
\left.\begin{aligned}
&\frac{\beta\gamma_2}{\alpha} = \frac{\beta_3\gamma}{\alpha} = \frac{\beta_1\gamma_3}{\alpha_1} = \frac{\beta_2\gamma_1}{\alpha_1} \\
&= \frac{\beta_1\gamma_2}{\alpha_2} = \frac{\beta_2\gamma}{\alpha_2} = \frac{\beta\gamma_3}{\alpha_3} = \frac{\beta_3\gamma_1}{\alpha_3} = a \\
&\frac{\alpha\gamma_1}{\beta} = \frac{\alpha_3\gamma}{\beta} = \frac{\alpha_1\gamma}{\beta_1} = \frac{\alpha_2\gamma_1}{\beta_1} \\
&= \frac{\alpha_1\gamma_2}{\beta_2} = \frac{\alpha_2\gamma_3}{\beta_2} = \frac{\alpha\gamma_3}{\beta_3} = \frac{\alpha_3\gamma_2}{\beta_3} = b \\
&\frac{\alpha\beta_1}{\gamma} = \frac{\alpha_2\beta}{\gamma} = \frac{\alpha_1\beta}{\gamma_1} = \frac{\alpha_3\beta_1}{\gamma_1} \\
&= \frac{\alpha\beta_2}{\gamma_2} = \frac{\alpha_2\beta_3}{\gamma_2} = \frac{\alpha_1\beta_3}{\gamma_3} = \frac{\alpha_3\beta_2}{\gamma_3} = c
\end{aligned}\right\} \tag{9}
$$

$$
\left.\begin{aligned}
&\alpha : \beta = \gamma : r_1 - r && \alpha_1 : \beta_1 = \gamma_1 : r_1 - r \\
&\beta : \alpha = \gamma : r_2 - r && \beta_1 : \alpha_1 = \gamma_1 : r_3 + r_1 \\
&\gamma : \alpha = \beta : r_3 - r && \gamma_1 : \alpha_1 = \beta_1 : r_1 + r_2 \\
&\alpha_2 : \beta_2 = \gamma_2 : r_2 + r_3 && \alpha_3 : \beta_3 = \gamma_3 : r_2 + r_3 \\
&\beta_2 : \alpha_2 = \gamma_2 : r_2 - r && \beta_3 : \alpha_3 = \gamma_3 : r_3 + r_1 \\
&\gamma_2 : \alpha_2 = \beta_2 : r_1 + r_2 && \gamma_3 : \alpha_3 = \beta_3 : r_3 - r
\end{aligned}\right\} \tag{10}
$$

$$
\left.\begin{aligned}
&\alpha_1 : \beta_2 = \gamma_3 : r_2 + r_3 && \alpha : \beta_3 = \gamma_2 : r_2 + r_3 \\
&\beta_2 : \alpha_1 = \gamma_3 : r_3 + r_1 && \beta_3 : \alpha = \gamma_2 : r_2 - r \\
&\gamma_3 : \alpha_1 = \beta_2 : r_1 + r_2 && \gamma_2 : \alpha = \beta_3 : r_3 - r \\
&\alpha_3 : \beta = \gamma_1 : r_1 - r && \alpha_2 : \beta_1 = \gamma : r_1 - r \\
&\beta : \alpha_3 = \gamma_1 : r_3 + r_1 && \beta_1 : \alpha_2 = \gamma : r_2 - r \\
&\gamma_1 : \alpha_3 = \beta : r_3 - r && \gamma : \alpha_2 = \beta_1 : r_1 + r_2
\end{aligned}\right\} \tag{11}
$$

$$
\left.\begin{aligned}
&\alpha^2 : bc = s_1 : s && \alpha_1^2 : bc = s : s_1 \\
&\beta^2 : ca = s_2 : s && \beta_1^2 : ca = s_3 : s_1 \\
&\gamma^2 : ab = s_3 : s && \gamma_1^2 : ab = s_2 : s_1 \\
&\alpha_2^2 : bc = s_3 : s_2 && \alpha_3^2 : bc = s_2 : s_3 \\
&\beta_2^2 : ca = s : s_2 && \beta_3^2 : ca = s_1 : s_3 \\
&\gamma_2^2 : ab = s_1 : s_2 && \gamma_3^2 : ab = s : s_3
\end{aligned}\right\} \tag{12}
$$

$$\left.\begin{aligned} \frac{\alpha^2}{bc}+\frac{\beta^2}{ca}+\frac{\gamma^2}{ab}&=1\\ \frac{\alpha_1^{\,2}}{bc}-\frac{\beta_1^{\,2}}{ca}-\frac{\gamma_1^{\,2}}{ab}&=1\\ -\frac{\alpha_2^{\,2}}{bc}+\frac{\beta_2^{\,2}}{ca}-\frac{\gamma_2^{\,2}}{ab}&=1\\ -\frac{\alpha_3^{\,2}}{bc}-\frac{\beta_3^{\,2}}{ca}+\frac{\gamma_3^{\,2}}{ab}&=1 \end{aligned}\right\}\quad(13)$$

$$\left.\begin{aligned} \frac{bc}{\alpha_1^{\,2}}+\frac{ca}{\beta_2^{\,2}}+\frac{ab}{\gamma_3^{\,2}}&=1\\ \frac{bc}{\alpha^2}-\frac{ca}{\beta_3^{\,2}}-\frac{ab}{\gamma_2^{\,2}}&=1\\ -\frac{bc}{\alpha_3^{\,2}}+\frac{ca}{\beta^2}-\frac{ab}{\gamma_1^{\,2}}&=1\\ -\frac{bc}{\alpha_2^{\,2}}-\frac{ca}{\beta_1^{\,2}}+\frac{ab}{\gamma^2}&=1 \end{aligned}\right\}\quad(14)$$

$$\left.\begin{aligned} \alpha^2\left(\frac{1}{c}-\frac{1}{b}\right)+\beta^2\left(\frac{1}{a}-\frac{1}{c}\right)+\gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)&=0\\ \alpha_1^{\,2}\left(\frac{1}{b}-\frac{1}{c}\right)+\beta_1^{\,2}\left(\frac{1}{a}+\frac{1}{c}\right)-\gamma_1^{\,2}\left(\frac{1}{a}+\frac{1}{b}\right)&=0 \end{aligned}\right\}\quad(15$$

$$\left.\begin{aligned} a\alpha^2(b-c)+b\beta^2(c-a)+c\gamma^2(a-b)&=0\\ a\alpha_1^{\,2}(c-b)+b\beta_1^{\,2}(c+a)+c\gamma_1^{\,2}(a+b)&=0 \end{aligned}\right\}\quad(16)$$

$$\left.\begin{aligned} \frac{b-c}{a\alpha_1^{\,2}}+\frac{c-a}{b\beta_2^{\,2}}+\frac{a-b}{c\gamma_3^{\,2}}&=0\\ \frac{c-b}{a\alpha^2}+\frac{c+a}{b\beta_3^{\,2}}-\frac{a+b}{c\gamma_2^{\,2}}&=0 \end{aligned}\right\}\quad(17)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}
\alpha^2+\beta^2+\gamma^2 &= \frac{bcs_1+cas_2+abs_3}{s} = bc+ca+ab-\frac{3abc}{s}\\
\alpha_1^2+\beta_1^2+\gamma_1^2 &= \frac{bcs+cas_3+abs_2}{s_1} = bc-ca-ab+\frac{3abc}{s_1}\\
\alpha_2^2+\beta_2^2+\gamma_2^2 &= \frac{bcs_3+cas+abs_1}{s_2} = -bc+ca-ab+\frac{3abc}{s_2}\\
\alpha_3^2+\beta_3^2+\gamma_3^2 &= \frac{bcs_2+cas_1+abs}{s_3} = -bc-ca+ab+\frac{3abc}{s_3}
\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}
\alpha_1^2+\beta_2^2+\gamma_3^2 &= (r_1+r_2+r_3)^2+s^2\\
\alpha^2+\beta_3^2+\gamma_2^2 &= (r-r_3-r_2)^2+s_1^2\\
\alpha_3^2+\beta^2+\gamma_1^2 &= (r-r_1-r_3)^2+s_2^2\\
\alpha_2^2+\beta_1^2+\gamma^2 &= (r-r_2-r_1)^2+s_3^2
\end{aligned}\right\} \quad (19)$$

Compare (15) of the r formulae*

$$\left.\begin{aligned}
\alpha\beta\gamma : abc &= r : s\\
\alpha_1\beta_1\gamma_1 : abc &= r_1 : s_1\\
\alpha_2\beta_2\gamma_2 : abc &= r_2 : s_2\\
\alpha_3\beta_3\gamma_3 : abc &= r_3 : s_3
\end{aligned}\right\} \quad (20)$$

Other proportions may be obtained by substituting for abc its equivalents in (7). Matthes (p. 49) gives

$$\alpha_3\beta_1\gamma_2 : \alpha\beta\gamma = \Delta : r^2$$

which may be reduced to

$$abc : \alpha\beta\gamma = s : r$$

$$\left.\begin{aligned}
\alpha_1\beta_2\gamma_3 : abc &= s : r\\
\alpha\beta_3\gamma_2 : abc &= s_1 : r_1\\
\alpha_3\beta\gamma_1 : abc &= s_2 : r_2\\
\alpha_2\beta_1\gamma : abc &= s_3 : r_3
\end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned}
h_1h_2h_3\alpha\beta\gamma &= 8\Delta^2r^2 \qquad & h_1h_2h_3\alpha_1\beta_2\gamma_3 &= 8\Delta^2s^2\\
h_1h_2h_3\alpha_1\beta_1\gamma_1 &= 8\Delta^2r_1^2 \qquad & h_1h_2h_3\alpha\beta_3\gamma_2 &= 8\Delta^2s_1^2
\end{aligned}\right\} \quad (22)$$

and so on.

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 91 (1894).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned} a\alpha^2 + b\beta^2 + c\gamma^2 &= \\ a\alpha_1^2 - b\beta_1^2 - c\gamma_1^2 &= \\ -a\alpha_2^2 + b\beta_2^2 - c\gamma_2^2 &= \\ -a\alpha_3^2 - b\beta_3^2 + c\gamma_3^2 &= abc \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} \alpha\beta\gamma &= (r_1 - r)(r_2 - r)(r_3 - r) \\ \alpha_1\beta_1\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_1 + r_2) \\ \alpha_2\beta_2\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_1 + r_2) \\ \alpha_3\beta_3\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_3 - r) \\ \alpha_1\beta_2\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_1 + r_2) \\ \alpha\beta_3\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_3 - r) \\ \alpha_3\beta\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_3 - r) \\ \alpha_2\beta_1\gamma &= (r_1 - r)(r_2 - r)(r_1 + r_2) \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} a\beta\beta_3 &= (r_2 + r_3)(r_3 - r)(r_1 - r) \\ a\gamma\gamma_2 &= (r_2 + r_3)(r_1 - r)(r_2 - r) \\ b\gamma\gamma_1 &= (r_3 + r_1)(r_1 - r)(r_2 - r) \\ b\alpha\alpha_3 &= (r_3 + r_1)(r_2 - r)(r_3 - r) \\ c\alpha\alpha_2 &= (r_1 + r_2)(r_2 - r)(r_3 - r) \\ c\beta\beta_1 &= (r_1 + r_2)(r_3 - r)(r_1 - r) \\ a\beta_1\beta_2 &= (r_1 - r)(r_1 + r_2)(r_2 + r_3) \\ a\gamma_3\gamma_1 &= (r_1 - r)(r_2 + r_3)(r_3 + r_1) \\ b\gamma_2\gamma_3 &= (r_2 - r)(r_2 + r_3)(r_3 + r_1) \\ b\alpha_1\alpha_2 &= (r_2 - r)(r_3 + r_1)(r_1 + r_2) \\ c\alpha_3\alpha_1 &= (r_3 - r)(r_3 + r_1)(r_1 + r_2) \\ c\beta_2\beta_3 &= (r_3 - r)(r_1 + r_2)(r_2 + r_3) \end{aligned}\right\} \tag{25}$$

Weddle remarks that (24) and (25) exhibit the twenty products of every three of the six quantities

$$r_1 - r, \quad r_2 - r, \quad r_3 - r, \quad r_2 + r_3, \quad r_3 + r_1, \quad r_1 + r_2$$

$$\left.\begin{array}{ll}
\alpha\,\alpha_2 = (r_2 - r)c & \alpha_3\,\alpha_1 = (r_3 + r_1)c \\
\alpha\,\alpha_3 = (r_3 - r)b & \alpha_1\,\alpha_2 = (r_1 + r_2)b \\
\beta\beta_3 = (r_3 - r)a & \beta_1\beta_2 = (r_1 + r_2)a \\
\beta\beta_1 = (r_1 - r)c & \beta_2\beta_3 = (r_2 + r_3)c \\
\gamma\gamma_1 = (r_1 - r)b & \gamma_2\gamma_3 = (r_2 + r_3)b \\
\gamma\gamma_2 = (r_2 - r)a & \gamma_3\gamma_1 = (r_3 + r_1)a
\end{array}\right\} \quad (26)$$

$$\left.\begin{array}{ll}
\alpha\,b = \alpha_3(r_2 - r) & \alpha_2 b = \alpha_1(r_2 - r) \\
\alpha\,c = \alpha_2(r_3 - r) & \alpha_2 c = \alpha\,(r_1 + r_2) \\
\beta\,c = \beta_1(r_3 - r) & \beta_2 c = \beta_3(r_1 + r_2) \\
\beta\,a = \beta_3(r_1 - r) & \beta_2 a = \beta_1(r_2 + r_3) \\
\gamma\,a = \gamma_2(r_1 - r) & \gamma_2 a = \gamma\,(r_2 + r_3) \\
\gamma\,b = \gamma_1(r_2 - r) & \gamma_2\,b = \gamma_3(r_2 - r) \\
\alpha_1 b = \alpha_2\,(r_3 + r_1) & \alpha_3\,b = \alpha\,(r_3 + r_1) \\
\alpha_1 c = \alpha_3\,(r_1 + r_2) & \alpha_3\,c = \alpha_1\,(r_3 - r) \\
\beta_1\,c = \beta\,(r_1 + r_2) & \beta_3\,c = \beta_2(r_3 - r) \\
\beta_1 a = \beta_2(r_1 - r) & \beta_3 a = \beta\,(r_2 + r_3) \\
\gamma_1 a = \gamma_3(r_1 - r) & \gamma_3 a = \gamma_1(r_2 + r_3) \\
\gamma_1\,b = \gamma\,(r_3 + r_1) & \gamma_3 b = \gamma_2(r_3 + r_1)
\end{array}\right\} \quad (27)$$

$$\left.\begin{array}{l}
(\alpha_1 - \alpha)r = \alpha\,(r_1 - r) = \beta\,\gamma \\
(\alpha_1 - \alpha)r_1 = \alpha_1(r_1 - r) = \beta_1\gamma_1 \\
(\beta_2 - \beta)r = \beta\,(r_2 - r) = \gamma\,\alpha \\
(\beta_2 - \beta)r_2 = \beta_2(r_2 - r) = \gamma_2\alpha_2 \\
(\gamma_3 - \gamma)r = \gamma\,(r_3 - r) = \alpha\,\beta \\
(\gamma_3 - \gamma)r_3 = \gamma_3(r_3 - r) = \alpha_3\beta_3 \\
(\alpha_2 + \alpha_3)r_2 = \alpha_2(r_2 + r_3) = \beta_2\gamma_2 \\
(\alpha_2 + \alpha_3)r_3 = \alpha_3(r_2 + r_3) = \beta_3\gamma_3 \\
(\beta_3 + \beta_1)r_3 = \beta_3(r_3 + r_1) = \gamma_3\alpha_3 \\
(\beta_3 + \beta_1)r_1 = \beta_1(r_3 + r_1) = \gamma_1\alpha_1 \\
(\gamma_1 + \gamma_2)r_1 = \gamma_1(r_1 + r_2) = \alpha_1\beta_1 \\
(\gamma_1 + \gamma_2)r_2 = \gamma_2(r_1 + r_2) = \alpha_2\beta_2
\end{array}\right\} \quad (28)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$
\left.
\begin{aligned}
(\alpha_1 - \alpha)s_2 &= \alpha_3(r_1 - r) = \beta\gamma_1 \\
(\alpha_1 - \alpha)s_3 &= \alpha_2(r_1 - r) = \beta_1\gamma \\
(\beta_2 - \beta)s_3 &= \beta_1(r_2 - r) = \gamma\alpha_2 \\
(\beta_2 - \beta)s_1 &= \beta_3(r_2 - r) = \gamma_2\alpha \\
(\gamma_3 - \gamma)s_1 &= \gamma_2(r_3 - r) = \alpha\beta_3 \\
(\gamma_3 - \gamma)s_2 &= \gamma_1(r_3 - r) = \alpha_3\beta \\
(\alpha_2 + \alpha_3)s &= \alpha_1(r_2 + r_3) = \beta_2\gamma_3 \\
(\alpha_2 + \alpha_3)s_1 &= \alpha(r_2 + r_3) = \beta_3\gamma_2 \\
(\beta_3 + \beta_1)s &= \beta_2(r_3 + r_1) = \gamma_3\alpha_1 \\
(\beta_3 + \beta_1)s_2 &= \beta(r_3 + r_1) = \gamma_1\alpha_3 \\
(\gamma_1 + \gamma_2)s &= \gamma_3(r_1 + r_2) = \alpha_1\beta_2 \\
(\gamma_1 + \gamma_2)s_3 &= \gamma(r_1 + r_2) = \alpha_2\beta_1
\end{aligned}
\right\} \quad (29)
$$

$$
\left.
\begin{aligned}
(\alpha_1 - \alpha)h_1 &= 2\alpha r_1 = 2\alpha_1 r = 2\alpha_2 s_2 = 2\alpha_3 s_3 \\
(\beta_2 - \beta)h_2 &= 2\beta r_2 = 2\beta_2 r = 2\beta_1 s_1 = 2\beta_3 s_3 \\
(\gamma_3 - \gamma)h_3 &= 2\gamma r_3 = 2\gamma_3 r = 2\gamma_1 s_1 = 2\gamma_2 s_2 \\
(\alpha_2 + \alpha_3)h_1 &= 2\alpha s = 2\alpha_1 s_1 = 2\alpha_2 r_3 = 2\alpha_3 r_2 \\
(\beta_3 + \beta_1)h_2 &= 2\beta s = 2\beta_2 s_2 = 2\beta_1 r_3 = 2\beta_3 r_1 \\
(\gamma_1 + \gamma_2)h_3 &= 2\gamma s = 2\gamma_3 s_3 = 2\gamma_1 r_2 = 2\gamma_2 r_1
\end{aligned}
\right\} \quad (30)
$$

$$
\left.
\begin{aligned}
(\alpha_1 - \alpha)a &= (\alpha_2 + \alpha_3)(r_1 - r) \\
(\beta_2 - \beta)b &= (\beta_3 + \beta_1)(r_2 - r) \\
(\gamma_3 - \gamma)c &= (\gamma_1 - \gamma_2)(r_3 - r) \\
(\alpha_2 + \alpha_3)a &= (\alpha_1 - \alpha)(r_2 + r_3) \\
(\beta_3 + \beta_1)b &= (\beta_2 - \beta)(r_3 + r_1) \\
(\gamma_1 + \gamma_2)c &= (\gamma_3 - \gamma)(r_1 + r_2)
\end{aligned}
\right\} \quad (31)
$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)b &= (\beta_2 - \beta)\gamma_1 = (\beta_3 + \beta_1)\gamma \\
(\alpha_1 - \alpha)c &= (\gamma_3 - \gamma)\beta_1 = (\gamma_1 + \gamma_2)\beta \\
(\beta_2 - \beta)c &= (\gamma_3 - \gamma)\alpha_2 = (\gamma_1 + \gamma_2)\alpha \\
(\beta_2 - \beta)a &= (\alpha_1 - \alpha)\gamma_2 = (\alpha_2 + \alpha_3)\gamma \\
(\gamma_3 - \gamma)a &= (\alpha_1 - \alpha)\beta_3 = (\alpha_2 + \alpha_3)\beta \\
(\gamma_3 - \gamma)b &= (\beta_2 - \beta)\alpha_3 = (\beta_3 + \beta_1)\alpha \\[1ex]
(\alpha_2 + \alpha_3)b &= (\beta_3 + \beta_1)\gamma_2 = (\beta_2 - \beta)\gamma_3 \\
(\alpha_2 + \alpha_3)c &= (\gamma_1 + \gamma_2)\beta_3 = (\gamma_3 - \gamma)\beta_2 \\
(\beta_3 + \beta_1)c &= (\gamma_1 + \gamma_2)\alpha_3 = (\gamma_3 - \gamma)\alpha_2 \\
(\beta_3 + \beta_1)a &= (\alpha_2 + \alpha_3)\gamma_1 = (\alpha_1 - \alpha)\gamma_3 \\
(\gamma_1 + \gamma_2)a &= (\alpha_2 + \alpha_3)\beta_1 = (\alpha_1 - \alpha)\beta_2 \\
(\gamma_1 + \gamma_2)b &= (\beta_3 + \beta_1)\alpha_2 = (\beta_2 - \beta)\alpha_1
\end{aligned}\right\} \quad (32)$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)\beta &= (\gamma_3 - \gamma)(r_1 - r) \\
(\alpha_1 - \alpha)\gamma &= (\beta_2 - \beta)(r_1 - r) \\
(\alpha_1 - \alpha)\beta_1 &= (\gamma_1 + \gamma_2)(r_1 - r) \\
(\alpha_1 - \alpha)\gamma_1 &= (\beta_3 + \beta_1)(r_1 - r) \\[1ex]
(\beta_2 - \beta)\gamma &= (\alpha_1 - \alpha)(r_2 - r) \\
(\beta_2 - \beta)\alpha &= (\gamma_3 - \gamma)(r_2 - r) \\
(\beta_2 - \beta)\gamma_2 &= (\alpha_2 + \alpha_3)(r_2 - r) \\
(\beta_2 - \beta)\alpha_2 &= (\gamma_1 + \gamma_2)(r_2 - r) \\[1ex]
(\gamma_3 - \gamma)\alpha &= (\beta_2 - \beta)(r_3 - r) \\
(\gamma_3 - \gamma)\beta &= (\alpha_1 - \alpha)(r_3 - r) \\
(\gamma_3 - \gamma)\alpha_3 &= (\beta_3 + \beta_1)(r_3 - r) \\
(\gamma_3 - \gamma)\beta_3 &= (\alpha_2 + \alpha_3)(r_3 - r)
\end{aligned}\right\} \quad (33)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}
(\alpha_2+\alpha_3)\beta_2&=(\gamma_1+\gamma_2)(r_2+r_3)\\
(\alpha_2+\alpha_3)\gamma_2&=(\beta_2-\beta\ )(r_2+r_3)\\
(\alpha_2+\alpha_3)\beta_3&=(\gamma_3-\gamma\ )(r_2+r_3)\\
(\alpha_2+\alpha_3)\gamma_3&=(\beta_3+\beta_1)(r_2+r_3)\\[4pt]
(\beta_3+\beta_1)\gamma_1&=(\alpha_1-\alpha\ )(r_3+r_1)\\
(\beta_3+\beta_1)\,\alpha_1&=(\gamma_1+\gamma_2)(r_3+r_1)\\
(\beta_3+\beta_1)\,\gamma_3&=(\alpha_2+\alpha_3)(r_3+r_1)\\
(\beta_3+\beta_1)\,\alpha_3&=(\gamma_3-\gamma_1)(r_3+r_1)\\[4pt]
(\gamma_1+\gamma_2)\,\alpha_1&=(\beta_3+\beta_1)(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_1&=(\alpha_1-\ \alpha\ )(r_1+r_2)\\
(\gamma_1+\gamma_2)\,\alpha_2&=(\beta_2-\beta\ )(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_2&=(\alpha_2+\ \alpha_3)(r_1+r_2)
\end{aligned}\right\}\quad(34$$

$$\left.\begin{aligned}
(\alpha_1-\alpha\ )r_2&=(\alpha_2+\alpha_3)s_3=a\alpha_2=\beta_1\gamma_2=\beta_2\gamma\\
(\alpha_1-\alpha\ )r_3&=(\alpha_2+\alpha_3)s_2=a\alpha_3=\beta\ \gamma_3=\beta_3\gamma_1\\
(\beta_2-\beta\ )r_3&=(\beta_3+\beta_1)s_1=b\beta_3=\gamma_3\alpha\ =\gamma_2\alpha_3\\
(\beta_2-\beta\ )r_1&=(\beta_3+\beta_1)s_3=b\beta_1=\gamma\,\alpha_1=\gamma_1\alpha_2\\
(\gamma_3-\gamma\ )r_1&=(\gamma_1+\gamma_2)s_2=c\,\gamma_1=\alpha_1\beta\ =\alpha_3\beta_1\\
(\gamma_3-\gamma\ )r_2&=(\gamma_1+\gamma_2)s_1=c\,\gamma_2=\alpha\ \beta_2=\alpha_2\beta_3\\[4pt]
(\alpha_2+\alpha_3)r\ &=(\alpha_1-\alpha\ )s_1=a\alpha\ =\beta\ \gamma_2=\beta_3\gamma\\
(\alpha_2+\alpha_3)r_1&=(\alpha_1-\alpha\ )s\ =a\,\alpha_1=\beta_1\gamma_3=\beta_2\gamma_1\\
(\beta_3+\beta_1)r\ &=(\beta_2-\beta\ )s_2=b\beta\ =\gamma_1\alpha\ =\gamma\ \alpha_3\\
(\beta_3+\beta_1)r_2&=(\beta_2-\beta\ )s\ =b\beta_2=\gamma_2\alpha_1=\gamma_3\alpha_2\\
(\gamma_1+\gamma_2)r\ &=(\gamma_3-\gamma\ )s_3=c\,\gamma\ =\alpha\ \beta_1=\alpha_2\beta\\
(\gamma_1+\gamma_2)r_3&=(\gamma_3-\gamma\ )s\ =c\,\gamma_3=\alpha_1\beta_3=\alpha_3\beta_2
\end{aligned}\right\}\quad(35$$

$$\left.\begin{aligned}
(\alpha_1-\alpha\ )^2 &= a^2+(r_1-r\ )^2=\frac{a^2bcrr_1}{\Delta^2}=\frac{a^2bc}{ss_1}\\
&=(r_1-r)\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}\\
(\beta_2-\beta\ )^2 &= b^2+(r_2-r\ )^2=\frac{ab^2crr_2}{\Delta^2}=\frac{ab^2c}{ss_2}\\
&=(r_2-r)\quad[\qquad\qquad\qquad]\\
(\gamma_3-\gamma\ )^2 &= c^2+(r_3-r\ )^2=\frac{abc^2rr_3}{\Delta^2}=\frac{abc^2}{ss_3}\\
&=(r_3-r)\quad[\qquad\qquad\qquad]\\
(\alpha_2+\alpha_3)^2 &= a^2+(r_2+r_3)^2=\frac{a^2bcr_2r_3}{\Delta^2}=\frac{a^2bc}{s_2s_3}\\
&=(r_2+r_3)\quad[\qquad\qquad\qquad]\\
(\beta_3+\beta_1)^2 &= b^2+(r_3+r_1)^2=\frac{ab^2cr_3r_1}{\Delta^2}=\frac{ab^2c}{s_3s_1}\\
&=(r_3+r_1)\quad[\qquad\qquad\qquad]\\
(\gamma_1+\gamma_2)^2 &= c^2+(r_1+r_2)^2=\frac{abc^2r_1r_2}{\Delta^2}=\frac{abc^2}{s_1s_2}\\
&=(r_1+r_2)\quad[\qquad\qquad\qquad]
\end{aligned}\right\}\qquad(36)$$

In Grunert's *Archiv*, XXIX., 436 (1857), Franz Unferdinger gives for $(\alpha_1-\alpha)^2$, etc., the values

$$r_1^2(r_2+r_3)\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{(r_2r_3+r_3r_1+r_1r_2)^2}\ \text{etc.}\qquad(37)$$

See (56) of the r formulae.*

$$\left.\begin{aligned}
&(\alpha_1-\alpha\ )^2+(\beta_2-\beta\ )^2+(\gamma_3-\gamma\ )^2\\
&+(\alpha_2+\alpha_3)^2+(\beta_3+\beta_1)^2+(\gamma_1+\gamma_2)^2\\
&\qquad=3(-r+r_1+r_2+r_3)^2
\end{aligned}\right\}\qquad(38)$$

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 98 (1894).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}
(\alpha_1-\alpha)^2 &= (\beta_3+\beta_1)\beta_1-(\beta_2-\beta)\beta \\
&= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma \\
(\beta_2-\beta)^2 &= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma \\
&= (\alpha_2+\alpha_3)\alpha_2-(\alpha_1-\alpha)\alpha \\
(\gamma_3-\gamma)^2 &= (\alpha_2+\alpha_3)\alpha_3-(\alpha_1-\alpha)\alpha \\
&= (\beta_3+\beta_1)\beta_3-(\beta_2-\beta)\beta_1 \\
(\alpha_2+\alpha_3)^2 &= (\beta_3+\beta_1)\beta_3+(\beta_2-\beta)\beta_2 \\
&= (\gamma_1+\gamma_2)\gamma_2+(\gamma_3-\gamma)\gamma_3 \\
(\beta_3+\beta_1)^2 &= (\gamma_1+\gamma_2)\gamma_1+(\gamma_3-\gamma)\gamma_3 \\
&= (\alpha_2+\alpha_3)\alpha_3+(\alpha_1-\alpha)\alpha_1 \\
(\gamma_1+\gamma_2)^2 &= (\alpha_2+\alpha_3)\alpha_2+(\alpha_1-\alpha)\alpha_1 \\
&= (\beta_3+\beta_1)\beta_1+(\beta_2-\beta)\beta_2
\end{aligned}\right\}\quad(39)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)(\alpha_2+\alpha_3) &= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_3 \\
&= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma_1)\gamma_2 \\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_1 \\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma_1)\gamma_1 \\
(\beta_2-\beta)(\beta_3+\beta_1) &= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma)\gamma_1 \\
&= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_3 \\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma)\gamma_2 \\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_2 \\
(\gamma_3-\gamma)(\gamma_1+\gamma_2) &= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_2 \\
&= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_1 \\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_3 \\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_3
\end{aligned}\right\}\quad(40)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma) : (\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2) &= r : s \\
(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_1+\gamma_2) : (\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma) &= r_1 : s_1 \\
(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_1+\gamma_2) : (\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma) &= r_2 : s_2 \\
(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_3-\gamma) : (\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2) &= r_3 : s_3
\end{aligned}\right\}\quad(41)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

By combining (41) with (8) and (20) other proportions may be obtained which it is needless to write down. T. S. Davies (in the *Ladies' Diary* for 1835, p. 53) gives one of them:

$$(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma):(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2)=\alpha\beta\gamma:abc \tag{42}$$

$$\left.\begin{aligned}&(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2)\\&\quad=(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma)+(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma)\\&\qquad+(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2)\end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned}&(\alpha_1-\alpha)\alpha s=(\beta_2-\beta)\beta s=(\gamma_3-\gamma)\gamma s\\&=(\alpha_2+\alpha_3)\alpha r_1=(\beta_3+\beta_1)\beta r_2=(\gamma_1+\gamma_2)\gamma r_3\\&=(\alpha_1-\alpha)\alpha_1 s_1=(\beta_2-\beta)\beta_1 r_3=(\gamma_3-\gamma)\gamma_1 r_2\\&=(\alpha_2+\alpha_3)\alpha_1 r=(\beta_3+\beta_1)\beta_1 s_1=(\gamma_1+\gamma_2)\gamma_1 s_1\\&=(\alpha_1-\alpha)\alpha_2 r_3=(\beta_2-\beta)\beta_2 s_2=(\gamma_3-\gamma)\gamma_2 r_1\\&=(\alpha_2+\alpha_3)\alpha_2 s_2=(\beta_3+\beta_1)\beta_2 r=(\gamma_1+\gamma_2)\gamma_2 s_2\\&=(\alpha_1-\alpha)\alpha_3 r_2=(\beta_2-\beta)\beta_3 r_1=(\gamma_3-\gamma)\gamma_3 s_3\\&=(\alpha_2+\alpha_3)\alpha_3 s_3=(\beta_3+\beta_1)\beta_3 s_3=(\gamma_1+\gamma_2)\gamma_3 r_3\\&=abc\end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned}\frac{1}{(\alpha_1-\alpha)^2}+\frac{1}{(\alpha_2+\alpha_3)^2}&=\frac{1}{a^2}\\\frac{1}{(\beta_2-\beta)^2}+\frac{1}{(\beta_3+\beta_1)^2}&=\frac{1}{b^2}\\\frac{1}{(\gamma_3-\gamma)^2}+\frac{1}{(\gamma_1+\gamma_2)^2}&=\frac{1}{c^2}\end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned}\frac{\alpha}{\alpha_1}+\frac{\beta}{\beta_2}+\frac{\gamma}{\gamma_3}=1 &\qquad \frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta_3}-\frac{\gamma_1}{\gamma_2}=1\\-\frac{\alpha_2}{\alpha_3}+\frac{\beta_2}{\beta}-\frac{\gamma_2}{\gamma_1}=1 &\qquad -\frac{\alpha_3}{\alpha_2}-\frac{\beta_3}{\beta_1}+\frac{\gamma_3}{\gamma}=1\end{aligned}\right\} \tag{46}$$

These equalities are merely particular cases of more general ones stated by Gergonne in his *Annales*, IX., 116, 284 (1818-9).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}
&\frac{\alpha_1-\alpha}{\alpha_1}+\frac{\beta_2-\beta}{\beta_2}+\frac{\gamma_3-\gamma}{\gamma_3}=2\\
-&\frac{\alpha_1-\alpha}{\alpha}+\frac{\beta_3+\beta_1}{\beta_3}+\frac{\gamma_1+\gamma_2}{\gamma_2}=2\\
&\frac{\alpha_2+\alpha_3}{\alpha_3}-\frac{\beta_2-\beta}{\beta}+\frac{\gamma_1+\gamma_2}{\gamma_1}=2\\
&\frac{\alpha_2+\alpha_3}{\alpha_2}+\frac{\beta_3+\beta_1}{\beta_1}-\frac{\gamma_3-\gamma}{\gamma}=2
\end{aligned}\right\}\qquad(47)$$

The first of these equations is a particular case of a theorem given by Vecten in Gergonne's *Annales*, IX., 277–9 (1819).

$$\left.\begin{aligned}
&\frac{1}{\alpha^2}+\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}+\frac{1}{\alpha_3^2}=\frac{4}{h_1^2}\\
&\frac{1}{\beta^2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}+\frac{1}{\beta_3^2}=\frac{4}{h_2^2}\\
&\frac{1}{\gamma^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\gamma_3^2}=\frac{4}{h_3^2}
\end{aligned}\right\}\qquad(48)$$

$$\Sigma\left(\frac{1}{\alpha^2}\right)+\Sigma\left(\frac{1}{\beta^2}\right)+\Sigma\left(\frac{1}{\gamma^2}\right)=\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\qquad(49)$$

See (35) of the r formulae.*

$$\left.\begin{aligned}
4\Delta_0 &= 2(\alpha_2+\alpha_3)\,\alpha_1 = 2(\beta_3+\beta_1)\,\beta_2 = 2(\gamma_1+\gamma_2)\,\gamma_3\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_1 &= 2(\alpha_2+\alpha_3)\,\alpha = 2(\beta_2-\beta)\,\beta_3 = 2(\gamma_3-\gamma)\,\gamma_2\\
&= -(\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_2 &= 2(\alpha_1-\alpha)\,\alpha_3 = 2(\beta_3+\beta_1)\,\beta = 2(\gamma_3-\gamma)\,\gamma_1\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_2)-(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_3 &= 2(\alpha_1-\alpha)\,\alpha_2 = 2(\beta_2-\beta)\,\beta_1 = 2(\gamma_1+\gamma_2)\,\gamma\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)-(\gamma_3-\gamma)(\gamma_1+\gamma_2)
\end{aligned}\right\}\qquad(50)$$

where Δ_0 Δ_1 Δ_2 Δ_3 denote

triangles $I_1I_2I_3$ II_3I_2 I_3II_1 I_2I_1I.

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 94 (1894).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

Historical Notes.

In 1841 the *Ladies' Diary*, which first appeared in 1704, and the *Gentleman's Diary*, which first appeared in 1741, were united and published under the title of the *Lady's and Gentleman's Diary*, which came to an end in 1871. This title will in the notes be shortened to *Diary*.

(1) The values of $\alpha_1 \; \beta_2 \; \gamma_3$ are given by J. Lowry in the *Ladies' Diary* for for 1836, p. 52; T. S. Davies adds six more in the *Diary* for 1842, p. 79; and Weddle completes the dozen by giving the values of $\alpha \; \beta \; \gamma$ in the *Diary* for 1843, p. 80.

(2) C. J. Matthes in his *Commentatio de Proprietatibus Quinque Circulorum*, pp. 46, 49 (1831).

(3) Weddle in the *Diary* for 1843, p. 81.

(4) Lhuilier in his *Élémens d'Analyse*, p. 215 (1809). The values of $\alpha\alpha_1 \; \beta\beta_2 \; \gamma\gamma_3$ were however given by J. Lowry in Leybourn's *Mathematical Repository*, old series, I. 394 (1799).

(5) T. T. Wilkinson in *Mathematical Questions from the Educational Times*, XIX. 107 (1873).

(6) C. Adams in *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 36 (1846).

(7)—(12) Weddle in the *Diary* for 1843, pp. 81, 82, 88. The first three proportions of (12) are however implicitly given by Matthes in his *Commentatio*, p. 46 (1831).

(13) The first property was proposed for proof at the *Concours Académique de Clermont*, 1875; the others were given by Mr H. Van Aubel in *Nouvelle Correspondance Mathématique*, IV. 364 (1878).

(14), (17) First property given in Todhunter's *Plane Trigonometry*, Chap. XVI., Ex. 37 (1859).

(15), (18) First property given in Hind's *Trigonometry*, 4th ed., pp. 304, 309 (1841).

(19) First property given in a slightly different form by Adams in his *Eigenschaften des...Dreiecks*, p. 40 (1846).

(20) First property given by C. F. A. Jacobi in his *De Triangulorum Rectilineorum Proprietatibus*, p. 10 (1825).

(21) First proportion given by J. Lowry in the *Ladies' Diary* for 1836, p. 52.

(22) First property on the left side given by Adams in his *Eigenschaften des ...Dreiecks*, p. 62 (1846).

(23) The first property was proposed for proof at the *Concours Académique de Clermont*, 1875. A geometrical solution of it occurs in Bourget's *Journal de Mathématiques Élémentaires*, II. 54-5 (1878).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

(24)—(26) Weddle in the *Diary* for 1845, p. 69.

(27)—(29) ,, ,, ,, ,, ,, ,, p. 70.

(30) ,, ,, ,, ,, ,, ,, p. 74.

(31)—(35) ,, ,, ,, ,, ,, ,, p. 71.

(36) The first values of $(a_1 - a)^2$, etc., occur in the *Diary* for 1847, pp. 49-50, in answer to a question proposed the previous year by the editor, W. S. B. Woolhouse. The second values are given by Matthes in his *Commentatio*, pp. 53-4 (1831); the third values by Weddle in the *Diary* for 1845, p. 74. The last values of $(a_2 + a_3)^2$, etc., are given by Franz Unferdinger in Grunert's *Archiv*, XXIX., 436 (1857).

(38) Weddle in the *Diary* for 1843, p. 83.

(39), (40) ,, ,, ,, ,, ,, 1845, p. 73.

(41) The first proportion is given by Adams in his *Eigenschaften des...Dreiecks*, p. 34 (1846). All four follow at once from eight expressions given by Weddle in the *Diary* for 1843, p. 82.

(43) Weddle in the *Diary* for 1843, p. 82.

(45) ,, ,, ,, ,, ,, ,, p. 83.

(46) ,, ,, ,, ,, ,, 1845, p. 76.

(47) J. W. Elliott in the *Diary* for 1847, p. 73.

(48) Weddle ,, ,, ,, ,, 1845, p. 75.

(49) ,, ,, ,, ,, ,, 1845, p. 76.

(50) ,, ,, ,, ,, ,, ,, pp. 72, 75.

§ 10.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with the sides.

The uniliteral notation for these bisectors

$$l_1 \quad l_2 \quad l_3 \quad \lambda_1 \quad \lambda_2 \quad \lambda_3$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77. In the expressions for them it has been assumed that the sides BC CA AB are in decreasing order of magnitude. Hence it will follow that

BL is less than CL, and BL′ is less than CL′
CM is greater than AM, and CM′ is greater than AM′
AN is less than BN, and AN′ is less than BN′.

The assumption "causes the sign of λ_2 (corresponding to the mean side b) to be contrary to those of λ_1 and λ_3. This must be borne in mind, otherwise the symmetry of the expressions in which these functions $(\lambda_1\ \lambda_2\ \lambda_3)$ are involved will not be seen." (Weddle in the *Diary* for 1848, p. 76.)

Two fundamental theorems* regarding two sides of a triangle and the bisectors of the angles between them give the following proportions:

$$b : c = u_2 : u_1 = u_2' : u_1'$$
$$bc = u_1u_2 + l_1^2 = u_1'u_2' - \lambda_1^2$$

Hence are derived

$$\left.\begin{aligned} b^2 &= u_2^2 + l_1^2 \cdot \frac{u_2}{u_1} = u_2'^2 - \lambda_1^2 \cdot \frac{u_2'}{u_1'} \\ c^2 &= u_1^2 + l_1^2 \cdot \frac{u_1}{u_2} = u_1'^2 - \lambda_1^2 \cdot \frac{u_1'}{u_2'} \end{aligned}\right\} \quad (1)$$

Segments of the sides in terms of the sides

$$\left.\begin{aligned} u_1 &= \frac{ca}{b+c} & v_1 &= \frac{ab}{c+a} & w_1 &= \frac{bc}{a+b} \\ u_1' &= \frac{ca}{b-c} & v_1' &= \frac{ab}{a-c} & w_1' &= \frac{bc}{a-b} \\ u_2 &= \frac{ab}{b+c} & v_2 &= \frac{bc}{c+a} & w_2 &= \frac{ca}{a+b} \\ u_2' &= \frac{ab}{b-c} & v_2' &= \frac{bc}{a-c} & w_2' &= \frac{ca}{a-b} \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} u_1' + u_1 &= u_2' - u_2 = \mathrm{LL}' = \frac{2abc}{b^2 - c^2} \\ v_2' + v_2 &= v_1' - v_1 = \mathrm{MM}' = \frac{2abc}{a^2 - c^2} \\ w_1' + w_1 &= w_2' - w_2 = \mathrm{NN}' = \frac{2abc}{a^2 - b^2} \end{aligned}\right\} \quad (3)$$

* The first is Euclid VI. 3 and its extension, which also was known to the Greeks, as is evident from Pappus's *Mathematical Collection*, VII. 39, second proof. The first part of the second fundamental theorem is given in Schooten's *Exercitationes Mathematicae*, p. 65 (1657).

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\frac{1}{LL'} - \frac{1}{MM'} + \frac{1}{NN'} = 0 \qquad (4)$$

$$\frac{a^2}{LL'} - \frac{b^2}{MM'} + \frac{c^2}{NN'} = 0 \qquad (5)$$

$$\frac{a}{LL'} - \frac{b}{MM'} + \frac{c}{NN'} = \frac{(b-c)(c-a)(a-b)}{2abc} \qquad (6)$$

The segments of the sides in terms of each other.

$$\left.\begin{array}{ll} u_1 = u_1'\dfrac{u_2' - u_1'}{u_2' + u_1'} & u_2 = u_2'\dfrac{u_2' - u_1'}{u_2' + u_1'} \\ \text{and so on.} & \\ u_1' = u_1\dfrac{u_2 + u_1}{u_2 - u_1} & u_2' = u_2\dfrac{u_2 + u_1}{u_2 - u_1} \\ \text{and so on.} & \end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l} u_1' + u_1 = u_2' - u_2 = LL' = \dfrac{2u_1u_2}{u_2 - u_1} \\ \text{and so on.} \end{array}\right\} \qquad (8)$$

$$LL'^2 = l_1^2 + \lambda_1^2 \qquad MM'^2 = l_2^2 + \lambda_2^2 \qquad NN'^2 = l_3^2 + \lambda_3^2 \qquad (9)$$

$$\left.\begin{array}{l} l_1 = \dfrac{2\sqrt{bcss_1}}{(b+c)} = \dfrac{2\Delta\sqrt{bcrr_1}}{(b+c)rr_1} \\ l_2 = \dfrac{2\sqrt{cass_2}}{(c+a)} = \dfrac{2\Delta\sqrt{carr_2}}{(c+a)rr_2} \\ l_3 = \dfrac{2\sqrt{abss_3}}{(a+b)} = \dfrac{2\Delta\sqrt{abrr_3}}{(a+b)rr_3} \end{array}\right\} \qquad (10)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}\lambda_1&=\frac{2\sqrt{bcs_2s_3}}{(b-c)}=\frac{2\Delta\sqrt{bcr_2r_3}}{(b-c)r_2r_3}\\ \lambda_2&=\frac{2\sqrt{cas_3s_1}}{(a-c)}=\frac{2\Delta\sqrt{car_3r_1}}{(a-c)r_3r_1}\\ \lambda_3&=\frac{2\sqrt{abs_1s_2}}{(a-b)}=\frac{2\Delta\sqrt{abr_1r_2}}{(a-b)r_1r_2}\end{aligned}\right\}\quad(11)$$

$$\left.\begin{aligned}\frac{l_1^2}{bc}+\frac{a^2}{(b+c)^2}&=1\\ \frac{l_2^2}{ca}+\frac{b^2}{(c+a)^2}&=1\\ \frac{l_3^2}{ab}+\frac{c^2}{(a+b)^2}&=1\end{aligned}\right\}\quad(12)\qquad\left.\begin{aligned}\frac{a^2}{(b-c)^2}-\frac{\lambda_1^2}{bc}&=1\\ \frac{b^2}{(a-c)^2}-\frac{\lambda_2^2}{ca}&=1\\ \frac{c^2}{(a-b)^2}-\frac{\lambda_3^2}{ab}&=1\end{aligned}\right\}\quad(13)$$

$$\left.\begin{aligned}l_1^2(b+c)^2+\lambda_1^2(b-c)^2&=4b^2c^2\\ l_2^2(c+a)^2+\lambda_2^2(a-c)^2&=4c^2a^2\\ l_3^2(a+b)^2+\lambda_3^2(a-b)^2&=4a^2b^2\end{aligned}\right\}\quad(14)$$

$$\left.\begin{aligned}\frac{l_1^2}{bc}(b+c)^2+\frac{l_2^2}{ca}(c+a)^2+\frac{l_3^2}{ab}(a+b)^2&=4s^2\\ \frac{l_1^2}{bc}(b+c)^2-\frac{\lambda_2^2}{ca}(a-c)^2-\frac{\lambda_3^2}{ab}(a-b)^2&=4s_1^2\\ -\frac{\lambda_1^2}{bc}(b-c)^2+\frac{l_2^2}{ca}(c+a)^2-\frac{\lambda_3^2}{ab}(a-b)^2&=4s_2^2\\ -\frac{\lambda_1^2}{bc}(b-c)^2-\frac{\lambda_2^2}{ca}(a-c)^2+\frac{l_3^2}{ab}(a+b)^2&=4s_3^2\end{aligned}\right\}\quad(15)$$

$$l_1^2bc\left(\frac{1}{b}+\frac{1}{c}\right)^2+l_2^2ca\left(\frac{1}{c}+\frac{1}{a}\right)^2+l_3^2ab\left(\frac{1}{a}+\frac{1}{b}\right)^2=4s^2\quad(16)$$

and so on.

$$\left.\begin{aligned}\frac{l_1^2}{a^2}\cdot\frac{1}{bc}\left(\frac{1}{b}+\frac{1}{c}\right)^2+\frac{l_2^2}{b^2}\cdot\frac{1}{ca}\left(\frac{1}{c}+\frac{1}{a}\right)^2+\frac{l_3^2}{c^2}\cdot\frac{1}{ab}\left(\frac{1}{a}+\frac{1}{b}\right)^2\\ =\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)^2\end{aligned}\right\}\quad(17)$$

and so on.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{b^2-c^2}{l_2^2l_3^2}+\left(\frac{1}{c}+\frac{1}{a}\right)\frac{c^2-a^2}{l_3^2l_1^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2-b^2}{l_1^2l_2^2}=0 \qquad (18)$$

$$\frac{\frac{1}{b^2}-\frac{1}{c^2}}{s_2s_3}\cdot\frac{b+c}{l_2^2l_3^2}+\frac{\frac{1}{c^2}-\frac{1}{a^2}}{s_3s_1}\cdot\frac{c+a}{l_3^2l_1^2}+\frac{\frac{1}{a^2}-\frac{1}{b^2}}{s_1s_2}\cdot\frac{a+b}{l_1^2l_2^2}=0 \qquad (19)$$

$$\left.\begin{aligned} u_1u_2+v_1v_2+w_1w_2 &= abc\left\{\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right\} \\ u_1'u_2'+v_1'v_2'+w_1'w_2' &= abc\left\{\frac{a}{(b-c)^2}+\frac{b}{(a-c)^2}+\frac{c}{(a-b)^2}\right\} \end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned} u_1u_2+v_1v_2+w_1w_2+(l_1^2+l_2^2+l_3^2) &= bc+ca+ab \\ u_1'u_2'+v_1'v_2'+w_1'w_2'-(\lambda_1^2+\lambda_2^2+\lambda_3^2) &= bc+ca+ab \end{aligned}\right\} \qquad (21)$$

$$l_1\alpha+l_2\beta+l_3\gamma=av_1+bw_1+cu_1 \qquad (22)$$

$$\left.\begin{aligned} &l_1(l_1-\alpha)+l_2(l_2-\beta)+l_3(l_3-\gamma) \\ &\quad=(a-v_1)v_2+(b-w_1)w_2+(c-u_1)u_2 \end{aligned}\right\} \qquad (23)$$

$$\left.\begin{aligned} &\alpha^2+\beta^2+\gamma^2-\{(l_1-\alpha)^2+(l_2-\beta)^2+(l_3-\gamma)^2\} \\ &\quad=(u_1+v_1+w_1)(u_2+v_2+w_2)-2(u_1v_2+v_1w_2+w_1u_2) \end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned} \frac{1}{u_1v_1w_1} &= \left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right) \\ \frac{1}{u_1v_1'w_1'} &= \left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right) \\ \frac{1}{u_1'v_1w_1'} &= \left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right) \\ \frac{1}{u_1'v_1'w_1} &= \left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right) \end{aligned}\right\} \qquad (25)$$

These may be put into the forms

$$u_1v_1w_1 : abc = abc : (b+c)(c+a)(a+b)$$

and so on ; or

$$\mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} : abc = abc : \mathrm{D_2D_3}\cdot\mathrm{E_3E_1}\cdot\mathrm{F_1F_2}$$

and so on.

$$\left.\begin{aligned}
u_1\, v_1\, w_1 = u_2\, v_2\, w_2 &= \frac{4\Delta Rr}{h_1 + h_2 + h_3 - r}\\
u_1\, v_1' w_1' = u_2\, v_2' w_2' &= \frac{4\Delta Rr_1}{h_1 - h_2 - h_3 + r_1}\\
u_1' v_1\, w_1' = u_2' v_2\, w_2' &= \frac{4\Delta Rr_2}{h_1 - h_2 + h_3 - r_2}\\
u_1' v_1' w_1 = u_2' v_2' w_2 &= \frac{4\Delta Rr_3}{-h_1 - h_2 + h_3 + r_3}
\end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned}
u_1' v_1' w_1' = u_2' v_2' w_2' &= \frac{a^2b^2c^2}{(b-c)(a-c)(a-b)}\\
u_1' v_1\, w_1 = u_2' v_2\, w_2 &= \frac{a^2b^2c^2}{(b-c)(c+a)(a+b)}\\
u_1\, v_1' w_1 = u_2\, v_2' w_2 &= \frac{a^3b^2c^2}{(b+c)(a-c)(a+b)}\\
u_1\, v_1\, w_1' = u_2\, v_2\, w_2' &= \frac{a^2b^2c^2}{(b+c)(c+a)(a-b)}
\end{aligned}\right\} \quad (27)$$

These may be put into the forms

$$BL' \cdot CM' \cdot AN' : abc = abc : DD_1 \cdot EE_2 \cdot FF_3$$

and so on.

$$\left.\begin{aligned}
l_1 l_2 l_3 &= \frac{8abcs\Delta}{(b+c)(c+a)(a+b)} & \lambda_1\lambda_2\lambda_3 &= \frac{8abcr\Delta}{(b-c)(a-c)(a-b)}\\
l_1\lambda_2\lambda_3 &= \frac{8abcs_1\Delta}{(b+c)(a-c)(a-b)} & \lambda_1 l_2 l_3 &= \frac{8abcr_1\Delta}{(b-c)(c+a)(a+b)}\\
\lambda_1 l_2 \lambda_3 &= \frac{8abcs_2\Delta}{(b-c)(c+a)(a-b)} & l_1 \lambda_2 l_3 &= \frac{8abcr_2\Delta}{(b+c)(a-c)(a+b)}\\
\lambda_1\lambda_2 l_3 &= \frac{8abcs_3\Delta}{(b-c)(a-c)(a+b)} & l_1 l_2 \lambda_3 &= \frac{8abcr_3\Delta}{(b+c)(c+a)(a-b)}
\end{aligned}\right\} \quad (28)$$

$$l_1 l_2 l_3 = \frac{32R\Delta^3}{r(b+c)(c+a)(a+b)} = \frac{8\Delta^2}{h_1 + h_2 + h_3 - r} \quad (29)$$

and so on.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\lambda_1\lambda_2\lambda_3 = \frac{32R\Delta}{s(b-c)(a-c)(a-b)}\right\} \quad (30)$$

and so on.

$$BL \cdot CM \cdot AN : l_1 l_2 l_3 = R : 2s \quad (31)$$

$$BL' \cdot CM' \cdot AN' : \lambda_1\lambda_2\lambda_3 = R : 2r \quad (32)$$

$$\left.\begin{aligned} &l_1l_2l_3(b+c)(c+a)(a+b) \\ &= 8\alpha\beta\gamma s^3 = 8\alpha_1\beta_1\gamma_1 s s_1^2 = 8\alpha_2\beta_2\gamma_2 s s_2^2 = 8\alpha_3\beta_3\gamma_3 s s_3^2 \\ &= 8\alpha_1\beta_2\gamma_3 s r^2 = 8\alpha\beta_3\gamma_2 s r_1^2 = 8\alpha_3\beta\gamma_1 s r_2^2 = 8\alpha_2\beta_1\gamma s r_3^2 \end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned} &\lambda_1\lambda_2\lambda_3(b-c)(a-c)(a-b) \\ &= 8\alpha_1\beta_2\gamma_3 r^3 = 8\alpha\beta_3\gamma_2 r r_1^2 = 8\alpha_3\beta\gamma_1 r r_2^2 = 8\alpha_2\beta_1\gamma r r_3^2 \\ &= 8\alpha\beta\gamma r s^2 = 8\alpha_1\beta_1\gamma_1 r s_1^2 = 8\alpha_2\beta_2\gamma_2 r s_2^2 = 8\alpha_3\beta_3\gamma_3 r s_3^2 \end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned} l_1l_2l_3\lambda_1\lambda_2\lambda_3 &= \frac{64a^2b^2c^2\Delta^3}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \\ &= \frac{1024R^2\Delta^5}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \end{aligned}\right\} \quad (35)$$

$$\frac{2a}{l_1\lambda_1} = \frac{h_3^2-h_2^2}{h_1h_2h_3} \qquad \frac{2b}{l_2\lambda_2} = \frac{h_3^2-h_1^2}{h_1h_2h_3} \qquad \frac{2c}{l_3\lambda_3} = \frac{h_2^2-h_1^2}{h_1h_2h_3} \quad (36)$$

$$\frac{2h_1}{l_1\lambda_1} = \frac{b^2-c^2}{abc} \qquad \frac{2h_2}{l_2\lambda_2} = \frac{a^2-c^2}{abc} \qquad \frac{2h_3}{l_3\lambda_3} = \frac{a^2-b^2}{abc} \quad (37)$$

$$l_1\lambda_1 = \frac{4bc\Delta}{b^2-c^2} \qquad l_2\lambda_2 = \frac{4ca\Delta}{a^2-c^2} \qquad l_3\lambda_3 = \frac{4ab\Delta}{a^2-b^2} \quad (38)$$

$$\frac{a}{l_1\lambda_1} - \frac{b}{l_2\lambda_2} + \frac{c}{l_3\lambda_3} = 0 \quad (39)$$

$$\frac{h_1}{l_1\lambda_1} - \frac{h_2}{l_2\lambda_2} + \frac{h_3}{l_3\lambda_3} = 0 \quad (40)$$

$$\frac{1}{al_1\lambda_1} - \frac{1}{bl_2\lambda_2} + \frac{1}{cl_3\lambda_3} = 0 \qquad (41)$$

$$\frac{1}{h_1l_1\lambda_1} - \frac{1}{h_2l_2\lambda_2} + \frac{1}{h_3l_3\lambda_3} = 0 \qquad (42)$$

$$\left.\begin{array}{l}
\alpha\ : l_1 = r\,(b+c) : 2\triangle = b+c\ : 2s \\
\beta\ : l_2 = r\,(c+a) : 2\triangle = c+a : 2s \\
\gamma\ : l_3 = r\,(a+b) : 2\triangle = a+b : 2s \\[4pt]
\alpha_1 : l_1 = r_1(b+c) : 2\triangle = b+c : 2s_1 \\
\beta_2 : l_2 = r_2(c+a) : 2\triangle = c+a\ : 2s_2 \\
\gamma_3 : l_3 = r_3(a+b) : 2\triangle = a+b : 2s_3 \\[4pt]
\alpha_2 : \lambda_1 = r_2(b-c)\ : 2\triangle = b-c\ : 2s_2 \\
\alpha_3 : \lambda_1 = r_3(b-c)\ : 2\triangle = b-c\ : 2s_3 \\[4pt]
\beta_3 : \lambda_2 = r_3(a-c)\ : 2\triangle = a-c : 2s_3 \\
\beta_1 : \lambda_2 = r_1(a-c)\ : 2\triangle = a-c : 2s_1 \\[4pt]
\gamma_1 : \lambda_3 = r_1(a-b) : 2\triangle = a-b\ : 2s_1 \\
\gamma_2 : \lambda_3 = r_3(a-b) : 2\triangle = a-b : 2s_2
\end{array}\right\} \qquad (43)$$

$$\left.\begin{array}{l}
\alpha\ : \mathrm{I\,L}\ = b+c\ : a \\
\beta\ : \mathrm{I\,M}\ = c+a\ : b \\
\gamma\ : \mathrm{I\,N}\ = a+b\ : c \\[4pt]
\alpha_1 : \mathrm{I_1L}\ = b+c\ : a \\
\beta_2 : \mathrm{I_2M}\ = c+a\ : b \\
\gamma_3 : \mathrm{I_3N}\ = a+b\ : c \\[4pt]
\alpha_2 : \mathrm{I_2L'} = b-c\ : a \\
\alpha_3 : \mathrm{I_3L'} = b-c\ : a \\[4pt]
\beta_3 : \mathrm{I_3M'} = a-c\ : b \\
\beta_1 : \mathrm{I_1M'} = a-c\ : b \\[4pt]
\gamma_1 : \mathrm{I_1N'} = a-b\ : c \\
\gamma_2 : \mathrm{I\ N'} = a-b\ : c
\end{array}\right\} (44)
\qquad
\left.\begin{array}{l}
l_1 : \mathrm{I\,L}\ = 2s\ : a \\
l_2 : \mathrm{I\,M}\ = 2s\ : b \\
l_3 : \mathrm{I\,N}\ = 2s\ : c \\[4pt]
l_1 : \mathrm{I_1L}\ = 2s_1 : a \\
l_2 : \mathrm{I_2M}\ = 2s_2 : b \\
l_3 : \mathrm{I_3N}\ = 2s_3 : c \\[4pt]
\lambda_1 : \mathrm{I_2L'}\ = 2s_2 : a \\
\lambda_1 : \mathrm{I_3L'}\ = 2s_3 : a \\[4pt]
\lambda_2 : \mathrm{I_3M'} = 2s_3 : b \\
\lambda_2 : \mathrm{I_1M'} = 2s_1 : b \\[4pt]
\lambda_3 : \mathrm{I_1N'}\ = 2s_1 : c \\
\lambda_3 : \mathrm{I_2N'}\ = 2s_2 : c
\end{array}\right\} (45)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

Values such as

$$\left.\begin{array}{l} l_1 = \dfrac{2\Delta}{r} \cdot \dfrac{a}{D_2D_3} = \dfrac{2\Delta}{r_1} \cdot \dfrac{a_1}{D_2D_3} \\ \cdots\cdots\cdots\cdots\cdots \\ \lambda_1 = \dfrac{2\Delta}{r_2} \cdot \dfrac{a_2}{DD_1} = \dfrac{2\Delta}{r_3} \cdot \dfrac{a_3}{DD_1} \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\} \quad (46)$$

$$\left.\begin{array}{ll} IL = \dfrac{a\sqrt{bcrr_1}}{(b+c)r_1} & I_1L = \dfrac{a\sqrt{bcrr_1}}{(b+c)r} \\ \cdots\cdots\cdots\cdots\cdots & \\ I_2L' = \dfrac{a\sqrt{bcr_2r_3}}{(b-c)r_3} & I_3L' = \dfrac{a\sqrt{bcr_2r_3}}{(b-c)r_2} \\ \cdots\cdots\cdots\cdots\cdots & \end{array}\right\} \quad (47)$$

need not be written out at length.

$$\left.\begin{array}{l} IL \cdot IM \cdot IN = \dfrac{16\Delta R^2r^2}{(b+c)(c+a)(a+b)} \\ I_1L \cdot I_2M \cdot I_3N = \dfrac{16\Delta R^2s^2}{(b+c)(c+a)(a+b)} \\ I_1L \cdot I_1M' \cdot I_1N' = \dfrac{16\Delta R^2r_1^2}{(b+c)(a-c)(a-b)} \\ IL \cdot I_3M' \cdot I_2N' = \dfrac{16\Delta R^2s_1^2}{(b+c)(a-c)(a-b)} \\ I_2L' \cdot I_2M \cdot I_2N' = \dfrac{16\Delta R^2r_2^2}{(b-c)(c+a)(a-b)} \\ I_3L' \cdot IM \cdot I_1N' = \dfrac{16\Delta R^2s_2^2}{(b-c)(c+a)(a-b)} \\ I_3L' \cdot I_3M' \cdot I_3N = \dfrac{16\Delta R^2r_3^2}{(b-c)(a-c)(a+b)} \\ I_2L' \cdot I_1M' \cdot IN = \dfrac{16\Delta R^2s_3^2}{(b-c)(a-c)(a+b)} \end{array}\right\} \quad (48)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= \frac{4\mathrm{R}r^3}{h_1+h_2+h_3-r} \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} &= \frac{4\mathrm{R}rs^2}{h_1+h_2+h_3-r} \end{aligned}\right\} \quad (49)$$

$$\left.\begin{aligned} \alpha\beta\gamma : \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= (b+c)(c+a)(a+b) : abc \\ &= abc : \mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} \\ &= h_1+h_2+h_3-r : r \\ &= \alpha_1\beta_2\gamma_3\cdot\mathrm{I_1L} : \mathrm{I_2M}\cdot\mathrm{I_3N} \end{aligned}\right\} \quad (50)$$

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} : l_1l_2l_3 &= \mathrm{R}r : 2s^2 \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} : l_1l_2l_3 &= \mathrm{R} \;\; : 2r \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} \frac{l_1}{\mathrm{LL'}}\cdot\frac{l_2}{\mathrm{MM'}}\cdot\frac{l_3}{\mathrm{NN'}} &= \frac{(b-c)(a-c)(a-b)}{16\mathrm{R}^2r} = \frac{h_1h_2h_3}{\lambda_1\lambda_2\lambda_3} \\ \frac{\lambda_1}{\mathrm{LL'}}\cdot\frac{\lambda_2}{\mathrm{MM'}}\cdot\frac{\lambda_3}{\mathrm{NN'}} &= \frac{(b+c)(c+a)(a+b)}{16\mathrm{R}^2s} = \frac{h_1h_2h_3}{l_1l_2l_3} \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} l_1^2 &= \frac{4rr_1(rr_1+r_2r_3)}{(r_1+r)^2} & \lambda_1^2 &= \frac{4r_2r_3(rr_1+r_2r_3)}{(r_2-r_3)^2} \\ l_2^2 &= \frac{4rr_2(rr_2+r_3r_1)}{(r_2+r)^2} & \lambda_2^2 &= \frac{4r_3r_1(rr_2+r_3r_1)}{(r_1-r_3)^2} \\ l_3^2 &= \frac{4rr_3(rr_3+r_1r_2)}{(r_3+r)^2} & \lambda_3^2 &= \frac{4r_1r_2(rr_3+r_1r_2)}{(r_1-r_2)} \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} \frac{1}{l_1^2}+\frac{1}{\lambda_1^2} &= \frac{1}{h_1^2} \\ \frac{1}{l_2^2}+\frac{1}{\lambda_2^2} &= \frac{1}{h_2^2} \\ \frac{1}{l_3^2}+\frac{1}{\lambda_3^2} &= \frac{1}{h_3^2} \end{aligned}\right\} \quad (54)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

$$\left.\begin{aligned}
\frac{l_1}{l_2 l_3} + \frac{l_1}{\lambda_2\lambda_3} &= \frac{\lambda_1}{\lambda_2 l_3} - \frac{\lambda_1}{l_2\lambda_3} = \frac{h_1}{h_2 h_3} = \frac{2R}{a^2} \\
\frac{l_2}{l_3 l_1} - \frac{l_2}{\lambda_3\lambda_1} &= \frac{\lambda_2}{l_3\lambda_1} + \frac{\lambda_2}{\lambda_3 l_1} = \frac{h_2}{h_3 h_1} = \frac{2R}{b^2} \\
\frac{l_3}{l_1 l_2} + \frac{l_3}{\lambda_1\lambda_2} &= \frac{\lambda_3}{l_1\lambda_2} - \frac{\lambda_3}{\lambda_1 l_2} = \frac{h_3}{h_1 h_2} = \frac{2R}{c^2}
\end{aligned}\right\} \quad (55)$$

$$\left.\begin{aligned}
\frac{l_1}{\lambda_1} &= \frac{l_2\lambda_3 - \lambda_2 l_3}{l_2 l_3 + \lambda_2\lambda_3} \\
\frac{l_2}{\lambda_2} &= \frac{l_3\lambda_1 + \lambda_3 l_1}{\lambda_3\lambda_1 - l_3 l_1} \\
\frac{l_3}{\lambda_3} &= \frac{\lambda_1 l_2 - l_1\lambda_2}{l_1 l_2 + \lambda_1\lambda_2}
\end{aligned}\right\} \quad (56)$$

Weddle remarks that the three preceding relations between $l_1\ l_2\ l_3\ \lambda_1\ \lambda_2\ \lambda_3$ all reduce to

$$l_1 l_2 l_3 = \lambda_1 l_2 \lambda_3 - \lambda_1\lambda_2 l_3 - l_1\lambda_2\lambda_3 \quad (57)$$

$$\left.\begin{aligned}
l_1 &= \frac{2\alpha\alpha_1}{\alpha+\alpha_1} & l_2 &= \frac{2\beta\beta_2}{\beta+\beta_2} & l_3 &= \frac{2\gamma\gamma_3}{\gamma+\gamma_3} \\
\frac{2}{l_1} &= \frac{1}{\alpha} + \frac{1}{\alpha_1} & \frac{2}{l_2} &= \frac{1}{\beta} + \frac{1}{\beta_2} & \frac{2}{l_3} &= \frac{1}{\gamma} + \frac{1}{\gamma_3} \\
\lambda_1 &= \frac{2\alpha_2\alpha_3}{\alpha_2-\alpha_3} & \lambda_2 &= \frac{2\beta_1\beta_3}{\beta_1-\beta_3} & \lambda_3 &= \frac{2\gamma_1\gamma_2}{\gamma_1-\gamma_2} \\
\frac{2}{\lambda_1} &= \frac{1}{\alpha_3} - \frac{1}{\alpha_2} & \frac{2}{\lambda_2} &= \frac{1}{\beta_3} - \frac{1}{\beta_1} & \frac{2}{\lambda_3} &= \frac{1}{\gamma_2} - \frac{1}{\gamma_1}
\end{aligned}\right\} \quad (58)$$

$$\left.\begin{aligned}
2AU &= \alpha_1 + \alpha & 2BV &= \beta_2 + \beta & 2CW &= \gamma_3 + \gamma \\
2AU' &= \alpha_2 - \alpha_3 & 2BV' &= \beta_1 - \beta_3 & 2CW' &= \gamma_1 - \gamma_2
\end{aligned}\right\} \quad (59)$$

$$\left.\begin{aligned}
AU\,(\alpha_2+\alpha_3) &= 2R(b+c)\\
BV\,(\beta_3+\beta_1) &= 2R(c+a)\\
CW\,(\gamma_1+\gamma_2) &= 2R(a+b)\\
AU'(\alpha_1-\alpha\,) &= 2R(b-c)\\
BV'(\beta_2-\beta\,) &= 2R(a-c)\\
CW'(\gamma_3-\gamma\,) &= 2R(a-b)
\end{aligned}\right\}\quad(60)$$

$$\left.\begin{aligned}
AU\cdot l_1 &= AU'\cdot\lambda_1 = bc\\
BV\cdot l_2 &= BV'\cdot\lambda_2 = ca\\
CW\cdot l_3 &= CW'\cdot\lambda_3 = ab
\end{aligned}\right\}\quad(61)$$

$$\left.\begin{aligned}
AU^2 &= \frac{bc(b+c)^2}{4ss_1} & AU'^2 &= \frac{bc(b-c)^2}{4s_2s_3}\\
BV^2 &= \frac{ca(c+a)^2}{4ss_2} & BV'^2 &= \frac{ac(a-c)^2}{4s_3s_1}\\
CW^2 &= \frac{ab(a+b)^2}{4ss_3} & CW'^2 &= \frac{ab(a-b)^2}{4s_1s_2}
\end{aligned}\right\}\quad(62)$$

$$\left.\begin{aligned}
UA' &= \frac{r^2}{h_1-2r} = \frac{r_1}{h_1+2r_1}\\
VB' &= \frac{r^2}{h_2-2r} = \frac{r_2^2}{h_2+2r_2}\\
WC' &= \frac{r^2}{h_3-2r} = \frac{r_3^2}{h_3+2r_3}
\end{aligned}\right\}\quad(63)$$

$$\frac{bc(b^2-c^2)}{4AU\cdot AU'} = \frac{ac(a^2-c^2)}{4BV\cdot BV'} = \frac{ab(a^2-b^2)}{4CW\cdot CW'} = \triangle \quad(64)$$

$$\left.\begin{aligned}
l_1(b+c) &= h_1(\alpha_2+\alpha_3) & \lambda_1(b-c) &= h_1(\alpha_1-\alpha)\\
l_2(c+a) &= h_2(\beta_3+\beta_1) & \lambda_2(a-c) &= h_2(\beta_2-\beta)\\
l_3(a+b) &= h_3(\gamma_1+\gamma_2) & \lambda_3(a-b) &= h_3(\gamma_3-\gamma)
\end{aligned}\right\}\quad(65)$$

$$\left.\begin{aligned}
\frac{4}{l_1^2}+\frac{4}{\lambda_1^2}&=\frac{1}{a^2}+\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}\\
\frac{4}{l_2^2}+\frac{4}{\lambda_2^2}&=\frac{1}{\beta^2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}+\frac{1}{\beta_3^2}\\
\frac{4}{l_3^2}+\frac{4}{\lambda_3^2}&=\frac{1}{\gamma^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\gamma_3^2}.
\end{aligned}\right\}\quad(66)$$

$$\left.\begin{aligned}
\frac{a}{l_1}+\frac{\beta}{l_2}+\frac{\gamma}{l_3}&=2\\
\frac{a_1}{l_1}-\frac{\beta_1}{\lambda_2}+\frac{\gamma_1}{\lambda_3}&=2\\
\frac{\beta_2}{l_2}+\frac{\gamma_2}{\lambda_3}-\frac{a_2}{\lambda_1}&=2\\
\frac{\gamma_3}{l_3}+\frac{a_3}{\lambda_1}+\frac{\beta_3}{\lambda_2}&=2
\end{aligned}\right\}\ (67)\qquad
\left.\begin{aligned}
\frac{a_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}+\frac{\gamma_3}{\lambda_3}&=0\\
\frac{a}{\lambda_1}+\frac{\beta_3}{l_2}-\frac{\gamma_2}{l_3}&=0\\
\frac{a_3}{l_1}+\frac{\beta}{\lambda_2}-\frac{\gamma_1}{l_3}&=0\\
\frac{a_2}{l_1}-\frac{\beta_1}{l_2}+\frac{\gamma}{\lambda_3}&=0
\end{aligned}\right\}\ (68)$$

Let AI BI CI meet MN NL LM
respectively at L_1 M_1 N_1

$$\left.\begin{aligned}
AL_1 : IL_1 &= AL : IL = h_1 : r\\
BM_1 : IM_1 &= BM : IM = h_2 : r\\
CN_1 : IN_1 &= CN : IN = h_3 : r
\end{aligned}\right\}\quad(69)$$

$$\left.\begin{aligned}
AL_1&=\frac{h_1 a}{h_1+r} & BM_1&=\frac{h_2\beta}{h_2+r} & CN_1&=\frac{h_3\gamma}{h_3+r}\\
IL_1&=\frac{ra}{h_1+r} & IM_1&=\frac{r\beta}{h_2+r} & IN_1&=\frac{r\gamma}{h_3+r}
\end{aligned}\right\}\quad(70)$$

Matthes (p. 47) gives the values

$$\left.\begin{aligned}
AL_1&=\frac{h_1\sqrt{bcrr_1}}{(h_1+r)r_1},\ \text{etc.,}\\
IL_1&=\frac{r\sqrt{bcrr_1}}{(h_1+r)r_1},\ \text{etc.}
\end{aligned}\right\}\quad(71)$$

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

Expressions for the sides of $\triangle LMN$.

$$\left.\begin{array}{l} MN^2 = \dfrac{abc}{(c+a)^2(a+b)^2} \times \\ (b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^2 - c^3 + 3abc) \end{array}\right\} \quad (72)$$

NL^2 and LM^2 can be obtained by cyclical permutations of the letters a b c.

These expressions can be put into shorter forms, by help of Landen's theorem that

$$I_1O^2 = R^2 + 2Rr_1$$

For $$4\triangle(R + 2r_1) = 4\triangle R + \frac{16\triangle^2}{2s_1}$$

$$= abc + (a+b+c)(a-b+c)(a+b-c)$$

$$= b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^3 - c^3 + 3abc$$

Hence

$$MN = \frac{4\triangle \cdot I_1O}{(c+a)(a+b)} \qquad NL = \frac{4\triangle \cdot I_2O}{(a+b)(b+c)} \qquad LM = \frac{4\triangle \cdot I_3O}{(b+c)(c+a)} \quad (73)$$

Matthes (p. 45) in transforming the ten-term factor which occurs in the expression for MN^2 does not appear to have observed the simplification that would result from introducing $R + 2r_1$. He introduces $R + 2r$, and obtains for MN the following value:

$$\frac{4\triangle}{(c+a)(a+b)r_2r_3}\sqrt{\{(R^2 + 2Rr)r_2r_3 + 2a\triangle R\}r_2r_3}$$

The points L' M' N' do not form the vertices of a triangle, but are collinear.

Expressions for the distances $M'N'$ $N'L'$ $L'M'$.

$$\left.\begin{array}{l} M'N'^2 = \dfrac{abc}{(a-c)^2(a-b)^2} \times \\ (-b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 + a^3 + b^3 + c^3 + 3abc) \end{array}\right\} \quad (74)$$

$N'L'^2$ and $L'M'^2$ can be obtained by cyclical permutations of the letters a b c.

https://doi.org/10.1017/S001309150003145X Published online by Cambridge University Press

These expressions can be put into shorter forms, by help of Chapple's theorem that

$$IO^2 = R^2 - 2Rr$$

For $$4\triangle(R-2r) = 4\triangle R - \frac{16\triangle^2}{2s}$$

$$= abc + (-a+b+c)(a-b+c)(a+b-c)$$
$$= -b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 + a^3 + b^3 + c^3 + 3abc$$

Hence

$$M'N' = \frac{4\triangle \cdot IO}{(a-c)(a-b)} \quad N'L' = \frac{4\triangle \cdot IO}{(a-b)(b-c)} \quad L'M' = \frac{4\triangle \cdot IO}{(b-c)(a-c)} \quad (75)$$

In deducing the expressions for M′N′ N′L′ L′M′ it has been assumed that a b c are in descending order of magnitude. If the figure do not correspond to this supposition, care must be taken in verifying the equation

$$L'M' = M'N' + N'L'$$

to affix the proper signs to the values of these magnitudes.

Historical Notes.

(3) Crelle's *Eigenschaften des...Dreiecks*, p. 39 (1816). The property is probably much older.

(4) Weddle in the *Diary* for 1843, p. 75.

(10) The first values of l_1 l_2 l_3 are given by Vecten in Gergonne's *Annales*, IX., 304 (1818-9); the second values by Matthes in his *Commentatio*, p. 42 (1831).

(11) The first values are given by Weddle in the *Diary* for 1848, p. 78; the second values by Matthes in his *Commentatio*, p. 58 (1831).

(12) Mr Robert E. Anderson.

(14) The first equality is given by Mr Launoy in Bourget's *Journal de Mathématiques Élémentaires*, III. 160 (1879).

(15) The first equality is given in J. A. Grunert's article "Dreieck" in *Supplemente zu Klügel's Wörterbuche der reinen Mathematik*, I. 709 (1833). In this article Grunert gives also (20).

(16)—(19) Mr Robert E. Anderson.

(21) First part in Jacobi's *De Triangulorum...Proprietatibus*, p. 8 (1825). Both parts certainly much older.

(22)—(24) Jacobi, p. 13 (1825).

(25) Jacobi, p. 12 (1825), gives the first equality in the first alternative form.

(26) First equality given by Matthes in his *Commentatio*, p. 42 (1831).

(27) First equality given by Marsano in his *Considerazioni sul Triangolo Rettilineo*, p. 29 (1863).

(28) The value of $l_1 l_2 l_3$ is given by Vecten in Gergonne's *Annales*, IX. 304 (1819); that of $\lambda_1 \lambda_2 \lambda_3$ by Weddle in the *Diary* for 1848, p. 78.

(29) The first value is given by Vecten in Gergonne's *Annales*, IX. 305 (1819).

(31), (32) J. W. Elliott in the *Diary* for 1851, p. 58.

(33) The first of these eight values is given by Jacobi, p. 10 (1825).

(35) Weddle in the *Diary* for 1848, p. 78.

(36) ,, ,, ,, ,, ,, ,, p. 81.

(37) ,, ,, ,, ,, ,, ,, p. 80.

(38) *Nouvelles Annales*, 2nd series, IX. 548 (1870).

(39)—(42) Weddle in the *Diary* for 1848, pp. 81-2.

(43)—(45) Matthes, pp. 46, 48, 50 (1831), gives several of these proportions, but they must all have been known long previously.

(46) Matthes, p. 58 (1831), gives the values of λ_1 λ_2 λ_3, but he does not seem to have observed the corresponding ones for l_1 l_2 l_3.

(47) Matthes, pp. 48, 51, gives the first half of these values, the first two of (48), and the first of (49).

(50) The first two proportions are given by Jacobi, pp. 11, 19 (1825); the last two by Matthes, pp. 48, 51 (1831).

(51) The last proportion is given by Matthes, p. 51.

(52) J. W. Elliott in the *Diary* for 1851, p. 58. The equality of the last two expressions is given by Vecten in Gergonne's *Annales*, IX. 305 (1819).

(53)—(58) Weddle in the *Diary* for 1848, pp. 76-78, 82. The values in (58) are probably much older than this.

(61) Weddle in the *Diary* for 1848, p. 82.

(62) Value of AU^2 is given by William Mawson in the *Diary* for 1845, p. 67.

(63) Adams's *Eigenschaften des...Dreiecks*, p. 75 (1846).

(65)—(68) Weddle in the *Diary* for 1848, p. 83. The first equality in (67) is given by Adams in his *Eigenschaften des...Dreiecks*, p. 61 (1846).

(69), (71), (72), (74), (75) Matthes, pp. 47, 44, 59 (1831).