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L_n SETS AND THE CLOSURES OF OPEN CONNECTED SETS

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1. Introduction. F. A. Valentine in [**4**] proved the following two theorems.

THEOREM 1. *Let S be a closed connected subset of R^d which has at most n points of local nonconvexity. Then S is an L_{n+1} set.*

THEOREM 2. *Let S be a closed connected subset of R^d whose points of local nonconvexity are decomposable into n closed convex sets. Then S is an L_{2n+1} set.*

These results have been extended by a number of authors, but always with stronger hypothesis. (See [**1**] and [**2**].) Using a minimal arc technique, new proofs of Theorems 1 and 2 were given in [**3**].

Valentine remarks in [**4**] that Theorem 2 might be improved in the case that S is the closure of an open connected set. The goal of this paper is to give such an improvement for sets satisfying a particular local connectivity property.

2. Statement of main result. Following two definitions, we present the main result to be proved in this paper.

Definition 1. A set S is said to be *locally starshaped at* x if there is a neighborhood N_x of x such that $S \cap N_x$ is starshaped with respect to x. A set is locally starshaped if it is *locally starshaped* at each of its points.

Definition 2. A set B in R^d is called *strongly locally convex connected* provided, given $x \in \bar{B}$ (the closure of B) and a neighborhood N_x of x, there exists a convex open neighborhood N_x' of x such that $N_x' \subset N_x$ and $N_x' \cap B$ is connected.

THEOREM 3. *Let S be a set in R^d which is the closure of an open connected, strongly locally convex connected set. Suppose S is locally starshaped and that the points of local nonconvexity of S are decomposable into n closed convex sets, $C_1, \ldots, C_n$. Then S is an L_{n+1} set.*

3. Conventions and notations. If S is a subset of R^d, $C(S)$ and $L(S)$ shall denote the points of local convexity of S and the points of local nonconvexity of S, respectively. The symbol int S shall denote the interior of S. If S is convex, aff S and dim S shall denote the linear variety generated by S and the dimension of that linear variety, respectively. If S is starshaped, the *convex*

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kernel of S is the set of all seeing points of S. If $a, b \in S$, ab shall denote the closed line segment from a to b and (ab), $[ab)$, $(ab]$ shall have the usual meanings.

The *join* of a point x and set S, denoted xS, is defined as $\{xs | s \in S\}$. Throughout the remainder of the paper we shall always assume S is the closure of an open connected, strongly locally convex connected subset of R^d.

If $L(S)$ is decomposable into n closed convex sets $C_1, \ldots, C_n$, we shall say $x \in \bigcup_{i=1}^{n} C_i$ is a *simple point* if x is contained in exactly one C_i. Otherwise, x shall be called a *juncture point*. It follows from the proofs in [**3**] that, given $x, y \in S$, where $L(S)$ is decomposable into n closed convex sets, that an arc from x to y in S of smallest arc length (a *minimal arc*) is a polygonal arc.

4. Preliminary results.

LEMMA 1. *Let S be a subset of R^d with $L(S) = C$ convex and suppose that $x \in C$, $y \in \text{int } S \sim \text{aff } C$ with $xy \subset S$. Then there exists a neighborhood N_y of y such that $xN_y \subset S$.*

Proof. Since $y \notin \text{aff } C$ there is a closed convex neighborhood N_y of y such that $N_y \subset S$ and for $z \in N_y$, $(xz] \cap \text{aff } C = \emptyset$, and so $(xN_y \sim \{x\}) \cap \text{aff } C = \emptyset$. Consider uz for $u \in (xy]$ and $z \in N_y$ and the component T of $S \cap uN_y$ containing $uy \cup yz \subset S$. Since $uN_y \subset xN_y \sim \{x\}$, we have $uN_y \cap C = \emptyset$; since uN_y is convex, any point of local nonconvexity of T is one in S, so T is closed, connected and locally convex and therefore convex by Tietze's theorem. Then $uz \subset T \subset S$ and hence $xz \subset S$. Therefore, $xN_y \subset S$.

LEMMA 2. *Suppose S is a locally starshaped subset of R^d with $L(S) = C$ convex and* $\dim C \leqq d - 2$. *If $x \in C$ and $y \in \text{int } S \sim \text{aff } C$, then $xy \subset S$.*

Proof. Let N_x be an open neighborhood of x such that $S \cap N_x$ is starshaped from x; there exists $u \in \text{int } S \cap N_x$ since $S = \text{int } S$. Let N_u be an open neighborhood about u such that $N_u \subset S \cap N_x$. Since $\dim (\text{aff } C) \leqq d - 2$, $N_u \not\subset \text{aff } C$ and so there exists $z \in N_u \sim \text{aff } C$. But $z \in \text{int } S$ so $z \in \text{int } S \sim \text{aff } C$. Since $\text{int } S$ is connected and $\dim (\text{aff } C) \leqq d - 2$, then $\text{int } S \sim \text{aff } C$ is connected and therefore polygonally connected. Hence, there exists a polygonal arc P in $\text{int } S \sim \text{aff } C$ from z to y. But $xz \subset S$, so suppose w is the last point on P which x can see via S. Now $w \in \text{int } S \sim \text{aff } C$ and so Lemma 1 asserts that w is not the last point which x can see. Therefore, $xy \subset S$.

LEMMA 3. *If S satisfies the hypothesis of Lemma* 2, *then C lies in the convex kernel of S.*

Proof. This follows from Lemma 2 and the fact that $\text{int } S \sim \text{aff } C$ is dense in S.

LEMMA 4. *If S is locally starshaped and $L(S) = C$ is convex, then* $\dim C \leqq d - 2$.

Proof. Suppose dim $C \geqq d - 1$; since C is convex and has empty interior, dim $C = d - 1$. Then $H =$ aff C is a hyperplane and let H^+ and H^- denote the two open half spaces determined by H. Let x be a point in the relative interior of C as a subset of H and let N_x be a neighborhood of x such that $S \cap N_x$ is starshaped with respect to x. Since $x \in L(S)$, there is a point y in (int $S \cap N_x) \sim H$ and without loss of generality, suppose $y \in$ int $S \cap H^+$. Since $xy \subset S$, let T be a maximal convex subset of S containing xy. We claim that dim $(T \cap H) = d - 1$. Note that $y \in$ int T. A polygonal arc lying in int S joining $u \in$ int $T \cap H^+ \subset$ int $S \cap H^+$ to a point $v \in$ int $S \cap H^-$ must meet int $S \cap H$ at some point w and we may assume that w is the first such point from u. By Tietze's theorem it follows that $uw \subset S$ and that if N_w is a convex neighborhood of w such that $N_w \subset S$, then $N_w \cap H^+ \subset T$. Hence, $N_w \cap H \subset T \cap H$ and dim $(T \cap H) = d - 1$. Choose x' in the relative interior of $C \cap (T \cap H) = C \cap T$. Since $x' \in L(S)$, there exists $y' \in$ int $S \cap H^-$ such that $x'y' \subset S$ and let T' be a maximal convex subset of S containing $x'y'$. As before T' meets H in a convex set of dimension $d - 1$ and we may locate an x'' in the relative interior of $C \cap T \cap T'$. But then $x'' \in$ int S, contradicting $x'' \in C = L(S)$. This proves dim $C \leqq d - 2$.

Lemmas 3 and 4 obviously imply the following result.

COROLLARY 1. *Let S be locally starshaped with $L(S) = C$ convex.* Then C lies in the convex kernel of S.

5. Proof of the main result.

Definition 3. A *single segment ab* is one having the property $(ab) \subset C(S)$. A *double segment ab* in S is one having the property $ab \subset L(S)$.

THEOREM 4. *Let S satisfy the hypothesis of Theorem 3. Let $x, y \in S$ and let l be a minimal arc from x to y in S. Let c be a vertex of l, c distinct from x and y. Suppose c is contained in both a single segment and double segment of l. Then c is a juncture point.*

Proof. Suppose $cd \cup ce \subset l$ and $cd \subset L(S)$, $(ce) \subset C(S)$. Assume c is a simple point and that $c \in C_i$. Since c is not a limit point of any C_j, $j \neq i$, there is an open convex set N_c such that $\overline{N_c} \cap C_j = \emptyset$ for $j \neq i$ and $N_c \cap$ int S is connected. Let $S' = \overline{N_c \cap \text{int}\, S}$. Then S' is the closure of an open connected set. Consider $L(S')$. Since $\overline{N_c}$ is convex, any point of local nonconvexity of S' will be a point of local nonconvexity of S, so $L(S') \subset \overline{N_c} \cap L(S) = \overline{N_c} \cap C_i$; we show that $u \in L(S')$. Let $u \in \overline{N_c} \cap C_i$. Then $uc \subset C_i$ and a sequence $\{u_j\} \subset N_c \cap C_i$ converging to u exists and thus sequences $\{v_j\}$ and $\{w_j\}$ in $N_c \cap$ int $S \subseteq S$ converging to u such that $v_j w_j \not\subset S$ exist, which implies $v_j w_j \not\subset \overline{N_c \cap \text{int}\, S} = S'$. Thus $u \in L(S')$ and hence $L(S') = \overline{N_c} \cap C_i$ is convex. Thus S' satisfies the hypothesis of Corollary 1. By Corollary 1, $L(S')$ is in the convex kernel of S'. Thus a point $f \in (cd)$ can see a point of (ce). This implies l could be shortened, a contradiction.

THEOREM 5. *Let S satisfy the hypothesis of Theorem 3. Let $x, y \in S$ and let l be a minimal arc in S joining x and y, where*

$$l = x_0x_1 \cup x_1x_2x \cup \ldots \cup x_kx_{k+1},$$

$x_0 = x$ and $x_{k+1} = y$. Then there exist k distinct integers $j_1, \ldots, j_k$ such that $x_i \in C_{j_i}$, $1 \leqq i \leqq k$.

Proof. We prove the theorem by induction on k. Suppose the theorem is true for integers $< k$ and that a choice for distinct integers $j_1, \ldots, j_{k-1}$ such that $x_i \in C_{j_i}$, $1 \leqq i \leqq k - 1$, has been made. Consider x_k. It is clear $x_k \notin C_{j_i}$ for any $i \leqq k - 2$ or else l could be shortened. If $x_k \in C_{j_{k-1}}$ and $x_k \notin C_i$ for $i \neq j_{k-1}$ then x_k is not a limit point for any other C_i. Thus $c \in (x_kx_{k+1})$ exists such that $(x_kc) \cap C_i = \emptyset$ for $i \neq j_{k-1}$. But $(x_kc) \cap C_{j_{k-1}} = \emptyset$ since otherwise again l could be shortened. Thus $(x_kc) \subset C(S)$ and $x_{k-1}x_k \subset C_{j_{k-1}}$, so x_k is contained in both a single and a double segment of l. By Theorem 4 x_k is a juncture point and $x_k \in C_i$ for some $i \neq j_{k-1}$. Take $j_k = i$; we have $x_k \in C_{j_k}$ with $j_k \neq j_{k-1}, j_1, \ldots, j_{k-1}$ and the induction carries. Thus, by the hypothesis of Theorem 3, $k \leqq n$, proving that S is an L_{n+1} set, which establishes our main theorem.

We remark that Theorem 4 is of interest in its own right since it gives insight into the relationship between vertices of a minimal arc and points of local nonconvexity.

6. An example. Consider in the plane one triangle T_1 contained in another T_2 and these two triangles have a common vertex c. See Figure 1. Define a set S as the set of points lying between T_1 and T_2 including T_1 and T_2. Obtain a set S' in R^3 by translating S' along a segment of length 1 starting at x perpendicular to the plane. Note that the line segment ab generated by the movement

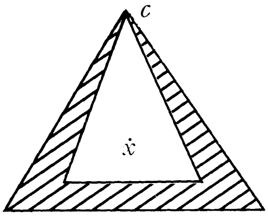

FIGURE 1

of c is contained in $L(S')$. Note that any point $y \in (ab)$ is contained in both a single and double segment of S' but y is not a juncture point. Note S' is not strongly locally convex connected. This shows that Theorem 4 fails unless strong local convex connectedness is assumed.

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