Can. J. Math., Vol. XXXI, No. 6, 1979, pp. 1269–1280

PROJECTIONS ON BERGMAN SPACES OVER PLANE DOMAINS

JACOB BURBEA

1. Introduction. Let D be a bounded plane domain and let $L_p(D)$ stand for the usual Lebesgue spaces of functions with domain D, relative to the area Lebesque measure $d\sigma(z) = dxdy$. The class of all holomorphic functions in D will be denoted by $H(D)$ and we write $B_p(D) = L_p(D) \cap H(D)$. $B_p(D)$ is called the *Bergman p-space* and its norm is given by

$$||f||_p = \left\{ \int_D |f(z)|^p d\sigma(z) \right\}^{1/p}, \quad 0 < p < \infty,$$

$$||f||_\infty = \operatorname{Sup}_{z \in D} |f(z)|.$$

Let $K_D(z, \bar{\zeta})$ be the Bergman kernel of D and consider the *Bergman projection*

$$(1.1) \quad (Pf)(\zeta) = (f, K_D(\ \ , \bar{\zeta})) = \int_D f(z) K_D(\zeta, \bar{z}) d\sigma(z).$$

It is well known that P is not bounded on $L_p(D)$, $p = 1, \infty$, and moreover, it can be shown that there are no bounded projections of $L_\infty(\Delta)$ onto $B_\infty(\Delta)$. Here and throughout this paper Δ stands for the unit disk $\{z: |z| < 1\}$. Bers [**3**], by replacing the Lebesgue measure with the Poincaré measure $\lambda_D^{-2}(z) d\sigma(z)$, where $\lambda_D(z)$ is the Poincaré metric for D, was able to show that $L_1(D)$ is continuously projected onto $B_1(D)$. It is impossible, however, to deduce from Bers result or its modification the existence of bounded projection from $L_p(D)$ onto $B_p(D)$ for $1 < p < \infty$.

Zaharjuta and Judovič [**14**], using the Calderòn-Zygmund theory of singular integrals, showed that P is bounded on $L_p(\Delta)$ for $1 < p < \infty$ and Stein [**11**] extended this result to the unit ball in $\mathbf{C}^n$.

Our main contribution in this paper is in showing that for a multiply connected domain D, with some smoothness requirements to be specified later, the Bergman projection P is bounded on $L_p(D)$ for $1 < p < \infty$. As in [**14**] we also exploit the Calderòn-Zygmund theory of singular integrals. However, our method proceeds in a different direction by first showing that an operator involving the "adjoint" of the Bergman kernel [**2**] is bounded on $L_p(D)$, $1 < p < \infty$. This operator behaves like the Hilbert transform and thus has the required singularity of the Calderòn-Zygmund theory. This property is not shared by the operator P.

Quite recently Bekollé and Bonami [**1**] have characterized the weighted

Received July 19, 1977 and in revised form November 2, 1978 and January 19, 1979.

https://doi.org/10.4153/CJM-1979-105-6 Published online by Cambridge University Press

measures ω on the unit disk Δ for which the Bergman projection P is bounded on $L_p(\Delta:\omega)$, $1 < p < \infty$. Our method can be also applied to this situation and even extend their result to the multiply connected case. This and other related results, however, will be elaborated elsewhere.

In § 2 we review some results from the theory of singular integrals which are needed in our work, and § 3 is devoted to a brief discussion on the various kernels of a domain. In § 4 we introduce some concepts relevant to the degree of smoothness of the domain. We prove two propositions associated with these concepts (Propositions 3 and 4) and we define the crucial class W_p. The main theorem (Theorem 1) is proved in § 5. There we also prove Theorem 2. In § 6 we discuss weak convergence in $B_p(D)$.

Finally, we wish to thank the referee and J. E. Brennan for their valuable comments.

2. Singular integrals. Let D be a bounded domain and set

$$C_p(D) = \{A \in \mathbf{R}^+ : A = k_1 p + k_2 (p-1)^{-1}\},\ 1 < p < \infty,$$

where k_1 and k_2 are positive constants depending only on the shape of D. We consider the following familiar transforms; the Hilbert transform

$$(T_p f)(\zeta) = \frac{1}{\pi} \int_D \frac{1}{\overline{(z-\zeta)}^2} f(z) d\sigma(z)$$

and the Riesz transform

$$(R_D f)(\zeta) = \frac{1}{2\pi} \int_D \frac{\overline{z-\zeta}}{|z-\zeta|^3} f(z) d\sigma(z),$$

where the integrals are taken in the principal value sense. These transforms are singular integrals of the Calderòn-Zygmund type. Therefore, they are bounded on $L_p(D)$ and in fact (cf. [**10**, p. 22])

$$\|T_D\| \leqq A_p,\ \|R_D\|_p \leqq A_p;\ A_p \in C_p(D).$$

The usefulness of the Riesz transform follows from the following well known proposition [**10**, p. 59]:

Proposition 1. *If* $f_{\bar z} \in L_p(D)$ *then* $f_z = -R_D{}^2 f_{\bar z}$ *and therefore*

$$\|f_z\|_p \leqq A_p \|f_{\bar z}\|,\ A_p \in C_p(D).$$

Here $f_z = \partial f/\partial z$ *and* $f_{\bar z} = \partial f/\partial \bar z$.

Let ω be a positive locally integrable function in D. ω is said to belong to $M_p(D)$ $(1 < p < \infty)$ if it satisfies the Muckenhoupt condition:

$$\operatorname*{Sup}_V \left[|V|^{-1} \int_V \omega(z) d\sigma(z) \right] \left[|V|^{-1} \int_V \omega(z)^{-1/(p-1)} d\sigma(z) \right]^{p-1} < \infty,$$

where the supremum is taken over all sectors $V \subset D$ and $|V| = \sigma(V)$. For ready reference we record the following proposition which is due to Coifman and Fefferman [**5**]:

PROPOSITION 2. *Let ω be a positive locally integrable function in D. Then T_D is a bounded operator on $L_p(D:\omega)$ if and only if $\omega \in M_p(D)$.*

3. The Bergman kernel. Let $G = G_D(z, \zeta)$ be the customary Green's function of the domain D. We write

$$G_D(z, \zeta) = H(z, \zeta) - \log |z - \zeta|,$$

where $H = H(z, \zeta)$ is symmetric and harmonic in $(z, \zeta) \in D \times D$. It is well known (see [**2**]) that

$$(3.1) \quad K_D(z, \bar{\zeta}) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{\zeta}}$$

and that its "adjoint" is given by

$$(3.2) \quad L_D(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \zeta}.$$

Here

$$L_D(z, \zeta) = \frac{1}{\pi} \frac{1}{(z - \zeta)^2} - l_D(z, \zeta)$$

where

$$l_D(z, \zeta) = \frac{2}{\pi} \frac{\partial^2 H}{\partial z \partial \zeta},$$

is symmetric and holomorphic in $(z, \zeta) \in D \times D$. We note that the "correction term" $l_D(z, \zeta)$ is identically zero when D is a disk. Also, if ∂D is analytic then $l_D(z, \zeta)$ is holomorphic in $(z, \zeta) \in \bar{D} \times \bar{D}$ (cf. [**2**, p. 211]). If ϕ is a conformal mapping of D onto Ω then

$$G_D(z, \zeta) = G_\Omega(\phi(z), \phi(\zeta))$$

and therefore

$$(3.3) \quad K_D(z, \zeta) = K_\Omega(\phi(z), \overline{\phi(\zeta)})\phi'(z)\overline{\phi'(\zeta)},$$

and

$$(3.4) \quad L_D(z, \zeta) = L_\Omega(\phi(z), \phi(\zeta))\phi'(z)\phi'(\zeta).$$

We introduce the "Bergman-Schiffer transforms"

$$(3.5) \quad (Q_D f)(\zeta) = \int_D \overline{L_D(z, \zeta)} f(z) d\sigma(z)$$

and

$$(3.6) \quad (S_D f)(\zeta) = \int_D \overline{l_D(z, \zeta)} f(z) d\sigma(z)$$

where the first integral is taken in the principal value sense. Therefore

$$(3.7) \quad T_D = Q_D + S_D.$$

4. Smoothness conditions. We now make some assumptions on the smoothness of the domain D. We assume that D is bounded by n nondegenerate boundary components $C_1, C_2, \ldots, C_n$ where, say, C_1 is the outer boundary. Then D can be conformally mapped onto a domain Ω which is bounded by n closed analytic curves. More specifically, let $\phi: D \rightarrow \Omega$ be such a mapping. Then ϕ can be written as $\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$, where each factor ϕ_j is a conformal mapping of a simply connected domain D_j. For example, $\omega_1 = \phi_1(z)$ is conformal on the simply connected domain D_1 which is bounded by C_1 and contains D_1, and $\phi_1(D_1)$ is the unit disk. $\omega_j = \phi_j(\omega_{j-1})$ $(2 \leqq j \leqq n)$ is conformal on the simply connected domain D_j which is bounded by $\phi_{j-1} \circ \phi_{j-2} \circ \ldots \circ \phi_1(C_j)$ and contains $\phi_{j-1} \circ \phi_{j-2} \circ \ldots \circ \phi_1(D)$; $\phi_j(D_j)$ is the exterior of the unit disk. For additional properties of the factorization of a conformal mapping see [**6**]. We let $\psi = \phi^{-1}$ and $\psi_j = \phi_j^{-1}$ $(1 \leqq j \leqq n)$. We also write $F_j = \phi_j \circ \phi_{j-1} \circ \ldots \circ \phi_1$ and $G_j = F_j^{-1}$ $(1 \leqq j \leqq n)$. As far as the smoothness properties of ϕ_j are concerned, we note that they are exactly the same as those of ϕ_1, provided $F_{j-1}(C_j)$ is of the same degree of smoothness as that of C_1. For example, as we shall see later, $\int_{D_1} |\phi_1'(z)|^p d\sigma(z) < \infty$ for all $p < 3$ just because C_1 bounds the simply connected domain D_1. Therefore, for any disk R with a fixed radius $0 < r < \infty$ we have

$$\int_{R \cap D_j} |\phi_j'(\omega_{j-1})|^p d\sigma(\omega_{j-1}) < \infty \quad \text{for all } p < 3$$

and consequently the same is true when $R \cap D_j$ is replaced by $F_{j-1}(D)$.

We write

$$t_n(D) = \operatorname{Sup} \{ r \in \mathbf{R} \cup \{\infty\} : \|\phi'\|_r < \infty \}.$$

This definition is clearly independent of the particular choice of the analytic doman $\Omega = \phi(D)$ and it is also obvious that $t_n(D) \geqq 2$. Here, however, we can even say more. Indeed, Brennan [**4**] has shown that for any simply connected domain D, $t_1(D) \geqq 3 + \tau$, where τ is a positive constant which does not depend on the domain. For close-to-convex domains τ is equal to 1 and probably so in all cases. It is interesting that Brennan's theorem can be also extended to the multiply connected case. This is shown in Proposition 3. The fact that $t_1(D) \geqq 3$ is rather elementary as the following argument shows. Since $\psi(\omega)$ is univalent on Δ we have (cf. [**9**, p. 21])

$$|\psi'(\omega)| \geqq |\psi'(0)| \frac{1 - |\omega|}{(1 + |\omega|)^3} \geqq k(1 - |\omega|^2)$$

with $k = 16^{-1}|\psi'(0)|$. Hence, for $2 < r < 3$,

$$\int_D |\phi'(z)|^r d\sigma(z) = \int_\Delta |\psi'(\omega)|^{2-r} d\sigma(\omega) \leqq k^{2-r} \int_\Delta (1 - |\omega|^2)^{2-r} d\sigma(\omega)$$
$$= \pi(3-r)^{-1} k^{2-r} < \infty .$$

The theorem of Brennan coupled with a successive application of the Holder's inequality on the factorization of ϕ yields:

PROPOSITION 3. *Let D be an n-connected domain as before. Then $t_n(D) \geqq 3 + \tau$ where $\tau > 0$ is a constant independent of D.*

Proof. We use induction on the factors of $\phi = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$. Brennan's theorem shows that $\|\phi_1'\|_p < \infty$ for $p < 3 + \tau$. Assume that for $F_{n-1} = \phi_{n-1} \circ \ldots \circ \phi_1$ we have $\|F_{n-1}'\|_p < \infty$ for $p < 3 + \tau$. For $\phi = \phi_n \circ F_{n-1}$ we have to show that

$$||\phi'||_p^p = \int_D |\phi_n'(F_{n-1}(z))|^p |F_{n-1}'(z)|^p d\sigma(z)$$

is finite for $p < 3 + \tau$. To do so, one has only to check what happens near the boundary curves $\Gamma_{n-1} = C_1 + C_2 + \ldots + C_{n-1}$ and C_n. Near Γ_{n-1} we have

$$|\phi_n'(F_{n-1}(z))| \leqq M$$

and near C_n we have

$$0 < K^{-1} \leqq |F_{n-1}'(z)| \leqq K.$$

Let T_n be a tube near C_n and let T_{n-1} be the tubes near Γ_{n-1}. Then, by the induction assumption,

$$\int_{T_{n-1}} |\phi_n'(F_{n-1}(z))|^p |F_{n-1}'(z)|^p d\sigma(z) \leqq M^p \int_{T_{n-1}} |F_{n-1}'(z)|^p d\sigma(z) < \infty$$

if $p < 3 + \tau$. On the other hand, by Brennan's theorem,

$$\int_{T_n} |\phi_n'(F_{n-1}(z))|^p |F_{n-1}'(z)|^p d\sigma(z) \leqq K^{p-2} \int_{T_n} |\phi_n'(F_{n-1}(z))|^p$$
$$\times |F_{n-1}'(z)|^2 d\sigma(z) = K^{p-2} \int_{F_{n-1}(T_n)} |\phi_n'(\omega)|^p d\sigma(\omega) < \infty$$

if $p < 3 + \tau$. Here, $F_{n-1}(T_n)$ is a tube around $F_{n-1}(C_n)$. This concludes the proof.

In view of the above proposition $t_D \equiv t_n(D) \geqq 3 + \tau$. We can therefore define the interval

$$I(D) = \begin{cases} \left[\dfrac{t_D}{t_D - 1}, t_D\right] ; & ||\phi'||_{t_D} < \infty . \\ \left(\dfrac{t_D}{t_D - 1}, t_D\right) ; & ||\phi'||_{t_D} = \infty . \end{cases}$$

We also write

$$J(D) = I(D) - \{1, \infty\}.$$

Therefore, if $\|\phi'\|_\infty < \infty$ or if $\|\phi'\|_r < \infty$ for each $3 + \tau \leqq r < \infty$ we have $J(D) = (1, \infty)$. In the first case $I(D) = [1, \infty]$ and in the second $I(D) = (1, \infty)$.

If D is simply connected and ∂D is of class C^1 then it follows from a theorem of Warschawski [**13**] (see also [**4**]) that $\|\phi'\|_r < \infty$ for every $r < \infty$. This theorem can be extended to our setting by using the same arguments as those of Proposition 3. Therefore, if $\partial D \in C^1$ then $t_D = \infty$. In this case, however, it may happen that $\|\phi'\|_\infty = \infty$ as the example of [**12**, p. 377] shows. On the other hand, if D is simply connected and ∂D is of class C^1 with a Dini continuous normal then it follows from yet another theorem of Warschawski (see [**9**, p. 298]) that there exist positive constants a and b such that

$$(4.1) \quad 0 < a \leqq |\phi'(z)| \leqq b < \infty, z \in D.$$

This is also true in the more general case when D is multiply connected by appealing to the above factorization of ϕ. Hence, $I(D) = [1, \infty]$ for D with ∂D being Dini-smooth. The last inequality could be also derived from a corresponding inequality for the derivatives of the Green's function. Indeed, if ∂D is of class C^1 with a Hölder continuous normal one has such an inequality (see [**7**]) and the same is true under the weaker assumption that ∂D is merely Dini-smooth.

From (3.3) follows that, for every $\zeta \in D$, $K_D(\ ,\bar{\zeta})$ is in $B_r(D)$ whenever $r \in I(D)$ and in fact:

PROPOSITION 4. *Let $p \in I(D)$. Then for each $f \in L_p(D)$, the Bergman projection* (1.1) *is in $H(D)$ and $Pf = f$ for every $f \in B_p(D)$.*

For a fixed $p \in J(D)$ we let $q = p/(p - 1)$ (of course $q \in J(D)$). D is said to belong class W_p if ϕ' satisfies

$$\operatorname*{Sup}_U \left(\frac{1}{||\phi'||_{2:U}{}^2} \cdot ||\phi'||_{p:U} ||\phi'||_{q:U} \right) < \infty,$$

where the supremum is taken over all sectors $U \subset D$ and

$$||f||_{k:U} = \left[\int_U |f(z)|^k d\sigma(z) \right]^{1/k}.$$

Obviously, the definition of $D \in W_p$ is independent of the particular choice of the analytic domain $\Omega = \phi(D)$. It is also clear that always $D \in W_2$ and that $D \in W_p$ if and only if $D \in W_q$. If ∂D is Dini-smooth then it follows from (4.1) that $D \in W_p$ for all p. Note also that the above definition is exactly the previously mentioned $M_p(\Omega)$ condition for the weight $\lambda = |\psi'|^{2-p}$ and where $U = \psi(V)$.

We do not know whether $D \in W_p$, $p \neq 2$, when ∂D is merely of class C^1.

5. The Bergman projection. The following lemma is crucial.

LEMMA 1. *Let $p \in J(D)$. The operator Q_D is bounded on $L_p(D)$ if and only if D is in W_p; and in this case $\|Q_D\|_p \leqq A_p$, $A_p \in C_p(D)$.*

Proof. For $z, \zeta \in D$ we write $\omega = \phi(z)$, $\tau = \phi(\zeta)$ with $\omega, \tau \in \Omega$. Also, for $f \in L_p(D)$ we let $g = (f \circ \psi) \cdot \psi'$. Using (3.4), (3.5), (3.6) and (3.7) we have

$$(Q_D f)(\zeta) = \overline{\phi'(\zeta)} \cdot \int_\Omega \overline{L_\Omega(\omega, \tau)} g(\omega) d\sigma(\omega) = \overline{\phi'(\zeta)} \cdot (Q_\Omega g)(\tau)$$
$$= \overline{\phi'(\zeta)} \cdot (T_\Omega g)(\tau) - \overline{\phi'(\zeta)} \cdot (S_\Omega g)(\tau).$$

Since $l_\Omega(\omega, \tau)$ is holomorphic for $(\omega, \tau) \in \bar{\Omega} \times \bar{\Omega}$ we have that $|l_\Omega(\omega, \tau)| \leqq A$ and therefore

$$\int_D |\phi'(\zeta)|^p |(S_\Omega g)(\tau)|^p d\sigma(\zeta) = \int_D |\phi'(\zeta)|^p \left| \int_\Omega \overline{l_\Omega(\omega, \tau)} g(\omega) d\sigma(\omega) \right|^p d\sigma(\zeta)$$
$$\leqq A^p \|\phi'\|_p^p \left[\int_D |f(z)| |\phi'(z)| d\sigma(z) \right]^p \leqq A^p \|\phi'\|_p^p \|\phi'\|_q^p \|f\|_p^p.$$

Consequently, since $p, q \in J(D)$, we have that the $L_p(D)$ boundedness of Q_D is equivalent to the inequality

$$\left\{ \int_D |\phi'(z)|^p |(T_\Omega g)(\omega)|^p d\sigma(z) \right\}^{1/p} \leqq A_p \|f\|_p.$$

The last inequality, however, is equivalent to

$$\left\{ \int_\Omega |(T_\Omega g)(\omega)|^p |\psi'(\omega)|^{2-p} d\sigma(\omega) \right\}^{1/p} \leqq A_p \left\{ \int_\Omega |g(\omega)|^p |\psi'(\omega)|^{2-p} d\sigma(\omega) \right\}^{1/p}.$$

Therefore, Q_D is bounded on $L_p(D)$ if and only if the Hilbert transform T_Ω is a bounded operator on $L_p(\Omega : |\psi'|^{2-p})$. An appeal now to Proposition 2 concludes the proof.

We are now in a position to state our main theorem. Its special case when D is the unit disk was resolved by a different method by Zaharjuta and Judovič [**14**].

THEOREM 1. *Let $p \in J(D)$. Then P is a bounded linear projection of $L_p(D)$ onto $B_p(D)$ if and only if $D \in W_p$; and in that case $\|P\|_p \leqq A_p$, $A_p \in C_p(D)$.*

Proof. In view of Proposition 4, we only have to prove the statement on the boundedness of P. For any $f \in L_p(D)$ we let

$$g(\zeta) = 2\pi^{-1} \int_D G_{\bar{z}}(z, \zeta) f(z) d\sigma(z).$$

From classical results of potential theory it is well known that g_ζ and $g_{\bar{\zeta}}$ exist a.e. in D, and they are given by

(5.1) $$g_\zeta(\zeta) = f(\zeta) + 2\pi^{-1} \int_D H_{\bar{z}\zeta}(z, \zeta) f(z) d\sigma(z)$$

and

$$(5.2) \quad g_{\bar{\zeta}}(\zeta) = 2\pi^{-1}\int_D G_{\bar{z}\bar{\zeta}}(z, \zeta)f(z)d\sigma(z).$$

According to (3.2) and (3.5), (5.2) can be written as

$$g_{\bar{\zeta}}(\zeta) = -(Q_D f)(\zeta).$$

Moreover, $H_{\bar{z}\zeta} = G_{\bar{z}\zeta}$, while by (1.1), (3.1) and (5.1)

$$g_{\zeta}(\zeta) = (I - P)f(\zeta),$$

where I is the identity operator on $L_p(D)$. According to Proposition 1, $g_\zeta = -R_D{}^2 g_{\bar{\zeta}}$ and therefore

$$I - P = R_D{}^2 Q_D.$$

The theorem now follows from Lemma 1 and the boundedness of the Riesz transform R_D.

Remark. According to the previously mentioned result of Bers [**3**] $L_1(D)$ is continuously projected onto $B_1(D)$. Therefore we can deduce, using [**8**], that $B_1(D)$, for any domain D whose boundary contains more than two points, is topologically isomorphic to l_1. In the same manner, Theorem 1 shows, for $p \in J(D)$ and $D \in W_p$, that $B_p(D)$ is topologically isomorphic to l_p.

Throughout the rest of this section we shall always assume that $p \in J(D)$ and $D \in W_p$. For $f \in L_p(D)$ and $g \in L_q(D)$ we set

$$(f, g) = \int_D f(z)\overline{g(z)}d\sigma(z).$$

Corollary 1. *The operator P is self-adjoint, and, in fact,*

$$(Pf, g) = (f, Pg) = (Pf, Pg); f \in L_p(D), \ g \in L_q(D),$$
$$\|P\|_p = \|P\|_q, \ \|P\|_2 = 1.$$

Proof. These follow from Fubini's theorem, Theorem 1 and Hölder's inequality.

Corollary 2. *We have the direct sum decomposition*

$$L_p(D) = B_p(D) \oplus B_q(D)^\perp.$$

Proof. For $f \in L_p(D)$, let $h = Pf$ and $h^\perp = (I - P)f$. Hence $f = h + h^\perp$ and by Theorem 1, $h \in B_p(D)$. Let $g \in B_q(D)$; then $Pg = g$ and by Corollary 1

$$(h^\perp, g) = ((I - P)f, g) = (f, g) - (Pf, g) = (f, g) - (f, Pg) = 0.$$

If $f \in B_p(D) \cap B_q(D)^{\perp}$, then $(f, g) = 0$ for all $g \in B_q(D)$. However, $K_D(\ , \bar{\zeta})$ is in $B_q(D)$, and so, using Proposition 4, $f(\zeta) = 0$ for all $\zeta \in D$.

We now generalize a result of [**14**] proved for the unit disk Δ.

THEOREM 2. *The projection P satisfies*

$$A_p^{(2)} \leqq \|P\|_p \leqq A_p^{(1)}; A_p^{(j)} \in C_p(D), j = 1, 2.$$

Proof. By Theorem 1 we have only to show that $\|P\|_p \geqq A_p^{(2)}$. We may assume, without any loss of generality, that $o \in D$. Let $a_0 \in C_1$ and therefore $a \equiv |a_o| > 0$. Consider the function

$$F_0(z) = g_0(z)[\log (1 - z/a_0) - \log (1 - \bar{z}/\bar{a}_0)],$$

where $g_0(z) = K_D(z, \bar{o})$. Clearly, $F_0 \in L_p(D)$ and $\|F_0\| \leqq M_0\|\phi'\|_p$, where $M_0 > 0$ depends only on D. Let

$$h_0(z) = g_0(z) \log (1 - \bar{z}/\bar{a}_0).$$

We shall show that $Ph_0 = 0$ or, in other words, that h_o belongs to the annihilator of $B_p(D)$. To this end we may also assume that $\partial D \in C^1$. Using Green's formula, we have

$$(Ph_0)(\zeta) = \int_D h_0(z)K_D(\zeta, \bar{z})d\sigma(z) = \tfrac{1}{2}i \int_{\partial D} [-2\pi^{-1}\, \partial G/\partial \bar{t}|_{t=0}] \times \log (1 - \bar{z}/\bar{a}_0)K_D(\zeta, \bar{z})d\bar{z}.$$

Here, we used the fact that $\partial/\partial z R_0(z) = h_0(z)$, where $R_0(z)$ is given by

$$R_0(z) = [-2\pi^{-1}\partial G/\partial \bar{t}|_{t=0}] \log (1 - \bar{z}/\bar{a}_0).$$

$(Ph_0)(\zeta) \equiv 0$, because $R_0(z)$ vanishes near ∂D, and therefore we need not make any assumption on the smoothness of ∂D, apart from $p \in J(D)$ and $D \in W_p$. Consequently,

$$f_0(z) = (PF_0)(z) = g_0(z) \log (1 - z/a_0),$$

and, by Theorem 1, $f_0 \in B_p(D)$. Consider the sector

$$D(\epsilon, \alpha) = \{z: |z - a_0| \leqq \epsilon, |\arg (a_0 - z) - \arg a_0| \leqq \alpha/2\},$$

where $0 \leqq \alpha < 1$, $0 < \epsilon < a$. Now, $K_D(z, \bar{o})$ has only a finite number of zeros in D, none of which is near ∂D. We choose $\epsilon > 0$ to be small enough so that $D(\epsilon, \alpha) \subset D$ and that there $|g_0(z)| = |K_D(z, \bar{o})| \geqq A > 0$. We can further restrict $\epsilon > 0$ to be within

$$e^{-Mp} < \epsilon < ae^{-p}\|\phi'\|_p A^{-1},$$

where $M > 0$ depends only on D, and is chosen to be large enough. Then,

$$\|f_0\|_p^p = \int_D |g_0(z)|^p \left|\log\left(1 - \frac{z}{a_0}\right)\right|^p d\sigma(z) \geqq \int_D |g_0(z)|^p \left|\log\left|1 - \frac{z}{a_0}\right|\right|^p$$

$$\times\, d\sigma(z) \geqq \int_{D(\epsilon,\alpha)} |g_0(z)|^p \left|\log \frac{|z - a_0|}{a}\right|^p d\sigma(z)$$

$$\geqq A^p \int_{D(\epsilon,\alpha)} \left|\log \frac{|z - a_0|}{a}\right|^p d\sigma(z) = A^p \int_0^{\epsilon} \int_{-\alpha/2}^{\alpha/2} \left|\log \frac{r}{a}\right|^p r dr d\theta$$

$$= A^p \alpha \int_0^{\epsilon} \left|\log \frac{r}{a}\right|^p r dr = A^p \alpha a^2 \int_0^{\epsilon/a} \left(\log \frac{1}{s}\right)^p s ds$$

$$= A^p \alpha \epsilon^2 \int_0^1 \left(\log \frac{a}{t\epsilon}\right)^p t dt > A^p \alpha \frac{\epsilon^2}{2} \left(\log \frac{a}{\epsilon}\right)^p > A^p \alpha \frac{\epsilon^2}{2} (p\|\phi'\|_p A^{-1})^p$$

$$= \frac{\alpha}{2} \epsilon^2 \|\phi'\|_p^p p^p > \frac{\alpha}{2} e^{-2Mp} \|\phi'\|_p^p p^p.$$

Therefore $\|f_0\|_p > M_1 \|\phi'\|_p p$, where $M_1 > 0$ depends only on D. Now,

$$\|P\|_p \geqq \frac{\|PF_0\|_p}{\|F_0\|_p} = \frac{\|f_0\|_p}{\|F_0\|_p} > \frac{M_1}{M_0} p,$$

and hence $\|P\|_p \geqq M_2 p$. From Corollary 1,

$$\|P\|_p = \|P\|_q > M_2 q > M_2/(p - 1)$$

and the theorem is proved.

Remark. The factor $g_0(z) = K_D(z, \bar{o})$ in the definition of $F_0(z)$ was needed to ensure that $h_0 \in B_q(D)^\perp$. If D was the unit disk Δ then $g_0(z) = K_\Delta(z, \bar{o}) \equiv \pi^{-1}$ (a_0 will be chosen as 1). This property is characteristic to all disks.

For $g \in B_q(D)$, we let $L_g(f) = (f, g)$ for all $f \in B_p(D)$. Using the previous assertions and a standard argument based on the Hahn-Banach theorem yields (cf. also [**14**]):

COROLLARY 3. *The mapping $T: B_q(D) \to (B_p)^*$ given by $T(g) = L_g$ is an anti-linear isomorphism of $B_q(D)$ onto the dual of $B_p(D)$, $(B_p)^*$. T is an isometry for $p = 2$, and, for $p \in J(D) - \{2\}$, the "isometry distortion", which is given by*

$$I_q = \operatorname{Sup}\{\|g\|_q / \|L_g\| : g \in B_q(D)\},$$

satisfies

$$A_p^{(2)} \leqq I_q \leqq A_q^{(1)};\ A_q^{(j)} \in C_q(D),\ j = 1, 2.$$

6. Weak convergence. Let $f_n, f \in B_p(D)$, $1 \leqq p \leqq \infty$. As usual, $f_n \to f$ weakly in $B_p(D)$ if $L(f_n) \to L(f)$ for each $L \in (B_p)^*$. The uniqueness of the weak limit, if it exists, is obvious in this case.

Assume now that $p \in I(D)$ and let $\{t_n\}$ be a dense sequence in the domain D. Consider the sequence of functions $\Phi_n(z) = K_D(z, \bar{t}_n)$, $n = 1, 2, \ldots$. In view of Proposition 4, for any $f \in B_p(D)$, $(f, \Phi_n) = 0$, $n = 1, 2, \ldots$, if and only if $f = 0$. We have the obvious:

LEMMA 2. *Let $p \in J(D)$ and $D \in W_p$. Then the linear envelope of the Φ_n's $N = [\Phi_n]$ is dense in $B_p(D)$.*

Proof. Suppose not, and let $f_0 \in B_p(D) - N$, $f_0 \neq 0$. The Hahn-Banach theorem implies the existence of $L \in (B_p)^*$ with $L(f_0) = 1$ and $L(N) = \{0\}$. According to Corollary 3, $L(f) = (f, g_L)$, $g_L \in B_q(D)$ and all $f \in B_p(D)$. Since $L(N) = \{0\}$, $L(\Phi_n) = (\Phi_n, g_L) = 0$, $n = 1, 2, \ldots$. Thus $g_L = 0$, contradicting $L(f_0) = 1$.

THEOREM 3. (i) *Suppose $f_n \rightarrow f$ weakly in $B_p(D)$, $1 \leqq p \leqq \infty$. Then $\{\|f_n\|_p\}$ is bounded and $f_n(z) \rightarrow f(z)$ uniformly on compacta of D.*

(ii) *Let $p \in J(D)$ and $D \in W_p$ and suppose that $\{\|f_n\|p\}$ is bounded, and that $f_n(z) \rightarrow f(z)$ for each $z \in D$. Then $f_n \rightarrow f$ weakly in $B_p(D)$.*

Proof. (i) $\{\|f_n\|_p\}$ is bounded because, in any normed space, the norms of a weakly convergent sequence are bounded. The subharmonicity of $|f(z)|^p$ in D implies now that $f_n(z) \rightarrow f(z)$ uniformly on compacta of D.

(ii) Assume $\|f_n\|_p \leqq M$. Hence $\{|f_n(z)|\}$ is uniformly bounded on compacta of D. Thus $f \in H(D)$ and $\|f\|_p \leqq M$. Since $f_n(t_m) \rightarrow f(t_m)$ as $n \rightarrow \infty$, we have $\lim_{n\rightarrow\infty}(f_n - f, \Phi_m) = 0$, $m = 1, 2, \ldots$. Let $L \in (B_p)^*$. According to Corollary 3, $L(f_n - f) = (f_n - f, g_L)$ for some $g_L \in B_q(D)$. Given $\epsilon > 0$, there is, in view of Lemma 2, an $h \in [\Phi_n]$, such that $\|g_L - h\|_q < \epsilon/4M$. Further, there is an integer $n(\epsilon)$ such that $|(f_n - f, h)| \leqq \epsilon/2$ for $n > n(\epsilon)$. Hence for $n > n(\epsilon)$

$$|L(f_n - f)| \leqq |(f_n - f, g_L - h)| + |(f_n - f, h)| \leqq \|f_n - f\|_p \|g_L - h\|_q + \epsilon/2 < \epsilon,$$

and $f_n \rightarrow f$ weakly in $B_p(D)$.

The fact that (ii) of Theorem 3 is not true for $p = 1$ can be seen from the following example: Let $f_n(z) = nz^n$, $n = 1, 2, \ldots$. Clearly $f_n \in B_1(\Delta)$ and $\|f_n\|_1 < 2\pi$ for each n. Next, $f_n(z) \rightarrow 0$ uniformly on compacta of Δ. Choose a function $g(z)$ in $L_\infty(\Delta)$ to be defined as follows: Let

$$[0, 1) = \cup_{k=0}^{\infty}[r_k, r_{k+1}); r_k = 1 - 2^{-k}, k = 0, 1, \ldots,$$

and set

$$g(re^{i\theta}) = e^{i2^{k+1}\theta} \text{ for } r \in [r_k, r_{k+1}).$$

Then, for

$$L(f) = \int_\Delta f(z)\overline{g(z)}d\sigma(z) = \sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} \int_0^{2\pi} f(re^{i\theta})re^{-i2^{k+1}\theta}d\theta dr,$$

$L \in B_1(\Delta)^*$. However,

$$\lim_{j\to\infty} L(f_{2^{j+1}}) = 2\pi(e^{-1} - e^{-2}) \neq 0,$$

and $\{f_n\}$ does not converge weakly.

COROLLARY 4. *Let* $p \in J(D)$ *and* $D \in W_p$. *Suppose* $f_n, f \in B_p(D)$ *with* $f_n(z) \to f(z)$ *for each* $z \in D$ *and* $\|f_n\|_p \to \|f\|_p$. *Then* $\|f_n - f\|_p \to 0$.

Proof. This follows from Theorem 3 (ii) and the fact that $B_p(D)$ is locally uniformly convex.

REFERENCES

1. D. Bekollé and A. Bonami, *Inégalités à poids pour le noyau de Bergman*, C. R. Acad. Sc. Paris *286* (1978), 775–778.
2. S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Math. *8* (1951), 205–249.
3. L. Bers, *Automorphic forms and Poincaré series for infinitely generated Fuchsian groups*, Amer. J. Math. *87* (1965), 196–214.
4. J. E. Brennan, *The integrability of the derivative in conformal mapping*, J. London Math. Soc. *18* (1978), 261–272.
5. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. *51* (1974), 241–250.
6. O. Hübner, *Die Faktorisierung konformer Abbildungen und Anwendungen*, Math. Z. *92* (1966), 95–109.
7. O. D. Kellogg, *Harmonic functions and Green's integral*, Trans. Amer. Math. Soc. *13* (1912), 109–132.
8. J. Lindenstrauss and A. Pełczyński, *Contributions to the theory of the classical Banach spaces*, J. Func. Analysis *8* (1971), 225–249.
9. Chr. Pommerenke, *Univalent functions* (Vandenhoeck and Ruprecht, Göttingen, 1975).
10. E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton Univ. Press, Princeton, 1970).
11. ——— *Singular integrals and estimates for the Cauchy-Riemann equations*, Bull. Amer. Math. Soc. *79* (1973), 440–445.
12. M. Tsuji, *Potential theory in modern function theory* (Maruzen Co., Ltd., Tokyo, 1959).
13. S. E. Warschawski, *On conformal mapping of regions bounded by smooth curves*, Proc. Amer. Math. Soc. *2* (1951), 254–261.
14. V. P. Zaharjuta and V. I. Judovič, *The general form of a linear functional in* H_p', (Russian) Uspehi. Mat. Nauk. *19* (1964), 139–142.

University of Pittsburgh,
Pittsburgh, Pennsylvania