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AMENABILITY FOR REAL C^*-ALGEBRAS

NHAN BAO HO

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Abstract

It is shown that the complexification of a positive linear map on a real C^*-algebra need not be positive whereas the complexification of a completely positive linear map is completely positive. It is further shown that a real C^*-algebra is amenable if and only if its complexification is amenable and that a completely positive linear map is amenable if and only if its complexification is. Finally, a real version of the Choi–Effros lifting theorem is established.

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1. Positive and completely positive maps

Let A be a complex C^*-algebra and let Φ be an involutory $*$-antiautomorphism of A. Then $A_\Phi = \{a \in A \mid \Phi(a) = a^*\}$ is a real C^*-algebra for which $A_\Phi \cap i A_\Phi = \{0\}$ and $A = A_\Phi + i A_\Phi$; that is, A is the complexification of A_Φ. By [1, Corollary 15.4] every real C^*-algebra arises in this way. If ϕ is a real-linear map between real C^*-algebras A_Φ and B_Ψ, then ϕ extends uniquely to a complex-linear map ϕ^C between A and B, called the complexification of ϕ. In the first results in this paper we obtain relations between positivity conditions for these two maps.

Recall that an element a in a complex C^*-algebra is said to be *positive* if it can be written in the form b^*b for some $b \in A$ and that a linear map is positive if it maps positive elements to positive elements. The same definition is given for real C^*-algebras in [1, Chapter 14], but the extra condition $\phi(x) = \phi(x^*)$ is imposed for real states (which are real maps from A to $\mathbb{R}$), thus excluding examples such as $\phi(a + ib) = a + b$ from $\mathbb{C}$ to $\mathbb{R}$. It therefore seems natural to demand, as is automatic for complex C^*-algebras, that a positive map ϕ between real C^*-algebras satisfies $\phi(x)^* = \phi(x^*)$ and this will be done here. Even with this extra condition, it is not true that the complexification of a positive map is positive, as the following example shows. On the other hand, if ϕ^C is positive then, for each positive $a \in A_\Phi$, $\phi(a)$ is in B_Ψ and is positive in B. It is therefore of the form $(b + ib')(b + ib')^* = bb^* + b'b'^*$, showing that ϕ is positive.

EXAMPLE 1. Let A be the algebra of 2×2 complex matrices and let

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then the real algebra

$$A_\Phi = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

is isomorphic to the algebra $\mathbb{H}$ of quaternions, for which the positive elements are the positive reals. Let the positive map $\phi : A_\Phi \to \mathbb{C}$ be defined by

$$\phi \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = a + 2ib.$$

The complexification of $\mathbb{C}$ is $\mathbb{C}^2$ with involutory *-antiautomorphism $\Psi(a, b) = (b, a)$ and corresponding real algebra $\{(a, \overline{a}) : a \in \mathbb{C}\}$. Therefore,

$$\phi^C \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\phi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = (1, 1) + i(2i, -2i) = (-1, 3),$$

so ϕ^C is not positive.

Despite the negative result above, it will now be shown that the complexification of a completely positive linear map between real C^*-algebras is completely positive. As with complex algebras, a completely positive map ϕ from A_Φ to B_Ψ is one for which the natural element-wise defined maps $\phi^{(n)}$ from $M_n(A_\Phi)$ to $M_n(B_\Psi)$ are all positive. As for positive maps, when ϕ^C is completely positive, then so is ϕ. The key to proving the converse is to use the $*$-isomorphism ψ from $\mathbb{C}$ into $M_2(\mathbb{R})$ defined by

$$\psi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

This gives rise to $*$-isomorphisms ψ_A from A into $M_2(A_\Phi)$ and ψ_B from B into $M_2(B_\Psi)$.

PROPOSITION 2. *Let A, B be C^*-algebras, let Φ, Ψ be involutory $*$-anti-automorphisms on A, B and let $\phi : A_\Phi \longrightarrow B_\Psi$ be a completely positive linear map. Then ϕ^C is also a completely positive linear map.*

PROOF. Let n be a positive integer and let $[a_{jk} + ib_{jk}] \in M_n(A)$ where, for each $1 \le j, k \le n$, $a_{jk}, b_{jk} \in A_\Phi$. Then

$$\psi_A^{(n)}([a_{jk} + ib_{jk}]) = [\psi_A(a_{jk} + ib_{jk})] = \left[\begin{bmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{bmatrix} \right] \in M_{2n}(A_\Phi)$$

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and so

$$\phi^{(2n)} \circ \psi_A^{(n)}([a_{jk} + i b_{jk}]) = \left[\begin{bmatrix} \phi(a_{jk}) & \phi(b_{jk}) \\ -\phi(b_{jk}) & \phi(a_{jk}) \end{bmatrix}\right] \in M_{2n}(B_\Psi).$$

Then

$$\begin{aligned} {\psi_B^{(n)}}^{-1} \circ \phi^{(2n)} \circ \psi_A^{(n)}([a_{jk} + i b_{jk}]) &= \left[{\psi_B}^{-1}\left(\begin{bmatrix} \phi(a_{jk}) & \phi(b_{jk}) \\ -\phi(b_{jk}) & \phi(a_{jk}) \end{bmatrix}\right)\right] \\ &= [\phi(a_{jk}) + i\phi(b_{jk})] \\ &= [\phi^c(a_{jk} + i b_{jk})] \\ &= \phi^{c(n)}([a_{jk} + i b_{jk}]). \end{aligned}$$

Therefore, $\phi^{c(n)} = {\psi_B^{(n)}}^{-1} \circ \phi^{(2n)} \circ \psi_A^{(n)}$, which is positive. □

2. Amenable algebras

A real or complex C^*-algebra A is said to be amenable if for all $\varepsilon > 0$ and for all finite subsets $\mathfrak{A} \subset A$ there exist a finite-dimensional real or complex C^*-algebra B and contractive completely positive linear maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that, for all $a \in \mathfrak{A}$,

$$\|a - \psi \circ \varphi(a)\| < \varepsilon.$$

If A_Φ is amenable and $\{a_1 + ib_1, \ldots, a_n + ib_n\} \subset A$ then, by applying the definition of amenability to $\mathfrak{A} = \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \subset A_\Phi$ and complexifying the resulting finite-dimensional algebra B and completely positive maps φ, ψ, it follows that A is also amenable. The following proposition establishes the converse.

Proposition 3. *Let A be a complex C^*-algebra and let Φ be an involutory $*$-antiautomorphism in A. Then $A_\Phi = \{a \in A \mid \Phi(a) = a^*\}$ is amenable.*

Proof. Let $\varepsilon > 0$ and let $\mathfrak{A} \subset A_\Phi$ be a finite subset. Since A is amenable, there exist a complex finite-dimensional C^*-algebra B and contractive completely positive maps $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that, for all $a \in \mathfrak{A}$, $\|a - \psi \circ \varphi(a)\| < \varepsilon$. Define $\psi' = (1/2)\,(\psi + \Phi \circ * \circ \psi)$.

Note that $\Phi \circ *$ is a real-linear automorphism and therefore ψ' is a real contractive completely positive linear map from the finite-dimensional real C^*-algebra B to A_Φ. Furthermore, for $a \in \mathfrak{A}$,

$$\begin{aligned} \|a - \psi' \circ \varphi(a)\| &= \|a - \tfrac{1}{2}(\psi + \Phi \circ * \circ \psi)\,(\varphi(a))\| \\ &\le \tfrac{1}{2}\|a - \psi(\varphi(a))\| + \tfrac{1}{2}\|a - \Phi \circ * \circ \psi(\varphi(a))\| \\ &= \tfrac{1}{2}\|a - \psi(\varphi(a))\| + \tfrac{1}{2}\|\Phi \circ *(a) - \Phi \circ * \circ \psi(\varphi(a))\| \\ &= \|a - \psi \circ \varphi(a)\| < \varepsilon, \end{aligned}$$

so A_Φ is amenable. □

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As in [2, Definition 5.4.1], a contractive completely positive linear map ϕ between two complex C^*-algebras A and B is said to be amenable if for any $\varepsilon > 0$ and any finite subset $\mathfrak{A} \subset A$, there are contractive completely positive linear maps $\varphi : A \longrightarrow M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \longrightarrow B$, for some $n > 0$, such that $\|\phi(a) - \psi \circ \varphi(a)\| < \varepsilon$ for all $a \in \mathfrak{A}$. A similar definition applies in the real case, with $M_n(\mathbb{C})$ replaced by $M_n(\mathbb{R})$. If $\phi : A_\Phi \to B_\Psi$ is amenable and $\{a_1 + ib_1, \ldots, a_n + ib_n\} \subset A$, it then follows by applying the definition of amenability to $\mathfrak{A} = \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \subset A_\Phi$ that ϕ^C is also amenable. The next main result establishes the converse.

LEMMA 4. *Let $\sigma : \mathbb{C} \to M_2(\mathbb{R})$ be defined by*

$$\sigma(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and let $P : M_2(\mathbb{R}) \to \mathbb{C}$ be defined by

$$P \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}(a + d) + \frac{1}{2} i(b - c).$$

Then σ, P are completely positive maps with $P \circ \sigma$ equal to the identity map.

PROOF. It is immediate that the $*$-isomorphism σ is completely positive and that $P \circ \sigma$ is equal to the identity map. The complexification of P maps

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{to} \quad \frac{1}{2}((a + d) + i(b - c), (a + d) - i(b - c))$$

and therefore maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & |b|^2 + |d|^2 \end{bmatrix}$$

to (r_1, r_2), where

$$\begin{aligned} 2r_1 &= (|a|^2 + |c|^2 + |b|^2 + |d|^2) + i(\overline{a}b + \overline{c}d - \overline{b}a - \overline{d}c), \\ 2r_2 &= (|a|^2 + |c|^2 + |b|^2 + |d|^2) + i(\overline{b}a + \overline{d}c - \overline{a}b - \overline{c}d). \end{aligned}$$

Note that $i(\overline{a}b - \overline{b}a) = 2\,\mathrm{Im}(\overline{b}a)$ and $|a||b| \geq \mathrm{Im}(\overline{b}a)$ and so

$$\begin{aligned} (|a|^2 + |b|^2) + i(\overline{a}b - \overline{b}a) &= |a|^2 + |b|^2 + 2\,\mathrm{Im}(\overline{b}a) \\ &\geq |a|^2 + |b|^2 - 2|a||b| \geq 0. \end{aligned}$$

Similarly, $(|c|^2 + |d|^2) + i(\overline{c}d - \overline{d}c) \geq 0$. Therefore P^C is positive. Since $\mathbb{C}^2$ is commutative, then [2, Theorem 2.2.5] implies that P^C is completely positive and therefore so is P. □

PROPOSITION 5. *Let A, B be complex C^*-algebras and let Φ, Ψ be involutory $*$-antiautomorphisms on A, B respectively. If $\phi : A_\Phi \longrightarrow B_\Psi$ is a completely positive linear map such that ϕ^C is amenable, then ϕ is amenable.*

PROOF. Let $\varepsilon > 0$ and let $\mathfrak{A} \subset A_\Phi$ be a finite subset. Since ϕ is amenable, there exist $n > 0$ and contractive completely positive linear maps $\varphi : A \longrightarrow M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \longrightarrow B$ such that $\|\phi(a) - \psi \circ \varphi(a)\| < \varepsilon$ for each $a \in \mathfrak{A}$.

Define the contractive completely positive linear map $\varphi' : A \longrightarrow M_{2n}(\mathbb{R})$ by $\varphi' = \sigma^{(n)} \circ \varphi$ and the contractive completely positive linear map $\psi' : M_{2n}(\mathbb{R}) \to B_\Psi$ by $\psi' = (1/2)\,(\psi \circ P^{(n)} + \Psi \circ * \circ \psi \circ P^{(n)})$. By the lemma, $P^{(n)} \circ \sigma^{(n)}$ is the identity map on $M_n(\mathbb{C})$. Therefore,

$$\begin{aligned}\psi' \circ \varphi' &= \tfrac{1}{2}(\psi \circ P^{(n)} + \Psi \circ * \circ \psi \circ P^{(n)}) \circ (\sigma^{(n)} \circ \varphi)\\ &= \tfrac{1}{2}(\psi \circ \varphi + \Psi \circ * \circ \psi \circ \varphi).\end{aligned}$$

For $a \in \mathfrak{A}$, we then have

$$\begin{aligned}\|\phi(a) - \psi' \circ \varphi'(a)\| &= \|\phi(a) - \tfrac{1}{2}(\psi \circ \varphi(a) + \Psi \circ * \circ \psi \circ \varphi(a))\|\\ &\leq \tfrac{1}{2}\|\phi(a) - \psi \circ \varphi(a)\| + \tfrac{1}{2}\|\phi(a) - \Psi \circ * \circ \psi \circ \varphi(a)\|.\end{aligned}$$

Since $\phi(a) \in B_\Psi$, $\phi(a) = \Psi \circ *(\phi(a))$ and so

$$\begin{aligned}\|\phi(a) - \Psi \circ * \circ \psi \circ \varphi(a)\| &= \|\Psi \circ *(\phi(a)) - \Psi \circ * \circ \psi \circ \varphi(a)\|\\ &= \|\phi(a) - \psi \circ \varphi(a)\|.\end{aligned}$$

Therefore,

$$\|\phi(a) - \psi' \circ \varphi'(a)\| \leq \|\phi(a) - \psi \circ \varphi(a)\| < \varepsilon,$$

establishing that ϕ is amenable. □

The final result gives a real version of the Choi–Effros theorem, described in [2, Theorem 5.4.4].

THEOREM 6. *Let A, B be C^*-algebras with A separable, let Φ, Ψ be involutory $*$-antiautomorphisms of A, B and let I be an ideal of B_Ψ. If $\phi : A_\Phi \to B_\Psi / I$ is an amenable contractive completely positive linear map, then there exists a contractive completely positive linear map $\psi : A_\Phi \to B_\Psi$ such that $\pi \circ \psi = \phi$, where $\pi : B_\Psi \to B_\Psi / I$ is the quotient map.*

PROOF. If I^C is the complexification of I, let Ψ_I be the involutory $*$-antiautomorphism of B/I^C defined by $\Psi_I(b + I^C) = \Psi(b) + I^C$, for which $\pi^C \circ \Psi = \Psi_I \circ \pi^C$, where π^C is the quotient map associated with I^C. Note that the associated real algebra is the image of B_Ψ / I under the injection $\iota : b + I \mapsto b + I^C$. By the Choi–Effros theorem the complexification ϕ^C of ϕ lifts to a completely positive linear map $\alpha : A \to B$. Let $\psi = (1/2)(\alpha + \Psi \circ * \circ \alpha)$, which maps A, and hence A_Φ, into B_Ψ. Note that $\pi^C \circ \Psi \circ * \circ \alpha = \Psi_I \circ * \circ \phi^C$ and, hence, if

$$a \in A_\Phi, \quad \pi^C(\psi(a)) = \tfrac{1}{2}(\phi^C(a) + (\Psi_I \circ * \circ \phi^C)(a)) = \iota(\phi(a))$$

and thus $\pi \circ \psi = \phi$. □

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References

[1] R. K. Goodearl, *Notes on Real and Complex C^*-algebras* (Shiva, England, 1982).
[2] H. Lin, *An Introduction to the Classification of Amenable C^*-algebras* (World Scientific, River Edge, NJ, 2001).

NHAN BAO HO, Department of Mathematical and Statistical Sciences,
La Trobe University, Victoria 3086, Australia
e-mail: honhanbao@yahoo.com