Glasgow Math. J. **43** (2001) 103–111.

PRIMARY MODULES OVER COMMUTATIVE RINGS

PATRICK F. SMITH

Department of Mathematics, University of Glasgow, Glasgow, G12 8QW, Scotland, UK
e-mail: pfs@maths.gla.ac.uk

(Received 11 May, 1999; revised 18 January, 2000)

Abstract. The radical of a module over a commutative ring is the intersection of all prime submodules. It is proved that if R is a commutative domain which is either Noetherian or a *UFD* then R is one-dimensional if and only if every (finitely generated) primary R-module has prime radical, and this holds precisely when every (finitely generated) R-module satisfies the radical formula for primary submodules.

1991 *Mathematics Subject Classification*. 13C99, 13E05, 13F05, 13F15.

Prime submodules of modules over commutative rings have been studied by various authors. In particular, a number of papers have been devoted to trying to calculate the radical of a module; see, for example, [**2**]–[**6**]. Only in special cases is there a simple description of the radical. It is natural therefore to ask whether the radical of a primary module has a simple description. McCasland and Moore [**5**, Theorem 2.10] proved that if R is a PID (principal ideal domain) and M is the free R-module $R^{(n)}$, for some positive integer n, then the radical of any primary submodule of M is a prime submodule of M. We shall show that this is not true in general.

Given a commutative ring R, R-modules which satisfy the radical formula were first considered by McCasland and Moore [**5**]. In [**3**], commutative Noetherian rings R such that every R-module satisfies the radical formula, were characterized. In this paper, we characterize all commutative Noetherian rings R such that every R-module satisfies the radical formula for primary submodules and show that these are precisely the commutative Noetherian rings such that every primary R-module has prime radical.

Throughout this note all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. For any submodule N of M let $(N : M) = \{r \in R : rM \subseteq N\}$. A submodule N of M is called *prime* (respectively, *primary*) if $N \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in N$ then $m \in N$ or $r \in (N : M)$ (respectively, $r \in \sqrt{(N : M)}$). Note that for any ideal $\mathbf{a}$ of R,

$$\sqrt{\mathbf{a}} = \{r \in R : r^n \in \mathbf{a} \text{ for some positive integer } n\}.$$

If N is a primary submodule of M and $\mathbf{p} = \sqrt{(N : M)}$, then $\mathbf{p}$ is a prime ideal of R and we shall call N $\mathbf{p}$-primary. The module M will be called *primary* if its zero submodule is primary. For any submodule N of an R-module M, the *radical*, $\mathrm{rad}_M(N)$, of N is defined to be the intersection of all prime submodules of M containing N and $\mathrm{rad}_M(N) = M$ if N is not contained in any prime submodule of M. The *radical of the module* M is defined to be $\mathrm{rad}_M(0)$.

Given a ring R and a submodule N of an R-module M we define

$$E_M(N) = \{rm : r \in R, m \in M \text{ and } r^k m \in N \text{ for some positive integer } k\}.$$

https://doi.org/10.1017/S0017089501010084 Published online by Cambridge University Press

Then $RE_M(N)$ will denote the submodule of M generated by the non-empty subset $E_M(N)$ of M. Note that $RE_M(N)$ consists of all finite sums of elements of $E_M(N)$. McCasland and Moore [**5**] say that the module M *satisfies the radical formula* if $\text{rad}_M(N) = RE_M(N)$ for any submodule N of M. Now we say that the module M *satisfies the radical formula for primary submodules* if $\text{rad}_M(N) = RE_M(N)$ for every primary submodule N of M.

1. Radicals of primary modules. The first result is well known and the proof (which is easy in any case) is omitted.

LEMMA 1.1. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* M *be an* R*-module. Then a proper submodule* N *of* M *is* $\mathbf{p}$*-primary if and only if*
(a) $\mathbf{p} \subseteq \sqrt{(N : M)}$*, and*
(b) $cm \notin N$*, for all* $c \in R \backslash \mathbf{p}$, $m \in M \backslash N$.

It is clear that for any module M, prime submodules of M are primary. In particular, maximal submodules (being prime) are primary. Thus, non-zero modules which are either projective or finitely generated have primary submodules (see [**1**, Proposition 17.14]). The next result shows that free modules have a rich supply of primary submodules.

PROPOSITION 1.2. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* $\mathbf{a}$ *be a* $\mathbf{p}$*-primary ideal of* R*. Let* M *be a free* R*-module with basis* $\{m_i : i \in I\}$*. Let* $(a_{\lambda i})$ *be an* $\Lambda \times I$ *matrix with entries in* R *such that* $a_{\mu j} \notin \mathbf{a}$ *for some* $\mu \in \Lambda$, $j \in I$*. Let* N *be the subset of* M *consisting of all elements* m *in* M *such that* $m = \sum_{i \in I} r_i m_i$ *for some* $r_i \in R (i \in I)$*, where* $r_i \neq 0$ *for at most a finite number of elements* $i \in I$*, and* $\sum_{i \in I} a_{\lambda i} r_i \in \mathbf{a}$ *for all* $\lambda \in \Lambda$*. Then* N *is a* $\mathbf{p}$*-primary submodule of* M*.*

Proof. It is clear that N is a submodule of M and $m_j \notin N$ so that $N \neq M$. Let $p \in \mathbf{p}$. There exists a positive integer k such that $p^k \in \mathbf{a}$ and hence $p^k M \subseteq \mathbf{a}M \subseteq N$. Let $m \in M$ and $c \in R \backslash \mathbf{p}$ such that $cm \in N$. There exist elements $s_i \in R (i \in I)$ such that $s_i \neq 0$ for at most a finite number of elements $i \in I$ and $m = \sum_{i \in I} s_i m_i$. Then $cm = \sum_{i \in I} c s_i m_i \in N$ implies that $\sum_{i \in I} a_{\lambda i}(c s_i) \in \mathbf{a}$ $(\lambda \in \Lambda)$, i.e. $c(\sum_{i \in I} a_{\lambda i} s_i) \in \mathbf{a}$ $(\lambda \in \Lambda)$ and hence $\sum_{i \in I} a_{\lambda i} s_i \in \mathbf{a}$. Thus $m \in N$. By Lemma 1.1, N is $\mathbf{p}$-primary.

We now turn our attention to the radical of primary modules. Note first the following elementary fact whose proof is omitted.

LEMMA 1.3. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* N *be a* $\mathbf{p}$*-primary submodule of an* R*-module* M*. Then* $RE_M(N) = N + \mathbf{p}M \subseteq \text{rad}_M(N)$.

LEMMA 1.4. *Let* $\mathbf{m}$ *be a maximal ideal of a ring* R*, let* M *be an* R*-module and let* N *be an* $\mathbf{m}$*-primary submodule of* M*. Then* $\text{rad}_M(N) = N + \mathbf{m}M = RE_M(N)$*. Moreover* $N + \mathbf{m}M$ *is a prime submodule of* M *or* $M = N + \mathbf{m}M$*.*

Proof. Clearly $N \subseteq N + \mathbf{m}M$ and it is easy to check that $N + \mathbf{m}M$ is a prime submodule of M or $M = N + \mathbf{m}M$. Then $\text{rad}_M(N) = N + \mathbf{m}M = RE_M(N)$ by Lemma 1.3.

https://doi.org/10.1017/S0017089501010084 Published online by Cambridge University Press

COROLLARY 1.5. *Let R be a 0-dimensional ring. Then a primary R-module M has prime radical if and only if M contains a prime submodule.*

Proof. The necessity is clear. Conversely, suppose that M contains a prime submodule K. Let $\mathbf{p} = \sqrt{(0 : M)}$. Because R is 0-dimensional we know that $\mathbf{p}$ is a maximal ideal of R. Now Lemma 1.4 gives that $rad_M(0) = \mathbf{p}M$ and $\mathbf{p}M$ is a prime submodule of M because $\mathbf{p}M \subseteq K \neq M$.

Note that Sharif, Sharifi and Namazi [**6**, Theorem 2.8] proved that if R is a 0-dimensional ring then every R-module satisfies the radical formula. In Lemma 1.4, it is possible for M to equal $N + \mathbf{m}M$ for an $\mathbf{m}$-primary submodule N of M, as the following example shows.

EXAMPLE 1.6. *There exists a ring R having a nil idempotent maximal ideal* $\mathbf{m}$ *so that if M is the R-module* $\mathbf{m}$ *then 0 is an* $\mathbf{m}$*-primary submodule of M but* $M = \mathbf{m}M$.

Proof. For any prime p, let F be any field of characteristic p and let G be the Prüfer p-group $C(p^\infty)$. Let R denote the group algebra $F[G]$. Then R is a commutative ring whose augmentation ideal $\mathbf{m}$ has the desired properties; (for more details, see [**8**, Lemma 5.5]).

If R is a Noetherian ring in Lemma 1.4, then $N + \mathbf{m}M$ is a prime submodule of M for any $\mathbf{m}$-primary submodule N of the R-module M. For, in this case, $\mathbf{m}^n \subseteq (N : M)$ for some positive integer n and hence $M = N + \mathbf{m}M$ implies that $M = N + \mathbf{m}^n M = N$, a contradiction. Thus $M \neq N + \mathbf{m}M$. In view of this, to study rings with the property that every primary module has prime radical it is natural to restrict to Noetherian rings.

Let R be any ring and let M be an R-module. If N is a proper submodule of M, then K is a *minimal prime submodule* of N in M if K is a prime submodule of M, $N \subseteq K$ and whenever L is a prime submodule of M with $K \supseteq L \supseteq N$ then $K = L$.

LEMMA 1.7. *Let R be any ring, let* $\mathbf{p}$ *be a prime ideal of R and let N be a* $\mathbf{p}$*-primary submodule of a finitely generated R-module M. Let* $K = \{m \in M : cm \in N + \mathbf{p}M$ *for some* $c \in R\backslash\mathbf{p}\}$. *Then K is a minimal prime submodule of N in M.*

Proof. Clearly K is a submodule of M. Moreover $\mathbf{p}M \subseteq K$ and it is easy to check that the $(R/\mathbf{p})$-module M/K is torsion-free. Suppose that $M = K$. Because M is finitely generated, there exists $a \in R\backslash\mathbf{p}$ such that $aM \subseteq N + \mathbf{p}M$; i.e. $a(M/N) \subseteq \mathbf{p}(M/N)$. Using the usual determinant argument, we find that $b(M/N) = 0$ for some $b \in R\backslash\mathbf{p}$. But this implies that $bM \subseteq N$ and hence $b \in \mathbf{p}$, a contradiction. It follows that $M \neq K$ and K is a prime submodule of M.

Let L be a prime submodule of M such that $K \supseteq L \supseteq N$. Let $\mathbf{q} = (L : M)$. Suppose that $\mathbf{q} \not\subseteq \mathbf{p}$ and let $d \in \mathbf{q}\backslash\mathbf{p}$. Then $dM \subseteq L \subseteq K$ and hence $M = K$, a contradiction. Thus $\mathbf{q} \subseteq \mathbf{p}$. On the other hand, it is clear that $\mathbf{p}M \subseteq L$, so that $\mathbf{p} \subseteq \mathbf{q}$. Hence $\mathbf{p} = \mathbf{q}$. It follows that $N + \mathbf{p}M \subseteq L$ and that $K \subseteq L$. Thus $K = L$.

Let $\mathbf{q}$ be a prime ideal of a ring R and let n be a positive integer. Then the nth *symbolic power* $\mathbf{q}^{(n)}$ of $\mathbf{q}$ is defined by

$$\mathbf{q}^{(n)} = \{r \in R : rc \in \mathbf{q}^n \text{ for some } c \in R\backslash\mathbf{q}\}.$$

It is well known (and easy to check) that $\mathbf{q}^{(n)}$ is a $\mathbf{q}$-primary ideal of R.

LEMMA 1.8. *Let R be any ring such that every primary submodule of the R-module $R \oplus R$ has prime radical. Then $\mathbf{q} = \{r \in \mathbf{q} : rc \in \mathbf{pq} + \mathbf{q}^{(2)}$ for some $c \in R\backslash\mathbf{p}\}$ for all distinct prime ideals $\mathbf{p}$,$\mathbf{q}$ of R with $\mathbf{p} \supseteq \mathbf{q}$.*

Proof. Suppose not and that $\mathbf{p} \supsetneqq \mathbf{q}$ are prime ideals of R such that $\mathbf{q} \supsetneqq \mathbf{a}$, where $\mathbf{a} = \{r \in \mathbf{q} : rc \in \mathbf{pq} + \mathbf{q}^{(2)}$, for some $c \in R\backslash\mathbf{p}\}$. Let $a \in \mathbf{p}\backslash\mathbf{q}$ and $b \in \mathbf{q}\backslash\mathbf{a}$. Set $M = R \oplus R$ and let

$$N = \{m \in M : cm \in R(a, b) + \mathbf{q}^2 M, \text{ for some } c \in R\backslash\mathbf{q}\}.$$

Clearly $\mathbf{q}^2 M \subseteq N$ and N is $\mathbf{q}$-primary, by Lemma 1.1. Set $K = \mathrm{rad}_M(N)$. Then K is a prime submodule of M. By Lemma 1.7,

$$K = \{m \in M : cm \in R(a, b) + \mathbf{q}M \text{ for some } c \in R\backslash\mathbf{q}\}.$$

Note that

$$a(1, 0) = (a, 0) = (a, b) - (0, b) \in R(a, b) + \mathbf{q}M$$

and hence $(1, 0) \in K$. Let $x, y \in R$ such that $(x, y) \in N$. There exist $c \in R\backslash\mathbf{q}$, $r \in R$, $u, v \in \mathbf{q}^2$ such that

$$c(x, y) = r(a, b) + (u, v),$$

and hence

$$cx = ra + u \text{ and } cy = rb + v.$$

Note that $cy \in \mathbf{q}$ and hence $y \in \mathbf{q}$. Moreover

$$c(xb - ya) = ub - va \in \mathbf{q}^2,$$

so that $xb - ya \in \mathbf{q}^{(2)}$ and $xb \in \mathbf{pq} + \mathbf{q}^{(2)}$. By the choice of b, $x \in \mathbf{p}$. Thus $(x, y) \in \mathbf{p} \oplus \mathbf{p} = \mathbf{p}M$. It follows that $N \subseteq \mathbf{p}M$. Clearly $\mathbf{p}M$ is a prime submodule of M. Hence $K = \mathrm{rad}_M(N) \subseteq \mathbf{p}M$. However $(1, 0) \in K$ gives the contradiction $1 \in \mathbf{p}$.

THEOREM 1.9. *The following statements are equivalent for a Noetherian ring R with prime radical $\mathbf{n}$.*

(i) *Every primary R-module has prime radical.*
(ii) *The radical of every primary submodule of the R-module $R \oplus R$ is prime.*
(iii) *For every non-maximal prime ideal $\mathbf{q}$ of R there exists $c \in R\backslash\mathbf{q}$ such that $c\mathbf{q} = 0$.*
(iv) *Every non-maximal prime ideal $\mathbf{q}$ of R is the only $\mathbf{q}$-primary ideal of R.*
(v) *R is Artinian or R is one-dimensional and $\mathbf{n}$ is an Artinian R-module.*

https://doi.org/10.1017/S0017089501010084 Published online by Cambridge University Press

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii). Let $\mathbf{q}$ be any non-maximal prime ideal of R. There exists a maximal ideal $\mathbf{p}$ of R such that $\mathbf{p} \supsetneqq \mathbf{q}$. By Lemma 1.8, $\mathbf{q} = \{r \in \mathbf{q} : rd \in \mathbf{pq} + \mathbf{q}^{(2)}$ for some $d \in R\backslash\mathbf{p}\}$. The usual determinant argument gives $d'\mathbf{q} \subseteq \mathbf{q}^{(2)}$ for some $d' \in R\backslash\mathbf{p}$. Hence $\mathbf{q} = \mathbf{q}^{(2)}$. By [**7**, Exercise 8.37] $c\mathbf{q} = 0$ for some $c \in R\backslash\mathbf{q}$.

(iii) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Let M be any primary R-module; i.e. the zero submodule is $\mathbf{q}$-primary for some prime ideal $\mathbf{q}$. Suppose that $\mathbf{q}$ is a maximal ideal of R. Then $\mathbf{q}M$ is a prime submodule of M and $\mathrm{rad}_M(0) = \mathbf{q}M$, by Lemma 1.4. Now suppose that $\mathbf{q}$ is not maximal. There exists a positive integer n such that $\mathbf{q}^n M = 0$. By (iv) $\mathbf{q} = \mathbf{q}^{(n)}$ and hence $c\mathbf{q} \subseteq \mathbf{q}^n$ for some $c \in R\backslash\mathbf{q}$. Then $c\mathbf{q}M = 0$ and hence $\mathbf{q}M = 0$. By Lemma 1.1, M is a torsion-free $(R/\mathbf{q})$-module and hence 0 is a prime submodule of M. In any case, $\mathrm{rad}_M(0)$ is prime.

(iii) $\Rightarrow$ (v). Suppose that $\mathbf{p}_0 \supsetneqq \mathbf{p}_1 \supsetneqq \mathbf{p}_2$ are distinct prime ideals of R. By (iii), there exists $b \in R\backslash\mathbf{p}_1$ such that $b\mathbf{p}_1 = 0 \subseteq \mathbf{p}_2$ and hence $\mathbf{p}_1 \subseteq \mathbf{p}_2$, a contradiction. Thus R is 0-dimensional, and hence Artinian by [**7**, Proposition 8.38], or R is one-dimensional. Set $\mathbf{a} = \{r \in R : r\mathbf{n} = 0\}$. Let $\mathbf{p}$ be a prime ideal of R such that the ideal $\mathbf{a} \subseteq \mathbf{p}$. If $\mathbf{p}$ is not maximal, then $c\mathbf{p} = 0$ for some $c \in R\backslash\mathbf{p}$. Hence $c\mathbf{n} = 0$ and $c \in \mathbf{a} \subseteq \mathbf{p}$, a contradiction. Thus $\mathbf{p}$ is maximal. It follows that every prime ideal of the ring $R/\mathbf{a}$ is maximal and hence $R/\mathbf{a}$ is an Artinian ring by [**7**, Proposition 8.38] again. Now $\mathbf{n}$ is a finitely generated $(R/\mathbf{a})$-module so that $\mathbf{n}$ is an Artinian $(R/\mathbf{a})$-module and hence also an Artinian R-module.

(v) $\Rightarrow$ (iii). If R is Artinian, then (iii) is clearly true. Now suppose that R is a one-dimensional ring such that $\mathbf{n}$ is an Artinian R-module. Let $\mathbf{q}$ be a non-maximal prime ideal of R. Then $\mathbf{q}/\mathbf{n}$ is a minimal prime ideal of the semiprime Noetherian ring $R/\mathbf{n}$. There exists $g \in R\backslash\mathbf{q}$ such that $g\mathbf{q} \subseteq \mathbf{n}$. Note that the R-module $\mathbf{n}$ has a composition series and hence $\mathbf{p}_1 \ldots \mathbf{p}_k\mathbf{n} = 0$ for some positive integer k and maximal ideals $\mathbf{p}_i (1 \leq i \leq k)$ of R. Thus $\mathbf{p}_1 \ldots \mathbf{p}_k g\mathbf{q} = 0$ and hence $c\mathbf{q} = 0$ for some $c \in R\backslash\mathbf{q}$ because $\mathbf{p}_i \not\subseteq \mathbf{q}$ $(1 \leq i \leq k)$.

COROLLARY 1.10. *Let R be a semiprime Noetherian ring. Then every primary R-module has prime radical if and only if R is at most one-dimensional.*

Proof. This follows at once from Theorem 1.9.

The following example illustrates the last theorem.

EXAMPLE 1.11. *Let R be the polynomial ring $\mathbb{Z}[X]$, M the R-module $R \oplus R$ and N the submodule $R(2, X) + R(X, 0)$. Then N is a $\mathbf{p}$-primary submodule of M, where $\mathbf{p}$ is the prime ideal RX, and $\mathrm{rad}_M(N) = K \cap \mathbf{m}M$, where K is the prime submodule $R \oplus RX$ and $\mathbf{m}$ the maximal ideal $R2 + RX$.*

Proof. It is easy to check that N is $\mathbf{p}$-primary, because $N = \{(u, v) \in M : Xu - 2v \in \mathbf{p}^2\}$ and Proposition 1.2 applies, K and $\mathbf{m}M$ are prime submodules of M and $K = \{m \in M : cm \in N + \mathbf{p}M$ for some $c \in R\backslash\mathbf{p}\}$. Let L be a prime submodule of M such that $N \subseteq L$. Let $\mathbf{q} = (L : M)$. Note that $\mathbf{p} \subseteq \mathbf{q}$. If $\mathbf{p} = \mathbf{q}$, then $K \subseteq L$. Suppose that $\mathbf{p} \neq \mathbf{q}$. Then $\mathbf{q} = RX + Rq$ for some prime q in $\mathbb{Z}$. Next note that

$$K \cap \mathbf{m}M = \mathbf{m} \oplus RX \subseteq N + \mathbf{q}M$$

https://doi.org/10.1017/S0017089501010084 Published online by Cambridge University Press

because if $\mathbf{q} \neq \mathbf{m}$ then $N + \mathbf{q}M = R \oplus \mathbf{q}$. It follows that $\text{rad}_M(N) = K \cap \mathbf{m}M$.

Among non-Noetherian rings, the case of UFD's is of interest. For UFD's we have the following result.

THEOREM 1.12. *The following statements are equivalent for a UFD R.*
(i) *R is a PID.*
(ii) *Every primary R-module has prime radical.*
(iii) *The radical of every primary submodule of the R-module $R \oplus R$ is prime.*

Proof. (i) $\Rightarrow$ (ii). This follows from Corollary 1.10.
(ii) $\Rightarrow$ (iii). This is clear.
(iii) $\Rightarrow$ (i). It is sufficient to prove that R is one-dimensional. (See [**7**, Exercise 14.21].) Suppose not. Let $\mathbf{p} \supseteq \mathbf{q} \supseteq 0$ be distinct prime ideals of R, where $\mathbf{q}$ has height 1. Then $\mathbf{q} = Rx$ for some element x. By Lemma 1.8, there exists $c \in R \backslash \mathbf{p}$ such that $cx \in \mathbf{pq} + \mathbf{q}^{(2)} = \mathbf{p}x + Rx^2$ and hence $c \in \mathbf{p} + Rx \subseteq \mathbf{p}$, a contradiction. Thus R is one-dimensional.

2. The radical formula. In this section, our concern is with rings R such that every R-module satisfies the radical formula for primary submodules. An R-module M satisfies the radical formula for primary submodules if and only if $\text{rad}_M(N) = N + \mathbf{p}M$ for every prime ideal $\mathbf{p}$ of R and $\mathbf{p}$-primary submodule N of M (Lemma 1.3).

LEMMA 2.1. *Let R be a ring such that for every non-maximal prime ideal* $\mathbf{q}$, $c\mathbf{q} = 0$ *for some element* $c \in R \backslash \mathbf{q}$. *Then every R-module satisfies the radical formula for primary submodules.*

Proof. Let M be an R-module and let N be a $\mathbf{p}$-primary submodule of M, for some prime ideal $\mathbf{p}$ of R. Suppose first that $\mathbf{p}$ is a maximal ideal of R. By Lemma 1.4, $\text{rad}_M(N) = N + \mathbf{p}M = RE_M(N)$. Now suppose that $\mathbf{p}$ is not maximal. There exists $c \in R \backslash \mathbf{p}$ such that $c\mathbf{p} = 0$ and hence $c\mathbf{p}M = 0 \subseteq N$. It follows that $\mathbf{p}M \subseteq N$ and M/N is a torsion-free $(R/\mathbf{p})$-module. Thus N is a prime submodule of M and $\text{rad}_M(N) = N = RE_M(N)$. Therefore M satisfies the radical formula for primary submodules.

COROLLARY 2.2. *Let R be a one-dimensional domain. Then every R-module satisfies the radical formula for primary submodules.*

Proof. By Lemma 2.1.

Leung and Man [**3**, Theorem 1.1] showed that a Noetherian domain R is Dedekind if and only if every R-module satisfies the radical formula. In view of Corollary 2.2, we see that any one-dimensional Noetherian domain R that is not Dedekind (for example, $R = \mathbb{Z}[\sqrt{5}]$ or more generally $R = \mathbb{Z}[\sqrt{d}]$ for any square-free integer d with $d \equiv 1 (\text{mod } 4)$) has the property that every R-module satisfies the radical formula for primary submodules but the R-module $R \oplus R$ does not satisfy the radical formula (see [**3**, Corollary 5.2]).

We now turn our attention to trying to prove a converse of Lemma 2.1.

LEMMA 2.3. *Let R be a ring such that the R-module $R \oplus R$ satisfies the radical formula for primary submodules. Let $\mathbf{q}$ be any prime ideal of R. Then $a\mathbf{q} \subseteq a^2\mathbf{q} + \mathbf{q}^{(2)}$* for any element $a \in R \backslash \mathbf{q}$.

Proof. Let $a \in R\backslash\mathbf{q}$, $b \in \mathbf{q}$. Let $M = R \oplus R$ and let

$$N = \{(x, y) \in M : a^2x - by \in \mathbf{q}^{(2)}\}.$$

By Proposition 1.2, N is a $\mathbf{q}$-primary submodule of M. Note that if $(x, y) \in N$ then $a^2x \in by + \mathbf{q}^{(2)} \subseteq \mathbf{q}$, so that $x \in \mathbf{q}$. Let K be any prime submodule of M such that $N \subseteq K$. Then $\mathbf{q}^2M \subseteq N$ gives that $\mathbf{q}M \subseteq K$. Since $(b, a^2) \in N$ it follows that $(0, a^2) = (b, a^2) - (b, 0) \in K$, and hence $(0, a) \in K$. Thus $(0, a) \in \text{rad}_M(N) = N + \mathbf{q}M$ by Lemma 1.3

There exist $x, y \in R$, $u, v \in \mathbf{q}$ such that $(0, a) = (x, y) + (u, v)$, where $(x, y) \in N$; i.e. $a^2x - by \in \mathbf{q}^{(2)}$. Now $by = a^2x + z$ for some $z \in \mathbf{q}^{(2)}$ and $a = y + v$, so that

$$ab = yb + vb = a^2x + z + vb \in a^2\mathbf{q} + \mathbf{q}^{(2)}$$

because $x \in \mathbf{q}$ (see above).

THEOREM **2.4**. *The following statements are equivalent for a Noetherian ring R.*
(i) *Every R-module satisfies the radical formula for primary submodules.*
(ii) *The R-module $R \oplus R$ satisfies the radical formula for primary submodules.*
(iii) *For every non-maximal prime ideal $\mathbf{q}$ of R there exists $c \in R\backslash\mathbf{q}$ such that $c\mathbf{q} = 0$.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii). Let $\mathbf{q}$ be any non-maximal prime ideal of R. There exists a maximal ideal $\mathbf{p}$ of R such that $\mathbf{q} \subsetneqq \mathbf{p}$. Let $a \in \mathbf{p}\backslash\mathbf{q}$. By Lemma 2.3, $a\mathbf{q} \subseteq a^2\mathbf{q} + \mathbf{q}^{(2)}$ and hence $M = aM$, where M is the finitely generated R-module $(a\mathbf{q} + \mathbf{q}^{(2)})/\mathbf{q}^{(2)}$. Hence there exists $b \in R$ such that $(1 - ab)M = 0$; i.e. $(1 - ab)a\mathbf{a} \subseteq \mathbf{q}^{(2)}$. It follows that $\mathbf{q} = \mathbf{q}^{(2)}$ and hence $c\mathbf{q} = 0$ for some $c \in R\backslash\mathbf{q}$, by [**7**, Exercise 8.37].

(iii) $\Rightarrow$ (i). This follows from Lemma 2.1.

We have already remarked that there exist Noetherian domains R for which every R-module satisfies the radical formula for primary submodules but not every R-module satisfies the radical formula. The following result shows that this cannot happen for UFD's.

THEOREM 2.5. *The following statements are equivalent for a UFD R.*
(i) *R is a PID.*
(ii) *Every R-module satisfies the radical formula.*
(iii) *Every R-module satisfies the radical formula for primary submodules.*
(iv) *The R-module $R \oplus R$ satisfies the radical formula for primary submodules.*

Proof. (i) $\Rightarrow$ (ii). See [**2**, Theorem 9].

(ii)$\Rightarrow$ (iii) $\Rightarrow$ (iv). These implications are clear.

(iv) $\Rightarrow$ (i). Let $\mathbf{q}$ be any height 1 prime ideal of R. Then $\mathbf{q} = Rq$ for some (prime) element q of R. Moreover, $\mathbf{q}^{(2)} = Rq^2$. Let $a \in R\backslash\mathbf{q}$. By Lemma 2.3, $a\mathbf{q} \in a^2\mathbf{q} + \mathbf{q}^{(2)} =$

$Ra^2q + Rq^2$ and hence $a \in Ra^2 + Rq$. There exist $r, s \in R$ such that $a = ra^2 + sq$ and hence $(1 - ra)a = sq \in \mathbf{q}$. It follows that $1 - ra \in \mathbf{q}$; i.e. $R = Ra + \mathbf{q}$. Therefore $\mathbf{q}$ is a maximal ideal of R. Hence R is a PID.

Combining Theorems 1.9, 1.12, 2.4 and 2.5 we have the following result without further proof.

Corollary 2.6. *Let R be a ring which is either Noetherian or a UFD. Then the following statements are equivalent.*

(i) *Every primary R-module has prime radical.*

(ii) *Every R-module satisfies the radical formula for primary submodules.*

It is natural to ask whether (i) and (ii) in Corollary 2.6 are always equivalent. We have the following special case.

Proposition 2.7. *Let S be a domain and let R be the polynomial ring $S[X]$. Then the following statements are equivalent for R.*

(i) *R is a PID*

(ii) *Every primary R-module has prime radical.*

(iii) *Every R-module satisfies the radical formula (for primary submodules).*

(iv) *S is a field.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 1.12.

(ii) $\Rightarrow$ (i). Let $\mathbf{q}$ denote the prime ideal RX of R. By the proof of (iii) $\Rightarrow$ (i) of Theorem 1.12, $\mathbf{q}$ is a maximal ideal of R. Hence S is a field and R is a PID.

(i) $\Rightarrow$ (iii). This follows from Theorem 2.5.

(iii) $\Rightarrow$ (i). Again using $\mathbf{q} = RX$, the proof of Theorem 2.5, (iv) $\Rightarrow$ (i), shows that $\mathbf{q}$ is a maximal ideal and hence R is a PID.

(i) $\Rightarrow$ (iv). This result is well-known.

Let R denote the polynomial ring $\mathbb{Z}[X]$ and $\mathbf{q}$ the prime ideal RX of R. Let $M = R \oplus R$ and let

$$N = \{(r, s) \in M : 4r - Xs \in RX^2\}.$$

By Proposition 1.2, N is a $\mathbf{q}$-primary submodule of M. It can easily be checked that $N = R(X, 4) + R(0, X) + X^2M$. In this case, $RE_M(N) = N + XM = R(0, 4) + XM$. However, it is also easy to check that $\mathrm{rad}_M(N) = R(0, 2) + XM \neq RE_M(N)$.

Acknowledgement. The author would like to thank the referee for various helpful comments and in particular for drawing Theorem 1.9(v) to his attention.

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and categories of modules* (Springer-Verlag, 1974).

2. J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1992), 3593–3602.

3. Kah Hin Leung and Shing Hing Man, On commutative Noetherian rings which satisfy the radical formula, *Glasgow Math. J.* **39** (1997), 285–293.

4. R. L. McCasland and M. E. Moore, On radicals of submodules of finitely generated modules, *Canad. Math. Bull.* **29** (1986), 37–39.

5. R. L. McCasland and M. E. Moore, On radicals of submodules, *Comm. Algebra* **19** (1991), 1327–1341.

6. H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, *Acta Math. Hungar.* **71** (1996), 103–108.

7. R. Y. Sharp, *Steps in commutative algebra*, London Mathematical Society Student Texts No 19 (Cambridge University Press, 1990).

8. P. F. Smith, On the dimension of group rings, *Proc. London Math. Soc. (3)* **25** (1972), 288–302.