Proceedings of the Edinburgh Mathematical Society (1986) **29**, 15–21 ©

BOUNDARY PROBLEMS FOR RICCATI AND LYAPUNOV EQUATIONS

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(Received 30th October 1984)

1. Introduction

The resolution problem of the system

$$d/dt\,U(t)=A+BU(t)+U(t)B^*+U(t)DU(t);\ U(t_0)=U_0$$

where $U(t)$, A, B, D and U_0 are bounded linear operators on H and B^* denotes the adjoint operator of B, arises in control theory, [**9**], transport theory, [**12**], and filtering problems, [**3**]. The finite-dimensional case has been introduced in [**6, 7**], and several authors have studied the infinite-dimensional case, [**4**], [**13**], [**18**]. A recent paper, [**17**], studies the finite dimensional boundary problem

$$\left.\begin{aligned} d/dt\,U(t)=A+BU(t)+U(t)B^*+U(t)DU(t)\\ U(b)-U(0)=G \end{aligned}\right\} \tag{1.1}$$

where $t\in[0,b]$. In this paper we consider the more general boundary problem

$$\left.\begin{aligned} d/dt\,U(t)=A+BU(t)-U(t)C-U(t)DU(t)\\ EU(b)-U(0)F=G \end{aligned}\right\} \tag{1.2}$$

where all operators which appear in (1.2) are bounded linear operators on a separable Hilbert space H. Note that we do not suppose $C=-B^*$ and the boundary condition in (1.2) is more general than the boundary condition in (1.1).

The idea of the present work is to reduce the boundary differential problem to an algebraic problem and to generate solutions for the algebraic problem.

We denote by $L(H)$ the algebra of all bounded linear operators defined on H. If $L\in L(H)$, we denote the numerical range of L by $w(L)=\{z\in\mathbb{C};\ z=\langle Lx,x\rangle, \|x\|=1\}$, and denote its spectrum by $\delta(L)$. In accordance with definition 3 given by H. Kuiper in [**8**], in a more general context, we say that L is of type (w,δ), $w\in R$, $0<\delta<\pi/4$, if

$$\sigma(L)\cup w(L)\subset \sum_{w,\delta}=\{z\in\mathbb{C};\ |\arg(w-z)|\leq\pi/2-\delta\}.$$

2. Boundary problems for operator Riccati equations

The first result is a necessary condition for the existence problem of a particular type of solution of the problem (1.2).

Theorem 1. *Let $U(t)$ be a solution of (1.2) such that for all t in $[0,b]$ satisfies the property:*

$$C+DU(t) \text{ is an operator of type } (w,\delta).$$

If we let

$$\exp\left(t\begin{bmatrix} C & D \\ A & B \end{bmatrix}\right)=\begin{bmatrix} S_1(t) & S_2(t) \\ S_3(t) & S_4(t) \end{bmatrix}$$

then $U=U(0)$ is a solution of the operator equation

$$M+NU-UP-UQU=0 \tag{2.1}$$

where

$$M=ES_3(b)-GS_1(b); \quad P=FS_1(b)$$

$$N=ES_4(b)-GS_2(b); \quad Q=FS_2(b).$$

Proof. Let $U(t)$ be a solution of (1.2) which satisfies the hypothesis of Theorem 1. From [**8**], p. 29–33, there exists only one solution $V(t)$ of the problem

$$d/dt\, V(t)=(C+DU(t))V(t); \quad V(0)=I$$

for all $t\in[0,b]$.

Let

$$Z(t)=U(t)V(t),$$

where $t\in[0,b]$. Then $Z(t)$ satisfies

$$d/dt\, Z(t)=(A+BU(t))V(t); \quad Z(0)=U(0)=U_0.$$

Thus the $L(H\oplus H)$ valued function $t\rightarrow\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}$, $t\in[0,b]$, is a solution of the problem

$$d/dt\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}=\begin{bmatrix} C & D \\ A & B \end{bmatrix}\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}; \quad \begin{bmatrix} V(0) \\ Z(0) \end{bmatrix}=\begin{bmatrix} I \\ U_0 \end{bmatrix}. \tag{2.3}$$

As the operator function $S(t,s)$, $s,t\in[0,b]$, defined by

$$S(t,s)=\exp\left(\begin{bmatrix} C & D \\ A & B \end{bmatrix}(t-s)\right) \tag{2.4}$$

is a fundamental solution of (2.3) such that

$$S(t,0)=\begin{bmatrix} S_1(t) & S_2(t) \\ S_3(t) & S_4(t) \end{bmatrix}, \tag{2.5}$$

it follows that for all $t\in[0,b]$,

$$\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}=\begin{bmatrix} S_1(t)+S_2(t)U_0 \\ S_3(t)+S_4(t)U_0 \end{bmatrix}. \tag{2.6}$$

By using the boundary condition satisfied by $U(t)$ in (1.2), with $U(0)=U_0$, we obtain from (2.2) that

$$EZ(b)=EU(b)V(b)=(G+U_0F)V(b).$$

From (2.6) it follows that

$$E(S_3(b)+S_4(b)U_0)=(G+U_0F)(S_1(b)+S_2(b)U_0) \tag{2.7}$$

that is,

$$(ES_3(b)-GS_1(b))+(ES_4(b)-GS_2(b))U_0-U_0FS_1(b)-U_0FS_2(b)U_0=0$$

and the result is proved.

The following theorem give us a sufficient condition for the existence of solutions to (1.2).

Theorem 2. *Let U_0 be a solution of* (2.1) *where M, N, P and Q are given as in Theorem* 1. *If for all t in $[0,b]$, it is verified that*

$$S_1(t)+S_2(t)U_0 \tag{2.8}$$

is invertible, there exists a solution of (1.2), *given by*

$$U(t)=(S_3(t)+S_2(t)U_0)(S_1(t)+S_2(t)U_0)^{-1},\quad t\in[0,b].$$

Proof. Let $V(t)$, $Z(t)$ be the $L(H)$ valued functions defined by

$$V(t)=S_1(t)+S_2(t)U_0;\quad Z(t)=S_3(t)+S_4(t)U_0,$$

then (2.2) implies that we must prove that $Z(t)(V(t))^{-1}$ is a solution of (1.2).

It is easy to prove from (2.6) that $\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix}$ satisfies the Cauchy problem (2.3), and a computation shows that

$$d/dt\, U(t) = [d/dt\, Z(t)](V(t))^{-1} - Z(t)(V(t))^{-1}[d/dt\, V(t)](V(t))^{-1}$$
$$= A + BU(t) - U(t)C - U(t)DU(t)$$

with $U(0) = U_0$. As U_0 is a solution of (2.1) satisfying (2.7), postmultiplying by $(V(b))^{-1}$, shows that

$$EU(b) - U(0)F = G.$$

Example 1. If we suppose that H is a finite-dimensional space and denote by W the operator on $H \oplus H$ defined by $W = \begin{bmatrix} P & Q \\ M & N \end{bmatrix}$, J_W the canonical form of W, and $WR = RJ_W$, where

$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \text{ and } R_1 \text{ is invertible,}$$

then by [**14**], the operator $U_0 = R_3 R_1^{-1}$, is a solution of (2.1).

A methodology for obtaining solutions of (2.1) for the infinite-dimensional case is studied in [**5**], by using annihilating analytic functions of the coefficient operators of (2.1). A lot of operators satisfy the property of being annihilated by an analytic function, see, for example, [**2**], [**19**].

3. Boundary problems for operator Lyapunov equations

Let us consider the boundary problem

$$\left.\begin{aligned} d/dt\, U(t) &= A + BU(t) - U(t)C \\ EU(b) - U(0)F &= G \end{aligned}\right\} \tag{3.1}$$

where all operators which appear in (3.1) are bounded linear operators on the Hilbert space H. Note that (3.1) is a particular case of (1.2), where $D = 0$.

Theorem 3. (i) *The problem* (3.1) *has a solution if and only if the algebraic operator equation*

$$M + NU - UP = 0 \tag{3.2}$$

is compatible, where the coefficient operators are given by the expressions

https://doi.org/10.1017/S0013091500017363 Published online by Cambridge University Press

$$\left.\begin{aligned} M &= -G\exp(bC) + E\int_0^b \exp((b-s)B)A\exp(sC)\,ds \\ N &= E\exp(bB);\quad P = F\exp(bC). \end{aligned}\right\} \tag{3.3}$$

(ii) *Under the hypothesis of* (i), *the problem* (3.1) *has only a solution, if and only if, the spectrums* $\sigma(N)$ *and* $\sigma(P)$ *have empty intersection.*

Proof. (i) Let $U(t)$ be a solution of (3.1). It is clear that $V(t)=\exp(tC)$, is a solution in $[0,b]$ of the problem

$$d/dt\, V(t) = CV(t);\quad V(0) = I.$$

The operator valued function $Z(t)=U(t)\exp(tC)$ is a solution of the following problem

$$d/dt\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ A & B \end{bmatrix}\begin{bmatrix} V(t) \\ Z(t) \end{bmatrix};\quad \begin{bmatrix} V(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U_0 \end{bmatrix} \tag{3.4}$$

where $U_0=U(0)$. An easy computation shows that a fundamental solution of (3.4) is given by

$$S(t,s) = \begin{bmatrix} \exp((t-s)C) & 0 \\ \int_0^t \exp((t-v)B)A\exp((v-s)C)\,dv & \exp((t-s)B) \end{bmatrix}.$$

Now, in an analogous way to the proof of Theorem 1, the result is proved. (ii) is a consequence of (i) and the Rosemblun Theorem, [11], p. 8.

Corollary 1. (*The finite-dimensional case*)

(i) *If H is finite-dimensional, then problem* (3.1) *is solvable if and only if the operators*

$$\begin{bmatrix} N & 0 \\ 0 & P \end{bmatrix};\quad \begin{bmatrix} N & -M \\ 0 & P \end{bmatrix}$$

are similar.

(ii) *Under the hypothesis of* (i) *and if* $p(\lambda)=\sum_{k=0}^{n} p_k\lambda^k$ *is the characteristic polynomial of* N, *then the only solution of* (3.2) *is given by*

$$U_0 = \left(\sum_{k=0}^{n} p_k N^k\right)^{-1}\left(-\sum_{k=1}^{n}\sum_{j=1}^{k} p_j N^{j-1} M P^{k-j}\right).$$

Proof. (i) Is a consequence of [16] and (ii) is a consequence of [15].

Corollary 2. *Under the hypothesis of compatibility of* (3.2), *the solution set of* (3.1) *is* *given by the expression*

$$U(t)=\exp(tB)U_0\exp(-tC)+\int_0^t \exp((t-s)B)A\exp((s-t)C)\,ds \tag{3.5}$$

where U_0 *is a solution of* (3.1).

Proof. From the proof of Theorem 3, it follows that

$$S(t,0)=\begin{bmatrix} \exp(tC) & 0 \\ \int_0^t \exp((t-s)B)A\exp(sC)\,ds & \exp(tB)\end{bmatrix}.$$

From Theorem 2, the expression for the solutions of (3.1) is given by (3.5).

Example 2. If N is a right invertible operator and P is a unilateral shift operator, a characterization is given in [1] of the operators M for which the equation (3.2) is solvable.

Example 3. Sufficient conditions in order that (3.2) has a solution in certain classes of operators are given in [**10**], when N and P are selfadjoint operators.

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