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NOTES ON A PAPER BY J. B. MILLER

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Abstract

Two sums given by J. B. Miller are evaluated in terms of classical hypergeometric results.

1. Introduction

In a recent paper [2] on the foliage density equation, J. B. Miller reproduces a proof due to G. A. Watterson, of the relation

$$\sum_{l=0}^{j}\binom{2k}{2l}\binom{k-l}{k-j}=\frac{k}{j}\binom{k+j-1}{k-j}2^{2j-1},\qquad 0\leqslant j\leqslant k. \tag{1}$$

He also considers the sum

$$S_r(j)={}_3F_2\left(\begin{matrix}-r,r+2j, & j\\ j-\frac{1}{2}, & 2j+2\end{matrix}\middle|1\right) \tag{2}$$

and says that by using a Burroughs B6700 computer B. J. Milne has obtained the following formulae, valid at least for $r=0,1,\ldots,12$ and all $j\geqslant 1$. There are separate forms for $r=2s+1$ and $r=2s$, with integral $s\geqslant 1$;

$S_{2s+1}(j)=$

$$-\frac{1^2\cdot 3^2\cdot 5^2\cdot\cdots\cdot(2s-1)^2(2s+1)}{2^{2s+1}\left(j^2-\frac{1}{4}\right)(j+1)\left(j+\frac{3}{2}\right)^2\left(j+\frac{5}{2}\right)^2\cdots\left(j+\frac{2s-1}{2}\right)^2\left(j+\frac{2s+1}{2}\right)}, \tag{3}$$

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$$S_{2s}(j) = S_{2s+1} \cdot \frac{2j^2 + (4s+1)j + 4s^2 - 1}{(2s-1)(2s+1)}, \qquad s = 1, 2, 3, \dots. \tag{4}$$

2. Some hypergeometric results

a) Note that (1) is a special case of Gauss's classical result

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad R(c-a-b) > 0.$$

More explicitly, for $0 \leqslant j \leqslant k$, we have

$$\sum_{l=0}^{j} \binom{2k}{2l}\binom{k-l}{k-j} = \binom{k}{j} {}_2F_1\left(\begin{matrix} -j, & \frac{1}{2}-k \\ & \frac{1}{2} \end{matrix}\middle| 1\right) = \frac{(k+j-1)!k}{j!(k-j)!(\frac{1}{2})_j},$$

where

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall n \in \{1, 2, \dots\}. \end{cases}$$

b) Instead of (2), consider the more general sum

$$S(a,b,c) = {}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b-1), & 2c+2 \end{matrix}\middle| 1\right). \tag{5}$$

Rainville [3], pp. 81–85, has given many relations involving contiguous functions, one of the simplest of which is

$$(\alpha - \beta + 1)F = \alpha F(\alpha+1) - (\beta-1)F(\beta-1).$$

Letting

$$F = {}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right),$$

then for $\alpha = c$ and $\beta = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}$, we obtain

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right) \\ &\quad = \frac{2c}{1+2c-a-b}\,{}_3F_2\left(\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right) \\ &\qquad - \frac{a+b-1}{1+2c-a-b}\,{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b-1), & 2c+2 \end{matrix}\middle| 1\right). \end{aligned}$$

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Also, when $\alpha = c$ and $\beta = 2c + 2$, we find that

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right) = \frac{2c+1}{c+1}\,{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \end{matrix}\middle| 1\right) - \frac{c}{c+1}\,{}_3F_2\left(\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right).
$$

Equating the right-hand sides of these last two relations yields

$$
\frac{a+b-1}{1+2c-a-b}S(a,b,c) = \frac{c(3+4c-a-b)}{(c+1)(1+2c-a-b)}\,{}_3F_2\left(\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right) - \frac{2c+1}{c+1}\,{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \end{matrix}\middle| 1\right). \tag{6}
$$

Finally, the left hand side can be easily transformed into the right hand side in the following relation:

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \end{matrix}\middle| 1\right) - {}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\middle| 1\right) = -\frac{ab}{(2c+1)(a+b+1)}\,{}_3F_2\left(\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c+2 \end{matrix}\middle| 1\right).
$$

Using this in the right side of (6) yields

$$
\begin{aligned}
S(a,b,c) = {} & \frac{c(3+4c-a-b)}{(c+1)(a+b-1)}\,{}_3F_2\left(\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+1), & 2c+2 \end{matrix}\middle| 1\right) \\
& - \frac{(2c+1)(1+2c-a-b)}{(c+1)(a+b-1)}\,{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\middle| 1\right) \\
& + \frac{ab(1+2c-a-b)}{(c+1)(a+b-1)(a+b+1)}\,{}_3F_2\left(\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c+2 \end{matrix}\middle| 1\right).
\end{aligned}
$$

The three ${}_3F_2$, on the right of the last relation, can be evaluated by Watson's classical summation formula [1], p. 16, 3.3.1:

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\middle| 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})}.
$$

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Thus

$$S(a,b,c) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2})\Gamma(c+\frac{3}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2})}{(c+1)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}$$

$$\times\left\{\frac{ab\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}b+1)\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+1)}\right.$$

$$\left.-\frac{(1-a)(1-b)+(1-a-b)c}{\Gamma(c-\frac{1}{2}a+\frac{3}{2})\Gamma(c-\frac{1}{2}b+\frac{3}{2})}\right\}, \quad (7)$$

$$R(2c-a-b) > -3.$$

In particular, from (2), (5) and (7):

$$S_{2s+1}(j) = S(-1-2s, 2s+2j+1, j)$$

$$= -\frac{(s+\frac{1}{2})(s+j+\frac{1}{2})(\frac{1}{2})_s(\frac{1}{2})_s}{(j+1)(j^2-\frac{1}{4})(j+\frac{3}{2})_s(j+\frac{3}{2})_s},$$

which is (3), and (4) is similarly obtained.

3. Some other sums

The expressions

$$u_1 = {}_4F_3\left(\begin{matrix} 3/2, & 1, & t, & -t \\ 1/2, & 3/2+t, & & 3/2-t \end{matrix}\middle|1\right).$$

and

$$u_2 = {}_4F_3\left(\begin{matrix} 3/2, & 1, & t+1, & -t \\ & 1/2, & 5/2+t, & 3/2-t \end{matrix}\middle|1\right)$$

are found in (45) and (46) of Miller's paper and they can be evaluated. These are terminating Saalschützian ${}_4F_3$ and they can be transformed by the following formula, [1], p. 56, 7.2.1:

$${}_4F_3\left(\begin{matrix} x, y, z, -n \\ u, v, w \end{matrix}\middle|1\right)$$

$$= \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left(\begin{matrix} u-x, & u-y, & z, & -n \\ 1-v+z-n, & 1-w+z-n, & u & \end{matrix}\middle|1\right).$$

With

$$x = 3/2, \quad y = 1, \quad z = t, \quad n = t, \quad u = 1/2, \quad v = 3/2+t, \quad w = 3/2-t,$$

the ${}_4F_3$ on the right will contain only two non zero terms and we find that

$$u_1 = \frac{(1-4t^2)(1-8t^2)}{1-16t^2}.$$

We find also, in exactly the same way, that

$$u_2 = -\frac{(2t-1)(2t+3)}{(4t+1)(4t+3)}(8t^2+8t+1).$$

References

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